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On the span of Hadamard products of vectors

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Abstract

Let $A_1, \ldots, A_k$ be positive semidefinite matrices and $B_1, \ldots, B_k$ arbitrary complex matrices of order n. We show that

$$\operatorname{span}\left\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) | x \in \mathbb{C}^n\right\} = \operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k)$$

and

$$\operatorname{span}\left\{(B_1x_1) \circ (B_2x_2) \circ \cdots \circ (B_kx_k) | x_j \in \mathbb{C}^n\right\} = \operatorname{range}\left((B_1B_1^*) \circ (B_2B_2^*) \circ \cdots \circ (B_kB_k^*)\right),$$

where $\circ$ means the Hadamard product. This generalizes two recent results of Sun, Du and Liu.

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1. Introduction

For two $m \times n$ matrices $A = (a_{ij})$, $B = (b_{ij})$, their Hadamard product (entrywise product) is defined to be $A \circ B = (a_{ij}b_{ij})$. Note that when $n = 1$ the matrices are column vectors. Given a positive integer k, the kth Hadamard power of A is $A^{(k)} = (a_{ij}^k)$. The book [4] contains many

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results on the Hadamard product. Throughout we consider complex matrices and denote by A^* the conjugate transpose of A. We regard an $n \times n$ matrix A as a linear operator on $\mathbb{C}^n$, so that range(A) is the image of A. Sun et al. [5] have proved the following results.

Theorem 1. *For any $n \times n$ positive semidefinite matrix A and any positive integer k,*

$$\operatorname{span}\left\{(Ax)^{(k)} | x \in \mathbb{C}^n\right\} = \operatorname{range}\left(A^{(k)}\right).$$

Theorem 2. *For any $n \times n$ complex matrix B and any positive integer k,*

$$\operatorname{span}\left\{(Bx)^{(k)} | x \in \mathbb{C}^n\right\} = \operatorname{range}\left((BB^*)^{(k)}\right).$$

Theorem 2 follows immediately from Theorem 1. These two results were conjectured by Gorni and Tutaj-Gasinska [2] in their study related to the well-known Jacobian conjecture which states that if $f : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map and the determinant of the Jacobian matrix of f is a nonzero constant, then f is bijective.

In this note we will generalize Theorems 1 and 2 to the case of Hadamard product of different matrices. The basic ideas in our proof are similar to those in [5], but the proof here is simpler.

2. Main results

We need the following fact, which is known as the principal submatrix rank property [3]. For the sake of completeness, we give a short proof.

Lemma 3. *Let A, B, C be complex matrices such that*

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

is positive semidefinite. Then range$(B) \subseteq$ range(A).

Proof. Since the given block matrix is positive semidefinite, there exists a contraction W such that $B = A^{1/2}WC^{1/2}$ [6, p. 15]. Thus range$(B) \subseteq$ range$(A^{1/2})$ = range(A), where we used the fact that range(G) = range(GG^*) for any complex matrix G. □

Theorem 4. *Let A_j, $j = 1, \ldots, k$ be $n \times n$ positive semidefinite matrices. Then*

$$\operatorname{span}\left\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) | x \in \mathbb{C}^n\right\} = \operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k).$$

Proof. Let e_i be the vector in $\mathbb{C}^n$ whose only nonzero component is the ith component which is equal to 1. Then the ith column of $A_1 \circ \cdots \circ A_k$ is

$$(A_1 \circ \cdots \circ A_k)e_i = (A_1e_i) \circ \cdots \circ (A_ke_i), \quad i = 1, \ldots, n.$$

Therefore

$$\operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k) \subseteq \operatorname{span}\left\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) | x \in \mathbb{C}^n\right\}. \tag{1}$$

It remains for us to prove the reversed inclusion relation

$$\operatorname{span}\left\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) | x \in \mathbb{C}^n\right\} \subseteq \operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k). \tag{2}$$

Let

$$A_j = \left(a_1^{[j]}, a_2^{[j]}, \ldots, a_n^{[j]}\right), \quad j = 1, \ldots, k$$

and $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$. Then

$$(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) = \sum x_{i_1} x_{i_2} \cdots x_{i_k} a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]},$$

where the summation is taken over all tuples $(i_1, i_2, \ldots, i_k)$ with $1 \leqslant i_t \leqslant n$. Hence, to prove (2) it suffices to show

$$a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]} \in \operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k) \tag{3}$$

for all tuples $(i_1, i_2, \ldots, i_k)$ with $1 \leqslant i_t \leqslant n$. For each t with $1 \leqslant t \leqslant k$, there is a permutation matrix P_t such that $a_{i_t}^{[t]}$ is the first column of $A_t P_t$. So $a_{i_1}^{[1]} \circ a_{i_2}^{[2]} \circ \cdots \circ a_{i_k}^{[k]}$ is the first column of $(A_1P_1) \circ (A_2P_2) \circ \cdots \circ (A_kP_k)$. Now, (3) will follow from

$$\operatorname{range}\big((A_1P_1) \circ (A_2P_2) \circ \cdots \circ (A_kP_k)\big) \subseteq \operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k). \tag{4}$$

