Proceedings of the 5th World Congress on Intelligent Control
and Automation, June 15-19, 2004, Hangzhou, P.R. China

第五届全球智能控制与自动化大会
2004 年 6 月 15 － 19 日，中国杭州

Performance Limitation in Random Sinusoidal Signal Estimation

Weizhou Su, Wei Xing Zheng, Li Qiu

Abstract— **This paper studies the performance limitations of an LTI SISO discrete-time system in estimating a type of random input signal from its output signal. The input signal under consideration is a random sinusoidal signal. The estimation performance is measured by the energy of certain averaged error between the output of the estimator and the signal under estimation. Our purpose is to find the fundamental limit of the best attainable estimation performance under any estimator structure and parameters, in terms of the characteristics of the given system as well as the signal under estimation. It is shown that the fundamental limit depends on the interaction between the signal under estimation and the nonminimum phase zeros of the system. Moreover, this result is extended to optimal prediction and smoothing problems.**

Keywords—**Performance limitations, optimal estimation, prediction, smoothing, periodic random signals.**

I. Introduction

It has been known that all systems in the estimation problem have the best attainable performance under any possible design. This best attainable performance is often called the *performance limit* of systems in the literature. The knowledge on the performance limit of systems provides not only a deeper insight for the systems but also certain design benchmarks for the problems under consideration. In the estimation theory, the Cramér-Rao bound (see, for example, [15]) is the most famous performance limit of the optimal estimation. Recently, there appear many interesting works on performance limitations in various estimation problems. For instance, under certain criterion, Weiss *et al.* [16] obtained an explicit minimal estimation error of subband adaptive filters. In [2], the best achievable accuracy in estimating the parameters of a random-amplitude sinusoid from its sample covariances was derived. The performance limitations of optimal estimation are discussed for linear time-invariant (LTI) systems in [9] and [3]. As a common feature of systems, the performance limitation also appears in a variety of forms for other problems, such as Shannon capacity of a communication channel [4], Bode integral of a feedback system [10], and the optimal tracking performance limitations of an LTI system (see [13] and references therein).

In this paper, we address the performance limits in estimating the input of an LTI discrete-time system from its output. The input signal is a random sinusoidal signal (see, for example, [8] and [15]). The LTI system could be transmission channels of the random signal, measurement devices or any other possible practical systems. Indeed, random sinusoidal signals have many applications in areas such as communications, mechanical vibrating systems and signal processing (see [12], [1], [5] and references therein). During the last twenty years, various algorithms have been developed for parameter estimation of the signal (see [1]-[7], [11], [12] and references therein). The existing works usually assume that the signal is perfectly known. However, in real systems, the signals could be distorted by transmission channels, measurement devices, etc. To remove the distortions, certain signal estimation methods are employed. The problem under consideration in this paper is to find some fundamental limitations on the signal estimation from the system's characteristics and some signal's parameters under all possible estimator designs. Since random sinusoidal signals include many different types, we select a typical random sinusoidal signal as the design benchmark. In this case, the signal is a sinusoid with amplitude and phase related to two independent random walk processes. Our results show that the estimation performance limit of the system depends on the nonminimum phase zeros and the frequency of the sinusoid. Furthermore, the performance limits of prediction and smoothing are addressed. It is revealed that the prediction performance limit increases as the prediction step increases, while the smoothing performance limit approaches zero as the smoothing step tends to infinity.

The notation used throughout this paper is fairly standard. For any complex number denote their conjugate, real and imaginary part by $(\cdot)^*$, R $\cdot$) and I $\cdot$), respectively. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open unit disk and the unit circle be denoted by $\mathbb{D}$ and $\mathbb{T}$, respectively. The usual Lebesgue space of all possibly square integral functions on $\mathbb{T}$ is denoted by $\mathcal{L}_2$. The norm in the space $\mathcal{L}_2$ is denoted by $\|\cdot\|_2$, where the norm of a function $f(\lambda)$ in the space $\mathcal{L}_2$ is defined by

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{-j\quad})f(e^{-j\quad})d\quad\right)^{1/2}.$$

