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Optimal Symmetric $\mathcal{H}_2$ Controllers for Systems with Collocated Sensors and Actuators[1]

Guang-Hong Yang[2] Li Qiu[3]

Abstract

This paper addresses the problem of designing the optimal symmetric $\mathcal{H}_2$ controller for a plant with collocated sensors and actuators but with possibly asymmetric disturbance injection and performance specification. For such a control problem, the generalized plant has a symmetric block representing the transfer matrix from the control input to the measured output. A complete solution to the optimal $\mathcal{H}_2$ control problem with the symmetric structural constraint is given in terms of the optimal solution to a standard $\mathcal{H}_2$ model matching problem without the constraint.

Keywords: Linear systems; symmetric systems; symmetric controllers; $\mathcal{H}_2$ control; model matching problem.

1 Introduction

There are a large number of systems having symmetric transfer functions in diverse fields, for example, large-space structures with collocated sensors and actuators [8], circuit systems and chemical reactors [1, 11, 17]. Such systems are called symmetric systems, which have received a great deal of investigations [2, 3, 4, 5, 7, 9, 12, 13, 14, 15, 17, 18, 19]. The obtained results show that the qualitative property of symmetry often offers some advantages in the analysis and synthesis of such systems. The realization problem of symmetric systems is addressed in [17]. In [13], it has shown that every symmetric system admits a balanced realization which is parity symmetric. The robustness of symmetric systems under symmetric or diagonal perturbations is examined in [14]. The decentralized control problem for symmetric systems is investigated in [19]. The model reduction problem for a subclass of symmetric systems, namely state-space symmetric systems, is investigated in [12, 15, 18].

For large flexible space structures with collocated sensors and actuators, the problem of designing collocated (symmetric) stabilizing controllers has been addressed in [10], where collocated controllers consist of compatible pairs of sensors and actuators which may be distributed throughout the large flexible space structures. Two types of collocated controllers are considered: (a) collocated attitude controller (CAC), and (b) controllers using velocity feedback including collocated damping enhancement controllers (CDEC), and total velocity feedback controllers (TVFC). The CDEC is used to enhance the structural damping without affecting the rigid modes, while the TVFC additionally stabilizes the rigid motion in the sense that all rigid-body rates also tend to zero. This shows that symmetric controllers are desirable for symmetric systems.

Moreover, it has been shown in [9] that symmetric controllers are superior to asymmetric ones in robust stabilizations of uncertain symmetric systems. For a symmetric generalized plant, it has been shown that both the $\mathcal{H}_2$ optimal controller and the central $\mathcal{H}_\infty$ controller are symmetric [9]. In [4], an equivalent symmetric $\mathcal{H}_\infty$ controller design is presented in terms of solution to a nonlinear matrix equation, but it seems that computing the solution of such a nonlinear matrix equation is not an easy task. However, a system with collocated sensors and actuators does not in general lead to a symmetric generalized system. Rather, only the block in the generalized plant representing the transfer function matrix from the control input to the measured output is symmetric. In such a case, the corresponding $\mathcal{H}_2$ optimal controller and central $\mathcal{H}_\infty$ controller may not be symmetric any more, which brings about the problem of how to design an optimal symmetric controller for such a generalized plant.

This paper will be concerned with the problem of designing optimal symmetric $\mathcal{H}_2$ controllers for systems with the transfer matrix from the control input to the measured output being symmetric. A complete solution to the optimal $\mathcal{H}_2$ control problem with the sym-

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[2]Temasek Laboratories, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Email: tslygh@nus.edu.sg. He was with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore.

[3]Department of Electrical Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email: eeqiu@ee.ust.hk.

metric structural constraint is given in terms of the optimal solution to a standard $\mathcal{H}_2$ model matching problem without constraint.

