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Orthonormal Rational Functions via the Jury Table and Their Applications

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Abstract— **The Jury table is used to construct orthonormal rational functions. Applications of these orthonormal functions in the computation of $\mathcal{H}_2$ norm, the computation of the Hankel singular values and Schmidt pairs, the solutions to the Hankel norm approximation and the Nehari problem are given.**

I. INTRODUCTION

Various orthogonal functions play important roles in science and engineering. Examples include orthogonal polynomials, the standard basis functions in Fourier series or power series, wavelet functions. In this paper, we deal with orthogonal rational functions. The study of orthogonal rational functions has a long history. The idea of decomposing a linear system in term of orthogonal components, such as Laguerre functions, other than the functions in the standard Fourier series dates back to the work of Lee [10] and Wiener [14]. Kautz [9] formulated a more general class of orthogonal rational functions with two parameters. Heuberger et al. [6] developed a theory on construction of orthogonal rational functions using balanced realizations of inner transfer functions. The standard basis functions in power series, Laguerre functions and Kautz functions are special cases in this theory. A further generalization was presented by Ninness and Gustasson [11]. The studies in [6] and [11] are motivated by applications in system identification.

These recently developed orthogonal functions are generated through the balanced realization of inner transfer functions and hence rely on modern state space system theory. Some new investigation of the connection between advanced optimal and robust control problems and the classical tools for continuous time systems is recently carried out by Qiu [13]. It is shown that the Routh table can be used to form orthonormal rational functions, to compute the $\mathcal{H}_2$ norm of a stable transfer function and can also be used to find the Hankel singular values and vectors, hence yielding the solution to the Hankel approximation and the Nehari problems.

The Jury table and the Jury stability criterion are the counterparts of the Routh table and the Routh stability criterion in the discrete time case. In this paper, we will show that the Jury table can also be used to construct orthonormal rational functions, to compute the $\mathcal{H}_2$ norm, to find the Hankel singular values and the corresponding Schmidt pairs and to solve the Hankel approximation and the Nehari problems.

II. JURY STABILITY TEST AND ORTHONORMAL FUNCTIONS

Consider a polynomial

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$

where $a_i \in \mathbb{R}$ and $a_0 > 0$. It is said to be stable if all of its roots are inside the unit disk.

Construct the Jury table [7]

r_0	a_0^0	a_1^0	$\cdots$	a_{n-1}^0	a_n^0
r_0^*	a_n^0	a_{n-1}^0	$\cdots$	a_1^0	a_0^0
r_1	a_0^1	a_1^1	$\cdots$	a_{n-1}^1	
r_1^*	a_{n-1}^1	a_{n-2}^1	$\cdots$	a_0^1	
$\vdots$	$\vdots$				
r_{n-1}	a_0^{n-1}	a_1^{n-1}			
r_{n-1}^*	a_{n-1}^1	a_{n-1}^0			
r_n	a_n^0				

In the Jury table, the first row is copied from the coefficients of the polynomial,

$$a_0^0 = a_0,\ a_1^0 = a_1,\ \ldots,\ a_{n-1}^0 = a_{n-1},\ a_n^0 = a_n.$$

The row r_i^*, $i = 0, \cdots, n-1$, is obtained by writing the elements of the preceding row in the reverse order. The row r_i, $i = 1, \cdots, n$, is computed from its two preceding rows r_{i-1} and r_{i-1}^* as

$$a_j^{i+1} = \frac{1}{a_0^i} \begin{vmatrix} a_j^i & a_{n-i}^i \\ a_{n-i-j}^i & a_0^i \end{vmatrix}, \qquad (1)$$

for $i = 0, \ldots, n-1$, $j = 0, \ldots, n-i-1$.

Theorem 1 (Jury Stability Criterion) [7] *The following statements are equivalent:*
(1) $a(z)$ is stable.
(2) $a_0^i > 0$ for all $i = 1, \ldots, n$.
(3) $|a_0^i| > |a_{n-i}^i|$ for all $i = 0, 1, \cdots, n-1$.

In general, the Jury table cannot be completely constructed when $a_0^i = 0$ for some $1 \leq i < n$. In this case, there is no need to complete the rest of the table since we already know from the Jury stability criterion that the polynomial is unstable.