The set of those functions in $\mathcal{L}_2$, which are analytic on the complementary set of $\mathbb{D}\cup\mathbb{T}$, is denoted by $\mathcal{H}_2$, and the set of those functions in $\mathcal{L}_2$, which are analytic on $\mathbb{D}$ and vanish at the origin, is denoted by $\mathcal{H}_2^{\perp}$. It is well known that $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ are indeed orthogonally complementary to each other as subspaces of $\mathcal{L}_2$. $\mathcal{H}_\infty$ is the set of all stable functions (see, for example [17]). For any complex function $f(\lambda)$, denote the function $f^*(\frac{1}{\lambda^*})$ by $f^{\sim}(\lambda)$.

This work was supported in part by the Hong Kong Research Grants Council and in part by the Australian Research Council.

Weizhou Su and Wei Xing Zheng are with School of QMMS, University of Western Sydney, Penrith South DC, NSW 1797, Australia w.su@uws.edu.au, w.zheng@uws.edu.au

Li Qiu is with Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China eeqiu@ee.ust.hk

0-7803-8273-0/04/$20.00 ©2004 IEEE

II. Problem Formulation

In this section, we will formulate the problems to be studied. The system shown in Figure 1 is considered. Here

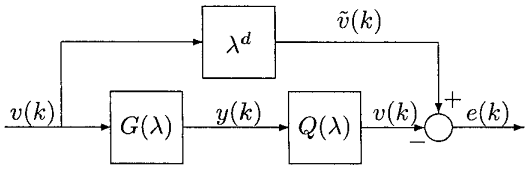

Fig. 1. An estimator structure

$\lambda = z^{-1}$ is a unit delay operator; $G(\lambda)$ is the given plant transfer function; and $Q(\lambda)$ is the estimator transfer function which is to be developed. The parameter d in the part of λ^d depends on the problems under consideration. For an estimation problem, d is selected as $d = 0$. For a prediction problem, this parameter is selected a negative integer (i.e., $d \quad 0$) and $|d|$ denotes the prediction step, while for a smoothing problem, d is selected a positive integer (i.e., $d \quad 0$) and d stands for the smoothing step. $y(k)$ is the measured signal, $v(k)$ is the transmitted signal, $\tilde{v}(k)$ is the signal under estimation and $\hat{v}(k)$ is its estimate. Denote the λ-transform of $v(k)$ by $V(\lambda)$.

The estimation performance is measured by the energy of estimation error:

$$J = \boldsymbol{E}\left\{\sum_{k=0}^{\infty} \|\tilde{v}(k) - \hat{v}(k)\|^2\right\} = \boldsymbol{E}\left\{\sum_{k=0}^{\infty} \|e(k)\|^2\right\}. \quad (1)$$

It follows from Parseval's relationship and the structure of the system that

$$\begin{aligned} J &= \boldsymbol{E}\left\{\|[\lambda^d - Q(\lambda)G(\lambda)\; V(\lambda)\|_2^2\right\} \\ &= \frac{1}{2\pi}\int_0^{2\pi} [e^{-j\ } \ - Q(e^{-j\ })G(e^{-j\ })\ S(e^{-j\ }) \\ &\quad \times [e^{-j\ } \ - Q(e^{-j\ })G(e^{-j\ })\ ^* d \end{aligned} \quad (2)$$

where $S(e^{-j\ })$ is the power spectrum of the signal $v(k)$. The signal to be estimated is given by

$$v(k) \quad a_{-1}(k)e^{-j\ _o k} + a_1(k)e^{j\ _o k} \quad (3)$$

where $a_{-1}(k)$ and $a_1(k)$ are random amplitudes and mutually independent. Furthermore, $a_{-1}(k)$ and $a_1(k)$ are generated by the autoregressive (AR) processes

$$a_l(k) \quad a_l(k-1) \quad u_l(k), \qquad l = -1, 1 \quad (4)$$

where $\{u_l(k)\}$ are independent and identically distributed (i.i.d.) normal random variables with

$$\boldsymbol{E}\{u_l(k)\} = 0, \quad r_{u_l}(\tau) \quad \boldsymbol{E}\{u_l(k)u_l(k+\tau)\} = \delta(\tau).$$

The amplitude model (4) generates a random walk process.

