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Performance Limitations in Estimation

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Abstract

In this paper, we address the performance limitation issues in estimation problems. Specifically, we study the performance limitations in four cases: (1) estimating the output of an LTI system under the corruption of a white noise, (2) estimating a Brownian motion input of an LTI system from the output measurement, (3) estimating the output of an LTI system under the corruption of a Brownian noise, (4) estimating a white noise input of an LTI system from the output measurement. In each case we find and characterize explicitly how the best achievable estimation errors may relate to certain simple, intrinsic system characteristics.

1 Introduction

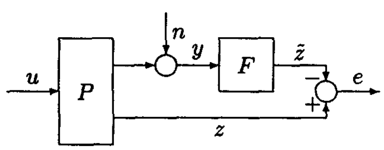

Figure 1: A general estimation problem

A standard estimation problem can often be schematically shown by Fig. 1. Here $P = \begin{bmatrix} G \\ H \end{bmatrix}$ is an LTI plant, u is the input to the plant, n is the measurement noise, z is the signal to be estimated, y is the measured signal, and $\tilde{z}$ is the estimate of z. Often u and n are modelled as stochastic processes with known statistical means and covariances. We shall assume, without loss of generality, that the means of the stochastic processes are zero. The objective is to design an LTI filter F so that the steady state error variance

$$V = \lim_{t\to\infty} \boldsymbol{E}[e(t)'e(t)]$$

is small. Clearly, for V to be finite for nontrivial u and n, it is necessary that $F \in \mathcal{RH}_\infty$ and $H - FG \in \mathcal{RH}_\infty$. This condition is also necessary and sufficient for the error to be bounded for arbitrary initial conditions of P and F, i.e., for the filter to be a bounded error estimator (BEE). There is an extensive theory for optimally designing the filter F to minimize V, see for example [1, 2, 6]. The optimal error variance is given by

$$V^* = \inf_{F,H-FG\in\mathcal{RH}_\infty} V.$$

Our interest in this paper is not to find the optimal filter F, which is addressed by the standard optimal filtering theory. Rather, we are interested in discovering how V^* may be related to the intrinsic characteristics of the plant P in some special, yet important cases. Since V^* gives a fundamental limit in achieving certain benchmark performance objectives in filtering problems, a simple relationship between V^* and the plant characteristics will not only provide a deep understanding and insightful knowledge on the estimation problems, but also can be used to assess the quality of different filter designs and to rule out impossible or unrealistic design objectives *a priori*.

The variance V gives an overall measure on the size of the steady state estimation error. Sometimes, we may wish to focus on some more detailed features of the error. For example we may be interested in a certain component of the estimation error projected to a certain direction, and henceforth the variance of the projection, which gives a measure of the error in the specific direction of concern. Assume that $z(t), \tilde{z}(t), e(t) \in \mathbb{R}^m$. Let $\xi \in \mathbb{R}^m$ be a vector of unit length representing a direction in $\mathbb{R}^m$. Then the projection of $e(t)$ to the direction represented by ξ is given by $\xi' e(t)$ and its steady state variance is given by

$$V_\xi = \lim_{t\to\infty} \boldsymbol{E}[(\xi' e(t))^2].$$

The best achievable error in the ξ direction is then determined by

$$V_\xi^* = \inf_{F,H-FG\in\mathcal{RH}_\infty} V_\xi.$$

The optimal or near-optimal filter in minimizing V_ξ in general depends on ξ. This very fact may limit the usefulness of V_ξ^*, since we are usually more interested in the directional error information under an optimal or near-optimal filter designed for all directions, i.e., designed to minimize V. Let $\{F_k\}$ be a sequence of filters satisfying $F_k, H - F_kG \in \mathcal{RH}_\infty$ such that the corresponding sequence of errors $\{e_k\}$ satisfies

$$V^* = \lim_{k\to\infty}\lim_{t\to\infty} \boldsymbol{E}[e_k(t)'e_k(t)].$$

Then we are more interested in

$$V^*(\xi) = \lim_{k\to\infty}\lim_{t\to\infty} \boldsymbol{E}[(\xi' e_k(t))^2].$$

In this paper, we will also derive the relations between V_ξ^*, $V^*(\xi)$ and the characteristics of the plant P for the same cases when that for V^* is considered.

One should note tht performance limitations in estimation have been studied recently in [4, 5, 9, 10] in various settings. In [4, 5, 9], the sensitivity and complimentary sensitivity functions of an estimation problem are defined and it is shown that they must satisfy certain integral constraints independent of filter design. In [10], a time domain technique is used to study the performance limitations in some special cases where one of n and u is diminishingly small and the other is either a white noise or a Brownian motion process.