Next we prove (4). Let I be the identity matrix. Choose an arbitrary but fixed real number r such that r is bigger than the spectral radius of A_j for all $1 \leqslant j \leqslant k$. Then by the Schur complement criterion [6, p. 5] we see that

$$\begin{pmatrix} A_j & A_jP_j \\ P_j^*A_j & rI \end{pmatrix}$$

is positive semidefinite. The Schur product theorem ([1, p. 23] or [6, p. 8]) asserts that the Hadamard product of two positive semidefinite matrices is positive semidefinite. So

$$\begin{aligned} &\begin{pmatrix} A_1 \circ A_2 \circ \cdots \circ A_k & (A_1P_1) \circ (A_2P_2) \circ \cdots \circ (A_kP_k) \\ (P_1^*A_1) \circ (P_2^*A_2) \circ \cdots \circ (P_k^*A_k) & r^kI \end{pmatrix} \\ &\quad = \begin{pmatrix} A_1 & A_1P_1 \\ P_1^*A_1 & rI \end{pmatrix} \circ \begin{pmatrix} A_2 & A_2P_2 \\ P_2^*A_2 & rI \end{pmatrix} \circ \cdots \circ \begin{pmatrix} A_k & A_kP_k \\ P_k^*A_k & rI \end{pmatrix} \end{aligned}$$

is positive semidefinite. Applying Lemma 3 we obtain (4). This completes the proof. □

Relations (1) and (3) in the proof of Theorem 4 yield the following result.

Theorem 5. *Let A_j, $j = 1, \ldots, k$ be $n \times n$ positive semidefinite matrices. Then*

$$\operatorname{span}\big\{(A_1x_1) \circ (A_2x_2) \circ \cdots \circ (A_kx_k) | x_j \in \mathbb{C}^n\big\} = \operatorname{range}(A_1 \circ A_2 \circ \cdots \circ A_k).$$

Combining Theorems 4 and 5 we get the following interesting conclusion: If A_j, $j = 1, \ldots, k$ are $n \times n$ positive semidefinite matrices then

$$\begin{aligned} &\operatorname{span}\big\{(A_1x_1) \circ (A_2x_2) \circ \cdots \circ (A_kx_k) | x_j \in \mathbb{C}^n\big\} \\ &\quad = \operatorname{span}\big\{(A_1x) \circ (A_2x) \circ \cdots \circ (A_kx) | x \in \mathbb{C}^n\big\}. \end{aligned}$$

The direct generalization of Theorem 2 would be

$$\operatorname{span}\big\{(B_1x) \circ (B_2x) \circ \cdots \circ (B_kx) | x \in \mathbb{C}^n\big\} = \operatorname{range}\big((B_1B_1^*) \circ (B_2B_2^*) \circ \cdots \circ (B_kB_k^*)\big)$$

for $n \times n$ complex matrices B_j, $j = 1, \ldots, k$. We point out that this is not true in general. Consider the example $n = 2$,

$$B_1 = I, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\operatorname{span}\big\{(B_1x)\circ(B_2x)|x\in\mathbb{C}^2\big\}=\big\{(\alpha,\alpha)^{\mathrm{T}}|\alpha\in\mathbb{C}\big\}\neq\mathbb{C}^2=\operatorname{range}\big((B_1B_1^*)\circ(B_2B_2^*)\big).$$

This example also shows that the condition that A_j, $j=1,\dots,k$ be positive semidefinite in Theorems 4 and 5 cannot be removed. The correct extension of Theorem 2 seems to be the following result.

Theorem 6. *Let B_j, $j=1,\dots,k$ be $n\times n$ complex matrices. Then*

$$\begin{aligned}&\operatorname{span}\big\{(B_1x_1)\circ(B_2x_2)\circ\cdots\circ(B_kx_k)|x_j\in\mathbb{C}^n\big\}\\&\quad=\operatorname{range}((B_1B_1^*)\circ(B_2B_2^*)\circ\cdots\circ(B_kB_k^*)).\end{aligned}$$

Proof. By Theorem 5 and the fact that $\operatorname{range}(BB^*)=\operatorname{range}(B)$ for any complex matrix B, we have

$$\begin{aligned}&\operatorname{range}\big((B_1B_1^*)\circ(B_2B_2^*)\circ\cdots\circ(B_kB_k^*)\big)\\&\quad=\operatorname{span}\big\{(B_1B_1^*y_1)\circ(B_2B_2^*y_2)\circ\cdots\circ(B_kB_k^*y_k)|y_j\in\mathbb{C}^n\big\}\\&\quad=\operatorname{span}\big\{(B_1x_1)\circ(B_2x_2)\circ\cdots\circ(B_kx_k)|x_j\in\mathbb{C}^n\big\}.\end{aligned}$$

This completes the proof. □

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