The autocorrelation function of the signal $v(k)$ in (3) is

$$\begin{aligned} r(\tau) \quad & \boldsymbol{E}\{a_{-1}(k)a_{-1}^*(k-\tau)\}\, e^{-j\ _o k}e^{j\ _o(k-\tau)} \\ & + \boldsymbol{E}\{a_1(k)a_1^*(k-\tau)\}\, e^{j\ _o k}e^{-j\ _o(k-\tau)} \\ & = r_{a_{-1}}(\tau)e^{-j\ _o\tau} + r_{a_1}(\tau)e^{j\ _o\tau}. \end{aligned} \quad (5)$$

Following the discussion in [1], its power spectrum is

$$\begin{aligned} & S(\lambda) \\ & = \frac{1}{1-\lambda\ ^{j\ _o}}\frac{1}{1-e\lambda^{-1}e^{-j\ _o}} + \frac{1}{1-\lambda\ ^{-j\ _o}}\frac{1}{1\ e\ \lambda^{-1}e^{j\ _o}}. \end{aligned} \quad (6)$$

A minimum phase $\hat{V}(\lambda)$ is obtained via spectral factorization such that

$$S(\lambda) \quad \hat{V}(\lambda)\hat{V}^{\sim}(\lambda) \quad (7)$$

and

$$\hat{V}(\lambda) \quad \begin{cases} \dfrac{e^{j(\frac{\omega_0}{2}-\frac{\pi}{4})}}{1-\lambda\ ^{j\ _o}} + \dfrac{e^{-j(\frac{\omega_0}{2}-\frac{\pi}{4})}}{1e-\lambda\ ^{-j\ _o}}, \quad e \qquad \omega_0 \in [0, \pi) \\ \dfrac{e^{j(\frac{\omega_0}{2}-\frac{3\pi}{4})}}{1-\lambda\ ^{j\ _o}} + \dfrac{e^{-j(\frac{\omega_0}{2}-\frac{3\pi}{4})}}{e-\lambda\ ^{-j\ _o}}, \quad e \qquad \omega_0 \in [\pi\ \ 2\pi). \end{cases} \quad (8)$$

The estimation performance J in (2) is now written as

$$J = \|[\lambda^d - Q(\lambda)G(\lambda)\ \hat{V}(\lambda)\|_2^2. \quad (9)$$

Noticing (2) and (7), the function $\hat{V}(\lambda)$ is an averaged function of $V(\lambda)$ in certain sense.

To guarantee its numerical stability, the estimator $Q(\lambda)$ is selected from $\mathcal{H}_\infty$. Our task is to find an explicit form of the smallest averaged estimation error, i.e., the performance limit of the system in estimating the random sinusoidal signal (3),

$$J_o \quad = \inf_{Q\in\mathcal{H}_\infty} J \quad (10)$$

Let $G(\lambda)$ be a real rational function representing the transfer function of an LTI discrete time system and $q_1, q_2, \ldots, q_m$ be the nonminimum phase zeros of $G(\lambda)$. Assume that each pair of complex conjugate zeros is ordered in adjacent order. Denote a Blaschke factor associated with nonminimum zero q_i by

$$G_i(\lambda) \quad \frac{\lambda - q_i}{q_i^*\lambda - 1}.$$

Then $G(\lambda)$ can be factorized as

$$G(\lambda) \quad G_1(\lambda)\cdot\cdot G_m(\lambda)G_o(\lambda). \quad (11)$$

where $G_0(\lambda)$ has no nonminimum phase zeros. This factorization is referred to as the inner-outer factorization. The product

$$G_i\ (\lambda) \quad G_1(\lambda)\cdot\cdot G_m(\lambda)$$

is called a Blaschke product.

III. Performance Limits of Optimal Estimation and Prediction

Let us go back to the setup shown in Figure 1. In order for the estimation problem to be meaningful and solvable, we make the following assumption throughout the paper.