In this paper, we will see that the utility of the Jury table goes much beyond testing the stability of a polynomial. In particular, it can be used to construct a set of orthonormal rational functions and these orthonormal functions can in turn be used to address various analysis and synthesis issues in system theory.

Let us first recall some frequently used function spaces [8]. Denote the open unit disk by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

$\mathcal{L}_2$ Space

$\mathcal{L}_2$ is the space of square integrable functions on the unit circle, i.e. functions $F(z)$ satisfying

$$\int_{-\pi}^{\pi} \overline{F(e^{j\omega})} F(e^{j\omega}) d\omega < \infty.$$

It is well-known that any $F(z) \in \mathcal{L}_2$ can be represented by

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}.$$

The inner product in this space is defined as

$$\langle F(z), G(z) \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{F(e^{j\omega})} G(e^{j\omega}) d\omega$$

for $F(z), G(z) \in \mathcal{L}_2$ and the induced norm is given by

$$\|F(z)\|_2 = \sqrt{\langle F(z), F(z) \rangle}.$$

$\mathcal{H}_2$ Space

The subspace of $\mathcal{L}_2$ with functions analytic outside of $\mathbb{D}$. It is well-known that any $F(z) \in \mathcal{H}_2$ can be represented by

$$F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}.$$

$\mathcal{H}_2^{\perp}$ Space

The subspace of $\mathcal{L}_2$ with functions analytic in $\mathbb{D}$ and vanish at 0. It is well-known that $\mathcal{H}_2^{\perp}$ is the orthogonal complement of $\mathcal{H}_2$ and any $F(z) \in \mathcal{H}_2^{\perp}$ can be represented by

$$F(z) = \sum_{k=-\infty}^{-1} f(k) z^{-k}.$$

The sets of real rational members of $\mathcal{L}_2, \mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ are denoted by $\mathcal{RL}_2, \mathcal{RH}_2$ and $\mathcal{RH}_2^{\perp}$ respectively. Let $a(z), b(z)$ be polynomials with real coefficients, then these spaces have the following characterizations:

$$\mathcal{RL}_2 = \left\{ \tfrac{b(z)}{a(z)} : \ a(z) \neq 0 \text{ for } z \in \mathbb{T} \right\}$$
$$\mathcal{RH}_2 = \left\{ \tfrac{b(z)}{a(z)} \in \mathcal{RL}_2 : \ a(z) \text{ stable}, \deg b(z) \leq \deg a(z) \right\}$$
$$\mathcal{RH}_2^{\perp} = \left\{ \tfrac{b(z)}{a(z)} \in \mathcal{RL}_2 : \ a(z) \text{ antistable}, \tfrac{b(0)}{a(0)} = 0 \right\}$$

where $a(z)$ being antistable means that all roots of $a(z)$ are outside the unit disk.

Since we are only interested in real rational functions, in the rest of this paper, we will assume that the polynomials considered all have real coefficients.

Let us now fix a stable polynomial

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \ a_0 > 0.$$

Consider the set of strictly proper rational functions with denominator $a(z)$

$$\mathcal{X}_a = \left\{ \frac{b(z)}{a(z)}, \ \deg b(z) < \deg a(z) \right\}. \tag{3}$$

Clearly, $\mathcal{X}_a$ is an n-dimensional subspace of $\mathcal{RH}_2$. In applications, as evidenced later in this paper, it is desirable to find a basis, or better an orthonormal basis of $\mathcal{X}_a$.

The most commonly used basis of $\mathcal{X}_a$ is the standard basis

$$\{F_i(z) = \frac{z^{i-1}}{a(z)}, \ i = 1, 2, \cdots, n.\}$$

In general, this basis is not orthonormal. Using this basis, an orthonormal basis can be constructed by using the Gram-Schmidt orthonormalization process:

$$E_i(z) = \frac{F_i(z) - \sum_{k=1}^{i-1} \langle E_k(z), F_i(z) \rangle E_k(z)}{\|F_i(z) - \sum_{k=1}^{i-1} \langle E_k(z), F_i(z) \rangle E_k(z)\|} \tag{4}$$

for $i = 1, 2, \cdots, n$. Carrying out this orthonormalization process requires the computation of the inner product $\langle E_k(z), F_i(z) \rangle$, which is cumbersome. We will see that this orthonormal basis can be obtained by using the Jury table.