The following notation will be used through out this paper. For transfer matrices $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and K, the LFT (linear fractional transformation) $\mathcal{F}(P, K)$ is defined by

$$\mathcal{F}(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

The $\mathcal{L}_2$ norm of a transfer matrix $T \in \mathcal{L}_2$ is defined by

$$\| T \|_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{tr} T(j\omega)^* T(j\omega) d\omega.$$

If T is in $\mathcal{H}_2$, then its $\mathcal{L}_2$ norm is also said to be $\mathcal{H}_2$ norm. $\mathcal{RH}_\infty$ denotes the set of stable proper transfer matrices. For $H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}$, $X = \operatorname{Ric}(H)$ means that X is the unique stabilizing solution, if one exists, to the Riccati equation

$$A^T X + XA + XRX + Q = 0.$$

For a transfer matrix P, its conjugate $P^\sim$ is defined by $P^\sim(s) = P^T(-s)$. A transfer matrix $P \in \mathcal{RH}_\infty$ is said to be inner if $P^\sim P = I$ and is said to be co-inner if $PP^\sim = I$.

2 Problem statement

An LTI system P is said to be symmetric if its transfer function matrix is symmetric, i.e.,

$$P(s)^T = P(s).$$

For a state space system

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

if there is a nonsingular symmetric matrix S such that

$$A^T S = SA, \quad C^T = SB, \tag{1}$$

Then it can be easily checked that P is symmetric. On the other hand, it was shown in [17] that if (A, B, C, D) is a minimal realization of P, then there must be a nonsingular symmetric matrix S such that (1) is satisfied.

Consider a feedback system shown in Figure 1. Here G is the so-called generalized plant, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^m$ is the measured output, $z \in \mathbb{R}^s$ is the output to be regulated, and $w \in \mathbb{R}^r$ is the disturbance input. The generalized plant is LTI given by

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2}$$

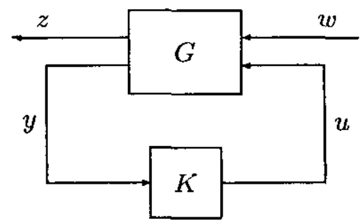

Figure 1: A feedback system

where A, B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, and D_{22} are real constant matrices of appropriate dimensions. In the following, we always assume that the transfer function matrix

$$G_{22}(s) = D_{22} + C_2(sI - A)^{-1}B_2 \tag{3}$$

is symmetric. In this case, we say that the generalized plant G has collocated sensors and actuators.

When applying a controller $u = Ky$ to the plant G, the closed-loop transfer matrix from w to z is given by

$$\mathcal{F}(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}. \tag{4}$$

In this paper, we consider the following optimal symmetric $\mathcal{H}_2$ control problem: Given the generalized plant G described by (2) with $G_{22}(s)$ symmetric, find, if possible, an internally stabilizing symmetric controller K_{opt} such that

$$\| \mathcal{F}(G, K_{opt}) \|_2 = \min\{\| \mathcal{F}(G, K) \|_2 \colon K \text{ is symmetric and internally stabilizes } G\}.$$

Without the requirement for K to be symmetric, the above $\mathcal{H}_2$ problem is reduced to the standard $\mathcal{H}_2$ control problem. When G is symmetric, Ikeda in [9] has shown that both the $\mathcal{H}_2$ optimal controller and the central $\mathcal{H}_\infty$ controller are symmetric. In [4], an equivalent symmetric $\mathcal{H}_\infty$ controller design is presented in terms of the solution to a nonlinear matrix equation, the method of effectively computing the solution of such a nonlinear matrix equation is not available yet. For the case in which it is only assumed that G_{22} is symmetric, the $\mathcal{H}_2$ optimal controller or the central $\mathcal{H}_\infty$ controller is not symmetric in general. The purpose of the paper is to seek an optimal $\mathcal{H}_2$ controller with symmetry for such a generalized plant.

In this paper, the following additional assumptions will be made:

A1: (A, B_2) is stabilizable, and (C_2, A) is detectable.

A2: D_{12} has orthonormal columns and D_{21} has full row rank.

A3: $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

A4: $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

A5: $D_{22} = 0$.

A6: (C_2, A, B_2) is in the form of Kalman Canonical Decomposition

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}, \ B_2 = \begin{bmatrix} B_{21} \\ B_{22} \\ 0 \\ 0 \end{bmatrix} \quad (5)$$

$$C_2 = \begin{bmatrix} C_{21} & 0 & C_{23} & 0 \end{bmatrix} \quad (6)$$

where (C_{21}, A_{11}, B_{21}) is minimal.