Assumption 1: $G(\lambda)$ has no zeros at $e^{-j\ _o}$ and $e^{j\ _o}$.

Now we are ready to discuss the minimal value of the estimation performance function J. When $d = 0$, a nice and explicit formula is obtained for the case where the periodic random signal $v(k)$ is

$$v(k) \quad a_{-1}(k)e^{-j \ _0 k} + a_1(k)e^{j \ _0 k}$$

where $a_{-1}(k)$ and $a_1(k)$ are the AR processes given by (4) and mutually independent. This result will then be extended to the prediction problem.

Theorem 1: Let $G(\lambda)$ have nonminimum phase zeros $q_1, \cdots, q_m$. Then the performance limit of optimal estimation is given by

$$J_o \ = \sum_{i=1}^{m} \left(\frac{1+q_i e^{-j \ _0}}{1-q_i e^{-j \ _0}} + \frac{1+q_i e^{j \ _0}}{1-q_i e^{j \ _0}} \right) + \frac{2}{|\mathrm{s} \quad \omega_0|} \mathrm{s} \quad ^2 2 \sum_{i=1}^{m} \left[\angle(\quad - q_i e^{-j \ _0}) \quad \frac{\omega_0}{2} \right].$$

Note that due to limited space, only the proof of Theorem 1 is presented in Appendix A. The proofs of the other theorems in the remainder of this paper are omitted, but the details can be found in [14].

We now discuss the performance limit of optimal prediction. In this problem, the parameter d in Figure 1 is selected as a negative integer (i.e., $d \quad 0$) while the prediction step is $|d| (> 0)$. The performance function is given by

$$\begin{aligned} J &= \|[\lambda^d - Q(\lambda)G(\lambda) \ \hat{V}(\lambda)\|_2^2 \\ &= \|[1 - \lambda^{|d|} Q(\lambda)G(\lambda) \ \hat{V}(\lambda)\|_2^2. \end{aligned} \qquad (12)$$

This equality shows that the prediction problem for the system is equivalent to an estimation problem of the system with extra $|d|$ step delays. Since a unit delay causes one nonminimum phase zero at the origin, the system $\lambda^{|d|}G(\lambda)$ has $|d|$ nonminimum phase zeros at the origin in addition to the nonminimum phase zeros of the system $G(\lambda)$. Therefore, straightforwardly applying Theorem 1 leads to following result.

Theorem 2: Let $G(\lambda)$ have nonminimum phase zeros $q_1, \cdots, q_m$. Then, the performance limit of $|d|$-step optimal prediction is given by

$$J_o \ = 2|d| + \sum_{i=1}^{m} \left(\frac{1+q_i e^{-j \ _0}}{1-q_i e^{-j \ _0}} + \frac{1+q_i e^{j \ _0}}{1-q_i e^{j \ _0}} \right) + \frac{2}{|\mathrm{s} \quad \omega_0|} \mathrm{s} \quad ^2 \left\{ |d|\omega_0 + 2 \sum_{i=1}^{m} \left[\angle(\quad - q_i e^{-j \ _0}) \quad \frac{\omega_0}{2} \right] \right\}.$$

It should be mentioned that the results in Theorem 1 and 2 can be extended to the case where the signal has multiple frequencies (see [14] for details).

IV. Performance Limit of Optimal Smoothing

In this section, the performance function under consideration is given by

$$J = \|[\lambda^d - Q(\lambda)G(\lambda) \ \hat{V}(\lambda)\|_2^2 \qquad (13)$$

where $d \quad 0$ is the delay steps in smoothing and the averaged λ-transform of the signal under estimation $\hat{V}(\lambda)$ is given in (8).