Recall the Jury table of $a(z)$ and for the rows r_i, $i = 1, 2, \ldots, n$, define polynomials

$$\begin{aligned} a_1(z) &= a_0^1 z^{n-1} + a_1^1 z^{n-2} + \cdots + a_{n-1}^1 \\ &\ \vdots \\ a_{n-1}(z) &= a_0^{n-1} z + a_1^{n-1} \\ a_n(z) &= a_0^n. \end{aligned} \tag{5}$$

Since $a(z)$ is stable, $a_0^i > 0$, $|a_0^i| > |a_{n-i}^i|$, for $i = 1, 2, \ldots, n$. We can define

$$\gamma_i = \sqrt{\frac{a_0^0}{a_0^i}}, \ k_i = a_{n-i}^i / a_0^i, \ i = 1, 2, \ldots, n.$$

Theorem 2 *The orthonormal functions $E_i(z)$ satisfy*

$$E_{n-i+1}(z) = \gamma_i \frac{a_i(z)}{a(z)}, \ i = 1, 2, \ldots, n.$$

An alternative orthonormal basis can be given in terms of the reverse versions of the polynomials in (5).

Corollary 1 *Let $\{\tilde{F}_i(z) = \frac{z^{n-i}}{a(z)}, \ i = 1, 2, \ldots, n\}$ be the "reversed" standard basis of $\mathcal{X}_a$ and $\{\tilde{E}_i(z), \ i = 1, 2, \ldots, n\}$ be the functions obtained from the orthonormalization of this basis via the Gram-Schmidt process. The functions $\tilde{E}_i(z), \ i = 1, 2, \ldots, n$, satisfy*

$$\tilde{E}_{n-i+1}(z) = \gamma_i \frac{z^{n-1} a_i(z^{-1})}{a(z)}, \ i = 1, \ldots, n.$$

An orthonormal basis of $\mathcal{H}_2$ can be extended from the orthonormal basis of $\mathcal{X}_a$.

Corollary 2 *Let*

$$G(z) = \frac{z^n a(z^{-1})}{a(z)}$$

be the inner function generated by a stable polynomial $a(z)$ *and* $E_i(z)$ *be the orthonormal functions in Theorem 2. The functions*

$$V_{k\times n+i}(z) = zE_i(z)G^k(z),\ i = 1, \ldots, n,\ k = 0, \cdots, \infty,$$

form an orthonormal basis of $\mathcal{H}_2$.

III. COMPUTATION OF THE $\mathcal{H}_2$ NORM

Consider a stable system

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}{a_0 z^n + a_1 z^{n-1} + \cdots + a_n},\ a_0 > 0.$$

Clearly, $G(z) \in \mathcal{R} \oplus \mathcal{X}_a$. If we let

$$b(z) = \beta_0 a(z) + \beta_1 a_1(z) \cdots + \beta_n a_n(z),$$

then we can expand $G(z)$ as

$$G(z) = \frac{\beta_0}{\gamma_0}E_0(z) + \frac{\beta_1}{\gamma_1}E_{n-1}(z) + \ldots + \frac{\beta_n}{\gamma_n}E_1(z), \quad (6)$$

where $E_0(z) = 1,\ \gamma_0 = 1$ and we can get

$$\|G(z)\|_2^2 = \sum_{i=0}^{n} \frac{\beta_i^2}{\gamma_i^2}.$$

Finding $\beta_i, i = 0, \ldots, n$, is simple. One only need to compare the coefficients in (6) and solve a set of linear equations. It turns out that these equations have special structure and we can obtain the orthonormal basis and these coefficients β_i simultaneously by using the following augmented Jury table.

a_0^0	$\cdots$	a_{n-1}^0	a_n^0	b_n^0	$\cdots$	b_1^0	b_0^0
a_n^0	$\cdots$	a_1^0	a_0^0	a_n^0	$\cdots$	a_1^0	a_0^0
a_0^1	$\cdots$	a_{n-1}^1		b_{n-1}^1	$\cdots$	b_0^1	
a_{n-1}^1	$\cdots$	a_0^1		a_{n-1}^1	$\cdots$	a_0^1	
$\vdots$				$\vdots$			
a_0^{n-1}	a_1^{n-1}			b_1^{n-1}	b_0^{n-1}		
a_1^{n-1}	a_0^{n-1}			a_1^{n-1}	a_0^{n-1}		
a_0^n				b_0^n			