Note that A1 is necessary for the considered system to be stabilizable. It is not reasonable to assume that (C_2, A, B_2, D_{22}) is a minimal realization since the matrix A may contain modes of the weighting functions in setting up the $\mathcal{H}_2$ optimal control problem and these modes are usually not controllable or observable from u or y respectively. Regarding A2, in the standard unconstraint $\mathcal{H}_2$ optimal control problem, D_{21} is also assumed to have orthonormal rows in addition to assuming D_{12} has orthonormal columns. This is reasonable because in the unconstrained case the control input and the measured output can be independently scaled. However in the case of systems with collocated sensors and actuators, the control input and the measured output cannot be independently scaled in order to keep the symmetric property of G_{22}. If a scaling in the control input is used to orthonormalize D_{12}, then it also determines the scaling in the measured output by virtue of the symmetry of the scaled G_{22}. A3 and A4 ensure the existence and the uniqueness of the optimal control. A5 is for simplicity and is usually satisfied. If A5 is not satisfied, the following development can be modified in the standard way. Assumption A6 is for the convenience of derivation and it does not lose generality.

3 Parameterization of all symmetric stabilizing controller

Consider the generalized plant G described by (2) with G_{22} symmetric, satisfying assumptions A1–A6. Then A6 implied that there exists a symmetric nonsingular matrix S such that

$$A_{11}^T S = S A_{11}, \quad C_{21}^T = S B_{21} \quad (7)$$

If

$$F = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix}$$

is a matrix such that $A + B_2 F$ is stable, then $A + B_{21} F_1$ is stable. This implies that $A + S^{-1} F_1^T C_{21}$ is stable. Hence there exists

$$L^T = \begin{bmatrix} L_1^T & L_2^T & L_3^T & L_4^T \end{bmatrix}$$

with $L_1 = S^{-1} F_1^T$ such that $A + LC_2$ is stable.

We have the parameterization of all symmetric stabilizing controllers of a generalized plant with collocated sensors and actuator given in the following theorem.

Theorem 1 *Consider the plant G described by (2) with $G_{22}(s)$ symmetric, satisfying assumptions A1–A6. Let*

$$F = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 \end{bmatrix}, L^T = \begin{bmatrix} L_1^T & L_2^T & L_3^T & L_4^T \end{bmatrix}$$

be such that $A + B_2 F$ and $A + LC_2$ are stable with $L_1 = S^{-1} F_1^T$. Then all symmetric controllers that internally stabilize G can be parameterized as

$$\{\mathcal{F}(J, Q) : \ Q \in \mathcal{RH}_\infty \text{ is symmetric}\} \quad (8)$$

where

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A + B_2 F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]. \quad (9)$$

Proof. By the standard theory, the set of all stabilizing controllers for G without the symmetrical structural constraint is given by

$$\{\mathcal{F}(J, Q) : \ Q \in \mathcal{RH}_\infty\}.$$

From (5)-(9), it follows that

$$J = \left[\begin{array}{c|cc} A_{11} + B_{21} F_1 + L_1 C_{21} & -L_1 & B_{21} \\ \hline F_1 & 0 & I \\ -C_{21} & I & 0 \end{array}\right]$$

and

$$\begin{aligned} (A_{11} &+ B_{21} F_1 + L_1 C_{21})^T (-S) \\ &= (-S)(A_{11} + B_{21} F_1 + L_1 C_{21}) \\ \begin{bmatrix} F_1 \\ -C_{21} \end{bmatrix} &= \begin{bmatrix} -L_1 & B_{21} \end{bmatrix}^T (-S) \end{aligned}$$

which implies that J is symmetric. Hence $K = \mathcal{F}(J, Q)$ is symmetric if Q is. Conversely, the mapping from Q to K defined by $K = \mathcal{F}(J, Q)$ is bijective and its inverse is defined by $Q = \mathcal{F}(\bar{J}, K)$, where

$$\bar{J} = \left[\begin{array}{c|cc} A & -L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right]$$

which can be easily shown to be symmetric. Hence if K is a symmetric stabilizing controller, then corresponding Q is also symmetric. $\square$

Theorem 1 indicates that all symmetric stabilizing controllers can be parameterized by stable symmetric transfer function matrices.