Theorem 3: Suppose that the system $G(\lambda)$ has only one nonminimum phase zero q_1. Then the smoothing performance limit is given by

$$J_o \ = q_1^d q_1^{*d} \left(\frac{1+q_1 e^{-j \ _0}}{1-q_1 e^{-j \ _0}} + \frac{1+q_1 e^{j \ _0}}{1-q_1 e^{j \ _0}} \right) + q_1^d q_1^{*d} \frac{2}{|\mathrm{s} \quad \omega_0|} \mathrm{s} \quad ^2 2 \left[\angle(\quad - q_1 e^{-j \ _0}) \quad \frac{\omega_0}{2} \right].$$

This theorem shows that the larger the smoothing step d is, the smaller the performance limit of the optimal smoother is. The performance limit approaches zero as $d \to \infty$. Intuitively, this is easily understandable. This statement also holds for a system with more than one nonminimum phase zeros.

Observation: Suppose that the system has more than one nonminimum phase zeros. Then the performance limit approaches zero as the smoothing step goes to infinity.

V. Conclusions

In this paper, the best achievable performance in estimating an input of an SISO discrete-time system from its output has been discussed. A simple and explicit formula has been obtained for the case where the input is a sinusoid with amplitude and phase related to two mutually independent random walk processes. This result shows that the performance limit depends on the nonminimum phase zeros of the system and the interaction between the frequency of the signal and these zeros. Furthermore, the best achievable performance has been discussed for optimal prediction and smoothing, and similar results have been obtained for the system. It has been shown that for the prediction problem, the performance limit increases as the prediction step increases. On the other hand, for the smoothing problem, the performance limit will reach zero when the smoothing step tends to infinity.

Appendix A
Proof of Theorem 1

In this problem, the signal under estimation is the current transmitted signal $v(k)$, i.e., $d = 0$. Following the discussion in Section II, the estimation performance function is given by

$$J = \|[1 - G(\lambda)Q(\lambda) \ \hat{V}(\lambda)\|_2^2$$

where the minimum phase solution of $\hat{V}(\lambda)$ is given in (8). Denote the inner-outer factorization of $G(\lambda)$ by

$$G(\lambda) \quad G_i \ (\lambda)G_o(\lambda)$$

and

$$G_i \ (\lambda) \quad G_1(\lambda) \cdots G_m(\lambda)$$

where $G_1(\lambda), \cdots, G_m(\lambda)$ are associated with nonminimum phase zeros $q_1, \cdots, q_m$, respectively.

For the case in which the frequency ω_0 of the transmitted signal belongs to $[0, \pi)$, the estimation performance J can be rewritten as

$$J = \left\| \hat{V}(\lambda) - Q(\lambda)G_{in}(\lambda)G_o(\lambda)\hat{V}(\lambda) \right\|_2^2$$
$$= \left\| G_{in}^{-1}(\lambda)\hat{V}(\lambda) - Q(\lambda)G_o(\lambda)\hat{V}(\lambda) \right\|_2^2$$
$$= \left\| \left[G_{in}^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right] + \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) - Q(\lambda)G_o(\lambda)\hat{V}(\lambda) \right\|_2^2.$$

where

$$\eta_{-\omega_0} = G_{in}^{-1}(e^{-j\omega_0})e^{j(\frac{\omega_0}{2}-\frac{\pi}{4})}$$

and

$$\eta_{\omega_0} = G_{in}^{-1}(e^{j\omega_0})e^{-j(\frac{\omega_0}{2}-\frac{\pi}{4})}.$$

It is easy to see that

$$G_{in}^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \in \mathcal{H}_2^{\perp}$$

and

$$\left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) - Q(\lambda)G_o(\lambda)\hat{V}(\lambda)$$

can be made to belong to $\mathcal{H}_2$ by properly choosing $Q(\lambda)$. It then follows that

$$J = \left\| \left[G_{in}^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2 + \left\| \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) - Q(\lambda)G_o(\lambda)\hat{V}(\lambda) \right\|_2^2$$

Due to the facts that $\hat{V}(\lambda)$ has minimum phase zeros and shares all the poles on the imaginary axis with

$$\left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right),$$

there exists a $Q(\lambda)$ such that

$$\left\| \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) - Q(\lambda)G_o(\lambda)\hat{V}(\lambda) \right\|_2^2 \to 0.$$