The augmented Jury table is formed by adding one block to the right of the usual Jury table, its first row is directly from the coefficients of $b(z)$:

$$b_0^0 = b_0,\ \ldots,\ b_{n-1}^0 = b_{n-1},\ b_n^0 = b_n.$$

The second, forth, sixth ... rows of the additional block are copied from the corresponding rows in the Jury table and the third, fifth ... rows are computed from its two preceding rows as

$$b_j^{i+1} = \frac{1}{a_0^i}\begin{vmatrix} b_{j+1}^i & b_0^i \\ a_{j+1}^i & a_0^i \end{vmatrix}, \quad (7)$$

for $i = 0, \ldots, n-1,\ j = 0, \ldots, n-i-1$.

In summary, the following algorithm gives the 2-norm of a stable proper transfer function.

Algorithm 1: Computation of the $\mathcal{H}_2$ norm

Step 1 Compute the augmented Jury table of $G(z)$.

Step 2 Set $\beta_i = \frac{b_0^i}{a_0^i},\ i = 0, \ldots, n$.

Step 3 $\|G(z)\|_2^2 = \frac{1}{a_0^0}\sum_{i=0}^{n} \beta_i^2 a_0^i$.

A similar method to compute the $\mathcal{H}_2$ norm also appeared in [2] where the augmented Jury table was defined in a "reverse" way. The method in [2] follows directly by expanding $G(z)$ in terms of the orthonormal basis $\{\tilde{E}_i(z)\}$ of $\mathcal{X}_a$ as in Corollary 1.

Example 1
Consider

$$G(z) = \frac{b(z)}{a(z)} = \frac{\sqrt{2}z + 1/2}{z^2 + \sqrt{2}z + 1/2}.$$

The augmented Jury table of $G(z)$ is

r_0	1	$\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{2}$	0
r_0^*	$\frac{1}{2}$	$\sqrt{2}$	1	$\frac{1}{2}$	$\sqrt{2}$	1
r_1	$\frac{3}{4}$	$\frac{\sqrt{2}}{2}$		$\frac{1}{2}$	$\sqrt{2}$	
r_1^*	$\frac{\sqrt{2}}{2}$	$\frac{3}{4}$		$\frac{\sqrt{2}}{2}$	$\frac{3}{4}$	
r_2	$\frac{1}{12}$			$-\frac{5}{6}$		

The orthonormal basis of $\mathcal{X}_a$ is given by

$$E_1(z) = \frac{\sqrt{12}\frac{1}{12}}{z^2 + \sqrt{2}z + 1/2} = \frac{\frac{\sqrt{3}}{6}}{z^2 + \sqrt{2}z + 1/2},$$

$$E_2(z) = \frac{\sqrt{4/3}(\frac{3}{4}z + \frac{\sqrt{2}}{2})}{z^2 + \sqrt{2}z + 1/2} = \frac{\frac{\sqrt{3}}{2}z + \frac{\sqrt{6}}{3}}{z^2 + \sqrt{2}z + 1/2},$$

and

$$\|G(z)\|_2^2 = 0 + \frac{4}{3}(\sqrt{2})^2 + 12(-\frac{5}{6})^2 = 11.$$

IV. HANKEL SINGULAR VALUES AND SCHMIDT PAIRS

Hankel operators find various applications in engineering problems such as in model reduction [8] and optimal control [17]. Analysis of the Hankel singular values and Schmidt pairs ([1], [3], [4], [15]) is the key for these applications. The recent developments are based on state space realizations, we try to find a new approach from the transfer function point of view by using the orthonormal functions constructed in

Theorem 2. Young [15] studied a similar problem by using a non-orthogonal basis.