It also follows from the standard theory that the set of all possible closed loop transfer matrices is given by

$$\{T_{11} + T_{12}QT_{21} : \ Q \in \mathcal{RH}_\infty \text{ is symmetric}\} \qquad (10)$$

where

$$T_{11} = \left[\begin{array}{cc|c} A + B_2F & -B_2F & B_1 \\ 0 & A + LC_2 & B_1 + LD_{21} \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} \end{array}\right]$$

$$T_{12} = \left[\begin{array}{c|c} A + B_2F & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array}\right]$$

$$T_{21} = \left[\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline C_2 & D_{21} \end{array}\right].$$

Assumptions A2–A4 ensures that $T_{12}(j\omega)$ has full column rank and $T_{21}(j\omega)$ has full row rank for all $\omega \in \mathbb{R} \cup \{\infty\}$.

4 Optimal symmetric $\mathcal{H}_2$ controller

In this section, we will provide a complete solution to the optimal symmetric $\mathcal{H}_2$ control problem.

By the development of the last section, the optimal symmetric $\mathcal{H}_2$ control problem can be reduced to the following model matching problem with symmetric structural constraint: Find a symmetric $Q_{opt} \in \mathcal{RH}_\infty$ such that

$$\| T_{11} + T_{12}Q_{opt}T_{21} \|_2 = \min\{\| T_{11} + T_{12}QT_{21} \|_2 : \ Q \in \mathcal{RH}_\infty \text{ is symmetric } \}.$$

In the following, we will show how the model matching problem with symmetric structural constraint can be solved. Let us start with some algebraic tools. For $X \in \mathbb{F}^{s\times r}$, where $\mathbb{F}$ is any field, define

$$\text{vec}(X) = [x_{11} \cdots x_{s1}\ x_{12} \cdots x_{s2} \cdots x_{sr}]^T.$$

It is well-known that for matrices U, V of compatible sizes,

$$\text{vec}(UXV) = (V^T \otimes U)\text{vec}(X)$$

where "$\otimes$" is the Kronecker product. Let $\mathcal{S}$ be the subspace of $\mathbb{F}^{m\times m}$ consisting of all symmetric matrices. Clearly operator vec is not a bijection between $\mathcal{S}$ and $\mathbb{F}^{m^2}$. A bijection between $\mathcal{S}$ and $\mathbb{F}^{m(m+1)/2}$ is given by

$$\Phi(X) = [x_{11} \cdots x_{m1}\ x_{22} \cdots x_{m2} \cdots x_{mm}]^T.$$

Let E_{ij} be a $m \times m$ matrix whose elements are all zeros except that the (i,j)-th and the (j,i)-th elements are 1. Then $\{E_{ij} : 1 \le j \le i \le m\}$ forms a basis of $\mathcal{S}$. Define $W \in \mathbb{R}^{m^2 \times m(m+1)/2}$ by

$$W = [\text{vec}E_{11} \ \cdots \ \text{vec}E_{m1} \ \text{vec}E_{22} \ \cdots \ \text{vec}E_{m2} \ \cdots \ \text{vec}E_{mm}].$$

Then it is easy to check that for $X \in \mathcal{S}$,

$$\text{vec}(X) = W\Phi(X)$$

and

$$\text{vec}(UXV) = (V^T \otimes U)W\Phi(X).$$

Lemma 1 *$(V^T \otimes U)W$ has full column rank if U has full column rank and V has full row rank.*

Proof: If U has full column rank and V has full row rank, let $U^\dagger$ be a right inverse of U and $V^\dagger$ be a left inverse of V. Then

$$(V^{T\dagger} \otimes U^\dagger)(V^T \otimes U) = I.$$

This shows that $V^T \otimes U$ is right invertible and hence has full column rank. Since W also has full column rank, it follows that $(V^T \otimes U)W$ has full column rank. $\square$

Back to the model matching problem, if we define

$$\tilde{T}_1 = \text{vec}(T_{11}), \quad \tilde{T}_2 = (T_{21}^T \otimes T_{12})W, \quad \tilde{Q} = \Phi(Q),$$

then

$$\begin{aligned} \|T_{11} + T_{12}QT_{21}\|_2 &= \|\text{vec}(T_{11} + T_{12}QT_{21})\|_2 \\ &= \|\tilde{T}_1 + \tilde{T}_2\tilde{Q}\|_2. \end{aligned}$$