Define

$$\eta_{-\omega_0 i} = \frac{e^{j\omega_0} - q_1^*}{q_1 e^{j\omega_0} - 1} \cdots \frac{e^{j\omega_0} - q_{i-1}^*}{q_{i-1} e^{j\omega_0} - 1} e^{j(\frac{\omega_0}{2}-\frac{\pi}{4})}$$
$$\eta_{\omega_0 i} = \frac{e^{-j\omega_0} - q_1^*}{q_1 e^{-j\omega_0} - 1} \cdots \frac{e^{-j\omega_0} - q_{i-1}^*}{q_{i-1} e^{-j\omega_0} - 1} e^{-j(\frac{\omega_0}{2}-\frac{\pi}{4})}$$
$$i = 2, \cdots, m$$

and

$$\eta_{-\omega_0 1} = e^{j(\frac{\omega_0}{2}-\frac{\pi}{4})}; \qquad \eta_{\omega_0 1} = e^{-j(\frac{\omega_0}{2}-\frac{\pi}{4})}.$$

Therefore, we have

$$J_{opt} = \left\| G_{in}^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2$$

It follows from the definition of $G_{in}(\lambda)$ that

$$J_{opt} = \left\| G_1^{-1}(\lambda)\hat{V}(\lambda) - G_2(\lambda)\cdots G_m(\lambda) \times \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2$$
$$= \left\| G_1^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) + \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) - G_2(\lambda)\cdots G_m(\lambda) \times \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2$$

Following the facts that

$$G_1^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) \in \mathcal{H}_2$$

and

$$\left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) - G_2(\lambda)\cdots G_m(\lambda) \times \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \in \mathcal{H}_2^{\perp},$$

it holds that

$$J_{opt} = \left\| G_1^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2 + \left\| \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) - G_2(\lambda)\cdots G_m(\lambda) \times \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2. \quad \text{(A-1)}$$

From the definition of $G_1(\lambda)$, it holds that

$$\frac{G_1^{-1}(\lambda)}{1-\lambda e^{j\omega_l}} - \frac{G_1^{-1}(e^{-j\omega_0})}{1-\lambda e^{j\omega_0}} = \frac{1-q_1}{1-q_1^*} \frac{1-q_1 q_1^*}{(1-q_1 e^{j\omega_0})(\lambda - q_1)}.$$

Then we have

$$\left\| G_1^{-1}(\lambda)\hat{V}(\lambda) - \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2 = (1-q_1 q_1^*) \left| \frac{\eta_{-\omega_0 1}}{1-q_1 e^{j\omega_0}} + \frac{\eta_{\omega_0 1}}{1-q_1 e^{-j\omega_0}} \right|^2. \quad \text{(A-2)}$$

Substituting (A-2) into (A-1) yields

$$J_{opt} = (1-q_1 q_1^*) \left| \frac{\eta_{-\omega_0 1}}{1-q_1 e^{j\omega_0}} + \frac{\eta_{\omega_0 1}}{1-q_1 e^{-j\omega_0}} \right|^2 + \left\| \left(\frac{\eta_{-\omega_0 2}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0 2}}{1-\lambda e^{-j\omega_0}} \right) - G_2(\lambda)\cdots G_m(\lambda) \times \left(\frac{\eta_{-\omega_0}}{1-\lambda e^{j\omega_0}} + \frac{\eta_{\omega_0}}{1-\lambda e^{-j\omega_0}} \right) \right\|_2^2.$$

Repeating the above steps, we get

$$J_{opt} = \sum_{i=1}^{m} (1-q_i q_i^*) \left| \frac{\eta_{-\omega_0 i}^*}{1-q_i e^{j\omega_0}} + \frac{\eta_{\omega_0 i}^*}{1-q_i e^{-j\omega_0}} \right|^2$$