Let $P_+ : \mathcal{L}_2 \to \mathcal{H}_2$ and $P_- : \mathcal{L}_2 \to \mathcal{H}_2^\perp$ denote the orthogonal projections such that

$$P_+\left(\sum_{k=-\infty}^{\infty} f(k)z^{-k}\right) = \sum_{k=0}^{\infty} f(k)z^{-k},$$
$$P_-\left(\sum_{k=-\infty}^{\infty} f(k)z^{-k}\right) = \sum_{k=-\infty}^{-1} f(k)z^{-k}.$$

Let $J : \mathcal{L}_2 \to \mathcal{L}_2$ denote the reversal operator and $S : \mathcal{L}_2 \to \mathcal{L}_2$ denote the backward shift operator such that

$$JF(z) = F(z^{-1})$$
$$SF(z) = zF(z).$$

Clearly J and S are both unitary operators. For any $F(z) = \frac{\alpha(z)}{a(z)} \in \mathcal{X}_a$, we have

$$JF(z) = F(z^{-1}) = \frac{\alpha^*(z)}{a^*(z)},$$

where $a^*(z) = z^n a(z^{-1})$ and $\alpha^*(z) = z^n \alpha(z^{-1})$.

Definition *Given a stable system with strictly proper transfer function $G(z)$, the associated Hankel operator $H_G : \mathcal{H}_2^\perp \to \mathcal{H}_2$ is defined by*

$$H_G U(z) = P_+(G(z)U(z)),\ U(z) \in \mathcal{H}_2^\perp.$$

It is well-known that H_G is a finite rank operator when $G(z)$ is rational.

Lemma 1 [3] *Let $G(z) = \frac{b(z)}{a(z)}$ be a strictly proper stable transfer function. Then*

$$ImH_G = S\mathcal{X}_a,$$
$$(KerH_G)^\perp = J\mathcal{X}_a.$$

The Hankel operator H_G is the orthogonal direct sum of a zero operator and a compression of H_G mapping $J\mathcal{X}_a$ into $S\mathcal{X}_a$. Everything interesting about it is contained in this compressed part.

This compressed Hankel operator can be represented by a matrix if we choose a basis in $(\mathrm{Ker}H_G)^\perp$ and a basis in $\mathrm{Im}H_G$. Note that both $(\mathrm{Ker}H_G)^\perp$ and $\mathrm{Im}H_G$ are isomorphic to $\mathcal{X}_a$, so we can use the orthonormal basis of $\mathcal{X}_a$

$$\{\ E_n(z),\ \ E_{n-1}(z),\ \ \cdots,\ \ E_1(z)\ \}$$

defined in Theorem 2 to form an orthonormal basis in $(\mathrm{Ker}H_G)^\perp$

$$\{\ E_n(z^{-1}),\ \ E_{n-1}(z^{-1}),\ \ \ldots,\ \ E_1(z^{-1})\ \}$$

and one in $\mathrm{Im}H_G$

$$\{\ zE_n(z),\ \ zE_{n-1}(z),\ \ \ldots,\ \ zE_1(z)\ \}.$$

The matrix representation under this basis is denoted by Γ_G. The singular values of Γ_G are called the Hankel singular values of $G(z)$ and are denoted by $\sigma_1, \sigma_2, \ldots, \sigma_n$. We assume that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n.$$

The largest singular value is called the Hankel norm of $G(z)$ and is denoted by $\|G(z)\|_H$. Let (u_i, v_i) be a left and right singular vectors of Γ_G corresponding to σ_i and let

$$U_i(z) = [\ zE_1(z)\ \ zE_2(z)\ \ \cdots\ \ zE_n(z)\ ]u_i$$
$$V_i(z) = [\ E_1(z^{-1})\ \ E_2(z^{-1})\ \ \cdots\ \ E_n(z^{-1})\ ]v_i.$$

Then $(U_i(z), V_i(z))$ is called a Schmidt pair of H_G corresponding to σ_i.

Since the matrix representation Γ_G depends on the choice of the basis, it seems that the Hankel singular values and the corresponding Schmidt pairs also depend on the choice of basis. Actually this is not the case. As long the basis is an orthonormal one, we will end up with the same singular values and Schmidt pairs.

We are interested in computing the Hankel singular values and Schmidt pairs of H_G, the key is to find Γ_G from $G(z) = \frac{b(z)}{a(z)}$.