Therefore, minimizing $\|T_{11}+T_{12}QT_{21}\|$ subject to symmetric Q in $\mathcal{RH}_\infty^{m\times m}$ is equivalent to minimizing $\|\tilde{T}_1 + \tilde{T}_2\tilde{Q}\|_2$ subject to $\tilde{Q}$ in $\mathcal{RH}_\infty^{m(m+1)/2}$. The latter problem is a standard $\mathcal{H}_2$ model matching problem without structural constraint, which can be solved by using the standard techniques as in [20]. Lemma 1 implies that $\tilde{T}_2(j\omega)$ has full column rank for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if $T_{12}(j\omega)$ has full column rank and $T_{21}(j\omega)$ has full row rank for all $\omega \in \mathbb{R} \cup \{\infty\}$. Hence the existence and uniqueness of this unconstrained model matching problem is ensured by assumptions A3 and A4.

For standard $\mathcal{H}_2$ optimal control problem, state space solutions based on Riccati equations render numerical advantages. In the rest of this section, we provide a state space solution to the optimal symmetric $\mathcal{H}_2$ control problem based on solutions to Riccati equations. Let $D_{12\perp}$ be a matrix such that $[D_{12} \quad D_{12\perp}]$ is unitary. Let

$$X = \text{Ric}\begin{bmatrix} A - B_2 D_{12}^T C_1 & -B_2 B_2^T \\ -C_1^T D_{12\perp} D_{12\perp}^T C_1 & -(A - B_2 D_{12}^T C_1)^T \end{bmatrix}.$$

and

$$F = -(B_2^T X + D_{12}^T C_1)$$

and denote

$$T_{12\perp} = \left[\begin{array}{c|c} A + B_2 F & -X^{+} C_1^T D_{12\perp} \\ \hline C_1 + D_{12} F & D_{12\perp} \end{array}\right].$$

Then $[T_{12} \; T_{12\perp}] \in \mathcal{RH}_\infty$ is square and inner. Thus,

$$\begin{aligned} &\| T_{11} + T_{12} Q T_{21} \|_2^2 \\ &\quad = \| [T_{12} \; T_{12\perp}]^\sim (T_{11} + T_{12} Q T_{21}) \|_2^2 \\ &\quad = \| T_{12}^\sim T_{11} + Q T_{21} \|_2^2 + \| T_{12\perp}^\sim T_{11} \|_2^2 . \quad (11) \end{aligned}$$

Therefore the problem of minimizing $\| T_{11} + T_{12} Q T_{21} \|_2$ is equivalent to minimizing $\| T_{12}^\sim T_{11} + Q T_{21} \|_2$, which is in turn equivalent to minimizing

$$\|\text{vec}(T_{12}^\sim T_{11}) + (T_{21}^T \otimes I) W \Phi(Q)\|.$$

Since T_{21} is given by

$$T_{21} = \left[\begin{array}{c|c} A + L C_2 & B_1 + L D_{21} \\ \hline C_2 & D_{21} \end{array}\right],$$

a realization of $(T_{21}^T \otimes I)W$ is given by

$$\left[\begin{array}{c|c} (A + L C_2)^T \otimes I & (C_2^T \otimes I) W \\ \hline (B_1 + L D_{21})^T \otimes I & (D_{21}^T \otimes I) W \end{array}\right] =: \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \tilde{T}_2.$$

Lemma 1 implies that assumptions A2 and A4 ensures that $\tilde{D}$ has full column rank and $\begin{bmatrix} \tilde{A} - j\omega I & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