Define

$$J_o = \sum_{i=1}^{m}\left[\frac{1-q_i^*q_i}{(\;-q_ie^{j\;\;o})\;\;-q_i^*e^{-j\;\;o})}\right.$$
$$\left.+\frac{1-q_i^*q_i}{(\;-q_ie^{-j\;\;o})\;\;-q_i^*e^{j\;\;o})}\right]$$

$$J_o = \sum_{i=1}^{m}\left[\frac{j(\;-q_i^*q_i)\eta_{-\omega_0 i}\eta_{\omega_0 i}^*}{(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}}\right.$$
$$\left.-\frac{j(\;-q_i^*q_i)\eta_{-\omega_0 i}^*\eta_{\omega_0 i}}{(\;-q_ie^{j\;\;o})\;\;-q_i^*e^{j\;\;o})e^{-j\;\;o}}\right].$$

Clearly, $J_o = J_o + J_o$. Simple algebra shows that

$$J_o = \sum_{i=1}^{m}\left(\frac{1+q_ie^{j\;\;o}}{1-q_ie^{j\;\;o}}+\frac{1+q_ie^{-j\;\;o}}{1-q_ie^{-j\;\;o}}\right).$$

In the remaining part of this proof, induction will be used to show

$$J_o = \frac{2}{\mathrm{s}\;\;\omega_0}\mathrm{s}\;\;^2 2\sum_{i=1}^{m}\left[\angle(\;-q_ie^{j\;\;o})\;\;\frac{\omega_0}{2}\right].$$

Denote the phase of $(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}$ by

$$\phi_i = \angle(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}$$

and notice the fact that

$$\mathrm{Im}[(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}] = (\;-q_i^*q_i)\;\;\;\omega_0.$$

Then, it follows that

$$|(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}|\,\mathrm{s}\;\;\phi_i = (\;-q_i^*q_i)\;\;\;\omega_0$$

and

$$(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}$$
$$= |(\;-q_i^*e^{-j\;\;o})\;\;-q_ie^{-j\;\;o})e^{j\;\;o}|e^{\phi_i}.$$

Hence, the first term of J_o is given by

$$\frac{(\;-q_1^*q_1)j}{(\;-q_1^*e^{-j\;\;o})\;\;-q_1e^{-j\;\;o})e^{j\;\;o}}$$
$$-\frac{(\;-q_1^*q_1)j}{(\;-q_1^*e^{j\;\;o})\;\;-q_1e^{j\;\;o})e^{-j\;\;o}}$$
$$= \frac{j}{\mathrm{s}\;\;\omega_0}(e^{-j\;\;1}\mathrm{s}\;\;\phi_1 - e^{j\;\;1}\mathrm{s}\;\;\phi_1)\;\;\;\frac{2}{\mathrm{s}\;\;\omega_0}\mathrm{s}\;\;^2\phi_1.$$

To carry out the induction, we assume that the sum of the first $k-1$ terms of J_o is

$$\frac{2}{\mathrm{s}\;\;\omega_0}\mathrm{s}\;\;^2(\phi_1+\cdots+\phi_{k-1}).$$

Notice that $\eta_{\omega_0 k}^*\eta_{-\omega_0 k} = e^{-j(\phi_1+\cdots+\phi_{k-1})}$. Then, we have

$$\frac{j(\;-q_k^*q_k)\eta_{-\omega_0 k}\eta_{\omega_0 k}^*}{(\;-q_k^*e^{-j\;\;o})\;\;-q_ke^{-j\;\;o})e^{j\;\;o}}$$
$$-\frac{j(\;-q_k^*q_k)\eta_{-\omega_0 k}^*\eta_{\omega_0 k}}{(\;-q_ke^{j\;\;o})\;\;-q_k^*e^{j\;\;o})e^{-j\;\;o}}$$
$$= \frac{2}{\mathrm{s}\;\;\omega_0}[\mathrm{s}\;\;^2(\phi_1+\cdots+\phi_k) - \mathrm{s}\;\;^2(\phi_1+\cdots+\phi_{k-1})\;.$$

Thus, the sum of the first k terms of J_o is given by

$$\frac{2}{\mathrm{s}\;\;\omega_0}\mathrm{s}\;\;^2(\phi_1+\cdots+\phi_k).$$

Following the similar steps as shown above and by induction, we can get

$$J_o = \frac{2}{\mathrm{s}\;\;\omega_0}\mathrm{s}\;\;^2(\phi_1+\cdots+\phi_m).$$