Theorem 3 *Construct the Jury table of $a(z)$. Define matrix A as in (9) and M as:*

$$M = \begin{bmatrix} \gamma_1 a_0^1 & 0 & \cdots & 0 \\ \gamma_1 a_1^1 & \gamma_2 a_0^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \gamma_1 a_{n-1}^1 & \gamma_2 a_{n-2}^2 & \cdots & \gamma_n a_0^n \end{bmatrix}.$$

Then

$$\Gamma_G = a^*(A)^{-1} b(A) M^{-1} \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & .\cdot\cdot & \vdots \\ 1 & \cdots & 0 \end{bmatrix} M. \qquad (8)$$

$$A = \begin{bmatrix} -k_0k_1 & \gamma_1/\gamma_2 & \cdots & 0 & 0 \\ -k_0k_2\gamma_1/\gamma_2 & -k_1k_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -k_0k_{n-1}\gamma_1/\gamma_{n-1} & -k_1k_{n-1}\gamma_2/\gamma_{n-1} & \cdots & -k_{n-2}k_{n-1} & \gamma_{n-1}/\gamma_n \\ -k_0k_n\gamma_1/\gamma_n & -k_1k_n\gamma_2/\gamma_n & \cdots & -k_{n-2}k_n\gamma_{n-1}/\gamma_n & -k_{n-1}k_n \end{bmatrix}. \qquad (9)$$

The adjoint Hankel operator $H_G^*: \mathcal{H}_2 \to \mathcal{H}_2^\perp$ is given by

$$H_G^* U(z) = P_-(G(z^{-1})U(z)),\ U(z) \in \mathcal{H}_2$$

and

$$\begin{aligned} \mathrm{Im} H_G^* &= J\mathcal{X}_a, \\ (\mathrm{Ker} H_G^*)^\perp. &= S\mathcal{X}_a. \end{aligned}$$

Corollary 3 *The adjoint Hankel operator H_G^* satisfies*

$$H_G^* = SJH_GSJ. \tag{10}$$

Remark: By definition, the matrix representation of H_G^* is the transpose of that of H_G. Hence Corollary 3 implies that Γ_G is symmetric.

Since Γ_G is symmetric, it is easy to show that

$$U_i(z) = \epsilon z V_i(z^{-1}) = \epsilon SJV_i(z) \tag{11}$$

where $\epsilon = \pm 1$. This fact may offer some simplification in the computation. We also give the following algorithm to find the Hankel matrix Γ_G, its singular values and corresponding Schmidt pairs.

Algorithm 2: Computation of Hankel matrix Γ_G, its singular values and corresponding Schmidt pairs.

Step 1 Construct the Jury table of $G(z)$.

Step 2 Construct matrices A and M as in Theorem 4.

Step 3 Use (8) to compute Γ_G.

Step 4 Use MATLAB command

$$\texttt{[u,s,v]=svd}(\Gamma_G)$$

to get the singular value decomposition of Γ_G.

Step 5 The singular values of Γ_G are given by

$$\sigma_i = s_{ii},\ i = 1, 2, \cdots, n,$$

where s_{ii} is the i-th diagonal element of s.

Step 6 The corresponding Schmidt pairs are given by

$$\begin{aligned} U_i(z) &= \left[\ zE_n(z) \ \ \cdots \ \ zE_1(z)\ \right] u_i, \\ V_i(z) &= \left[\ E_n(z^{-1}) \ \ \cdots \ \ E_1(z^{-1})\ \right] v_i, \end{aligned}$$

where u_i and v_i are the i-th column of u and v.

Example 2
Consider

$$G(z) = \frac{b(z)}{a(z)} = \frac{\sqrt{2}z + 1/2}{z^2 + \sqrt{2}z + 1/2}.$$

From Example 1, we can get

$$A = \begin{bmatrix} -\frac{\sqrt{2}}{3} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{2\sqrt{2}}{3} \end{bmatrix},\ M = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \sqrt{\frac{2}{3}} & \frac{1}{12} \end{bmatrix},$$

and

$$\Gamma_G = \begin{bmatrix} 1.8856 & -3.3333 \\ -3.3333 & 3.7712 \end{bmatrix}.$$

The singular values of Γ_G are

$$\sigma_1 = 6.2925,\ \sigma_2 = 0.6357,$$

and the corresponding singular vectors are

$$\begin{bmatrix} u_1 & u_2 \end{bmatrix} = \begin{bmatrix} -0.6033 & 0.7975 \\ 0.7975 & -0.6033 \end{bmatrix},$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} -0.6033 & -0.7975 \\ 0.7975 & -0.6033 \end{bmatrix}.$$