Following [20, Thereom 13.35], carry out inner-outer factorization of $\tilde{T}_2$ as

$$\tilde{T}_2 = N M^{-1}$$

and $[N \quad N_\perp] \in \mathcal{RH}_\infty$ is square and co-inner, where

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} \tilde{A} + \tilde{B}\tilde{F} & \tilde{B}(\tilde{D}^T \tilde{D})^{-1/2} \\ \hline \tilde{F} & (\tilde{D}^T \tilde{D})^{-1/2} \\ \tilde{C} + \tilde{D}\tilde{F} & \tilde{D}(\tilde{D}^T \tilde{D})^{-1/2} \end{array}\right]$$

with

$$\tilde{F} = -(\tilde{D}^T \tilde{D})^{-1}(\tilde{B}^T \tilde{X} + \tilde{D}^T \tilde{C}), \quad \tilde{R} = \tilde{D}^T \tilde{D} > 0$$

$$\tilde{X} = \text{Ric}\begin{bmatrix} \tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{D}^T \tilde{C} & -\tilde{B}\tilde{R}^{-1}\tilde{B}* \\ -\tilde{C}^*(I - \tilde{D}\tilde{R}^{-1}\tilde{D}^T) C & -(\tilde{A} - \tilde{B}\tilde{R}^{-1}\tilde{D}^T \tilde{C})^T \end{bmatrix}.$$

Then we have

$$\begin{aligned} \| T_{12}^\sim T_{11} + Q T_{21} \|_2^2 = &\| N^\sim \text{vec}(T_{12}^\sim T_{11}) + M^{-1}\Phi(Q) \|_2^2 \\ &+ \| N_\perp^\sim \text{vec}(T_{12}^\sim T_{11}) \|_2^2 . \end{aligned}$$

Let $N^\sim \text{vec}(T_{12}^\sim T_{11}) = T_s + T_a$, where $T_s \in \mathcal{RH}_\infty$ and $T_a \in \mathcal{H}_2{}^\perp$. Denote

$$\tilde{Q}_{opt} = -M T_s.$$

Then the optimal symmetric controller K is given by $K = \mathcal{F}(J, Q_{opt})$, where J is given by (9) and $Q_{opt} = \Phi^{-1}(\tilde{Q}_{opt})$.

5 A numerical example

Consider a generalized plant

$$\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] =$$

$$\left[\begin{array}{c|cc} \begin{bmatrix} -1 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & 0 \end{array}\right]$$

which satisfies (1) with

$$S = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The optimal $\mathcal{H}_2$ controller K without symmetric structural constraint is given by

$$K_{opt} = \frac{1}{p_0}\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$$

with

$$\begin{aligned} p_0(s) &= s^3 + 10.9296 s^2 + 33.1533 s + 31.6671, \\ q_{11}(s) &= -5.7753 s^2 - 19.1373 s - 19.83, \\ q_{12}(s) &= -6.7510 s^2 - 24.4303 s - 26.2066, \\ q_{21}(s) &= -5.7203 s^2 - 19.9098 s - 20.4966, \\ q_{22}(s) &= -6.6038 s^2 - 24.5814 s - 25.1588 \end{aligned}$$

which is not symmetric. The optimal $\mathcal{H}_2$ performance is 5.965. On the other hand, the optimal symmetric $\mathcal{H}_2$ controller is given by

$$K_{opt} = \frac{1}{p}\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$$

with

$$
\begin{aligned}
p(s) &= s^3 + 157.4177s^2 + 12206s + 621501 \\
q_{11}(s) &= -5.8916s^2 - 872.04s - 63613.5 \\
q_{12}(s) &= -8.2822s^2 - 1254s - 93567 \\
q_{22}(s) &= -2.129s^2 - 308.7429s - 21997.
\end{aligned}
$$

The optimal $\mathcal{H}_2$ performance with symmetric controller is 6.639. As expected, it is seen that the side effect of enforcing symmetry in the controller is that the optimal cost deteriorates by 11.3%. In this example, the order of the optimal symmetric controller is same as that of the optimal controller without symmetric structural constraint. However, the order of the optimal symmetric controller may become higher.

6 Conclusions

In this paper, we have investigated the optimal symmetric $\mathcal{H}_2$ control problem for a generalized plant with a symmetric block representing the transfer function matrix from the control input to the measured output. A complete solution to the optimal $\mathcal{H}_2$ control problem with symmetric structure constraint is given in terms of the optimal solution to a standard $\mathcal{H}_2$ model matching problem without the constraint. For the corresponding $\mathcal{H}_\infty$ control problem, the proposed approach here is not directly applicable; this constitutes a future research topic.

The same problem considered here was also studied in [16] together with a host of other control problems with controller structural constraints, motivated from various applications. The same approach was suggested to solve the problems.

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