Similarly, for the case $\omega_0 \in [\pi\;\;2\pi)$, we have

$$J_o = -\frac{2}{\mathrm{s}\;\;\omega_0}\mathrm{s}\;\;^2(\phi_1+\cdots+\phi_m).$$

Finally, the fact that

$$\sum_{i=1}^{m}\phi_i = 2\sum_{i=1}^{m}\left[\angle(\;-q_ie^{-j\;\;o})\;\;\frac{\omega_0}{2}\right]$$

leads to the result in this theorem. □

REFERENCES

[1] O. Besson, and P. Stoica, "Sinusoidal signals with random amplitude: Least-squares estimators and their statistical analysis," *IEEE Traps. Signal Processing*, vol. 43, no. 11, pp. 2733-2744, 1995.

[2] O. Besson, and P. Stoica, "Estimation of the parameters of a random amplitude sinusoid by correlation fitting," *IEEE Trans. Signal Processing*, vol. 44, no. 11, pp. 2911-2916, 1996.

[3] J. H. Braslavsky, M. M. Seron, D. G. Mayne, and P. V. Kokotovic, "Limiting performance of optimal linear filters", *Automatica*, vol 35, pp. 189–199, 1999.

[4] A. B. Carlson, P. B. Crilly, and J. C. Rutledge, *Communication System: An Introduction to Signals and Noise in Electrical Communication*, 4th Ed., McGraw-Hill, New York, 2002.

[5] A. V. Dandawate, and G. B. Giannakis, "Nonparametric polyspectral estimators for kth-order (almost) cyclostationary processes dandawate," *IEEE Trans. Information Theory*, vol. 40, no. 1, pp. 67-84, 1994.

[6] T. Frederick and N. Erdol, "Time-frequency estimation for cyclostationary signals," in *Proc. 1996 IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 5, Atlanta, GA, May 1996, pp. 2928-2931.

[7] H. Li and Q. Cheng, "Some estimation problems for harmonics in muliplicative and additive noise,", in *Proc. 3rd International Conference on Signal Processing*, vol. 1, Beijing, China, Oct. 1996, pp. 181-184.

[8] B. Picinbono, *Random Signals and Systems*. Prentice Hall: Englewood Cliffs, NJ, 1993.

[9] L. Qiu, Z. Ren and J. Chen, "Performance limitations in estimation", *Communications in Information and Systems*, vol. 2, pp. 371–384, 2002.

[10] M. M. Seron, J. H. Braslavsky and G. C. Goodwin, *Fundamental Limitations in Filtering and Control*, Springer-Verlag, London, 1997.

[11] J. M. Spanjaard and L. B. White, "Adaptive period estimation of a class of periodic random processes," in *Proc. 1995 IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 3, Detroit, MI, May 1995, pp. 1792-1795.

[12] P. Stoica, H. Li and J. Li, "Amplitude estimation of sinusoidal signals: Survey, new results, and an application," *IEEE Trans. Signal Processing*, vol. 48, no. 2, pp. 338-352, 2000.

[13] W. Su, L. Qiu and J. Chen, "Fundamental performance limitations in tracking sinusoidal signals", *IEEE Trans. Automat. Contr.*, vol. 48, no. 8, pp. 1371-1380, Aug. 2003.

[14] W. Su, W. X. Zheng and L. Qiu, "Performance limitations of linear discrete-time systems in estimating a periodic random processes", submitted to *IEEE Trans. Signal Processing*, 2003.

[15] C. W. Therrien, *Discrete Random Signals and Statistical Signal Processing*. Prentice Hall: Englewood Cliffs, NJ, 1992.

[16] S. Weiss, A. Stenger, R. W. Stewart, and R. Rabenstein, "Steay-state performance limitations of subband adaptive filiters," *IEEE Trans. Signal Processing*, vol. 49, no. 9, pp. 1982-1991, 2001.

[17] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: MIT Press, 1985.