The corresponding Schmidt pairs are given by

$$\begin{aligned} U_1(z) &= -\frac{0.52z^2 + 0.26z}{z^2 + \sqrt{2}z + 0.5} \\ V_1(z) &= -\frac{0.26z^2 + 0.52z}{0.5z^2 + \sqrt{2}z + 1} \\ U_2(z) &= \frac{0.69z^2 + 0.83z}{z^2 + \sqrt{2}z + 0.5} \\ V_2(z) &= -\frac{0.83z^2 + 0.69z}{0.5z^2 + \sqrt{2}z + 1}. \end{aligned}$$

V. HANKEL APPROXIMATION AND THE NEHARI PROBLEM

In this section, we first have a look at the theory of Hankel norm approximation problem. Given a stable system with strictly proper transfer function, we want to find a lower order system to approximate the high order system so that the Hankel norm of the error is minimized.

Theorem 4 *Let $(U_{k+1}(z), V_{k+1}(z))$ be the Schmidt pair of H_G corresponding to $(k+1)$-st Hankel singular value σ_{k+1}. Then*

$$\min_{\text{order } \tilde{G}(z) \le k} \|G(z) - \tilde{G}(z)\|_H = \sigma_{k+1},$$

and the unique minimizing $\tilde{G}(z)$ is given by

$$\tilde{G}(z) = G(z) - P_+\left[\sigma_{k+1}\frac{U_{k+1}(z)}{V_{k+1}(z)}\right].$$

Example 3
We wish to find the 1st order Hankel approximation $\tilde{G}(z)$ of

$$G(z) = \frac{\sqrt{2}z + 1/2}{z^2 + \sqrt{2}z + 1/2}.$$

From Example 2, we can get

$$\min_{\text{order } \tilde{G}(z) \leq k} \|G(z) - \tilde{G}(z)\|_H = \sigma_2(G(z)) = 0.6357$$

and the best approximation is given by

$$\begin{aligned} \tilde{G}(z) &= G(z) - P_+ \left[\sigma_2 \frac{U_2(z)}{V_2(z)} \right] \\ &= \frac{0.22z + 1.74}{0.83z + 0.69}. \end{aligned}$$

The Nehari problem [12] is another approximation problem with respect to the $\mathcal{L}_\infty$ norm: Given a stable strictly proper system $G(z) = \frac{b(z)}{a(z)}$, find $Q(z) \in \mathcal{H}_\infty$ to minimize

$$\|G(z^{-1}) - Q(z)\|_\infty.$$

Theorem 5 *Let* $(U_1(z), V_1(z))$ *be the Schmidt pair of* H_G *corresponding to the largest Hankel singular value* σ_1. *Then*

$$\min_{Q(z) \in \mathcal{H}_\infty} \|G(z^{-1}) - Q(z)\|_\infty = \sigma_1,$$

and the unique minimizing $Q(z)$ *is given by*

$$Q(z) = G(z^{-1}) - \sigma_1 \frac{U_1(z^{-1})}{V_1(z^{-1})}.$$

Example 4
For

$$G(z) = \frac{\sqrt{2}z + 1/2}{z^2 + \sqrt{2}z + 1/2},$$

We wish to find $Q(z) \in \mathcal{H}_\infty$ to minimize

$$\|G(z^{-1}) - Q(z)\|_\infty.$$

From Theorem 6 and Example 2, we can get

$$\min_{Q(z) \in \mathcal{H}_\infty} \|G(z^{-1}) - Q(z)\|_\infty = 6.2925,$$

and the unique minimizing $Q(z)$ is given by

$$\begin{aligned} Q(z) &= G(z^{-1}) - \sigma_1 \frac{U_1(z^{-1})}{V_1(z^{-1})} \\ &= -\frac{2.78z + 1.64}{0.52z + 0.26}. \end{aligned}$$

Theorem 4 and Theorem 5 are known as part of the AAK theory [1] and see also [3], [16]. The novalty here is that the required Schmidt pairs can be computed by means of the orthonormal functions generated from the Jury table.

VI. CONCLUSION

An algorithm of finding orthonormal rational functions from the Jury table is given in this paper. Applications of these orthonormal functions include the calculation of $\mathcal{H}_2$ norm, computation of the Hankel singular values and Schmidt pairs, the solutions to the Hankel norm approximation and the Nehari problem.

VII. ACKNOWLEDGMENTS

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