This paper addresses similar problems as in [10] and their extensions, but is pursued from a pure input-output point of view using frequency domain techniques. We investigate the issues in more detail by providing directional information on the best errors. Our results are dual to those in [3, 7] where the performance limitations of tracking and regulation problems are considered. The new investigation provides more insights into the performance limitations of estimation problems.

2 Preliminaries

Let G be a continuous time FDLTI system. We will use the same notation G to denote its transfer matrix. Assume that G is left invertible. The poles and zeros of G, including multiplicity, are defined according to its Smith-McMillan form. A zero of G is said to be nonminimum phase if it has positive real part. G is said to be minimum phase if it has no nonminimum phase zero; otherwise it is said to be nonminimum phase. A pole of G is said to be antistable if it has a positive real part. G is said to be semistable if it has no antistable pole; otherwise it is said to be strictly unstable. As usual, G is said to be stable if all of its poles have negative real parts; otherwise it is said to be unstable.

Suppose that G is stable and z is a nonminimum phase zero of G. Then, there exists a vector u of unit length such that

$$G(z)u = 0.$$

We call u a (right or input) zero vector corresponding to the zero z. Let the nonminimum phase zeros of G be ordered as $z_1, z_2, \ldots, z_\nu$. Let also η_1 be a zero vector corresponding to z_1. Define

$$G_1(s) = I - \frac{2\,\mathrm{Re}\, z_1}{s + z_1^*}\eta_1\eta_1^*.$$

Here η_1^* means the conjugate transpose of η_1. Note that G_1 is so constructed that it is inner, has only one zero at z_1 with η_1 as a zero vector. Now GG_1^{-1} has zeros $z_2, z_3, \ldots, z_\nu$. Find a zero vector η_2 corresponding to the zero z_2 of GG_1^{-1}, and define

$$G_2(s) = I - \frac{2\,\mathrm{Re}\, z_2}{s + z_2^*}\eta_2\eta_2^*.$$

It follows that $GG_1^{-1}G_2^{-1}$ has zeros $z_3, z_4, \ldots, z_\nu$. Continue this process until $\eta_1, \ldots, \eta_\nu$ and $G_1, \ldots, G_\nu$ are obtained. Then we have one vector corresponding to each nonminimum phase zero, and the procedure yields a factorization of G in the form of

$$G = G_0 G_\nu \cdots G_1, \tag{1}$$

where G_0 has no nonminimum phase zeros and

$$G_i(s) = I - \frac{2\,\mathrm{Re}\, z_i}{s + z_i^*}\eta_i\eta_i^*. \tag{2}$$

Since G_i is inner, has the only zero at z_i, and has η_i as a zero vector corresponding to z_i, it will be called a matrix Blaschke factor. Accordingly, the product

$$G_z = G_\nu \cdots G_1$$

will be called a matrix Blaschke product. The vectors $\eta_1, \ldots, \eta_\nu$ will be called zero Blaschke vectors of G corresponding to the nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$. These vectors depend on the order of the nonminimum phase zeros. One might be concerned with the possible complex coefficients appearing in G_i when some of the nonminimum phase zeros are complex. However, if we order a pair of complex conjugate nonminimum phase zeros adjacently, then the corresponding pair of Blaschke factors will have complex conjugate coefficient and their product is then real rational and this also leads to real rational G_0.

The choice of G_i as in (2) seems *ad hoc*, notwithstanding that G_i has to be unitary, has the only zero at z_i with η_i as a zero vector corresponding to z_i. Another choice, among infinite many possible ones, is

$$G_i(s) = I - \frac{2\,\mathrm{Re}\, z_i}{z_i}\frac{s}{s + z_i^*}\eta_i\eta_i^*, \tag{3}$$

and if this choice is adopted, the same procedure can be used to find a factorization of the form (1). In the latter case, the Blaschke vectors are not the same. We see that for the first choice $G_i(\infty) = I$, whereas for the second choice $G_i(0) = I$. We will use both constructions in the sequel. For purpose of distinction, we will call the factorization resulting from the first choice (2) that of Type I and the one from the second choice (3) of type II.

For an unstable G, there exist stable real rational matrix functions

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}, \begin{bmatrix} M & Y \\ N & X \end{bmatrix}$$

such that

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

and

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I.$$

This is called a doubly coprime factorization of G. Note that the nonminimum phase zeros of G are the nonminimum phase zeros of $\tilde{N}$ and the antistable poles of G are the nonminimum phase zeros of $\tilde{M}$. If we order the antistable poles of G as $p_1, p_2, \dots, p_\mu$ and the nonminimum phase zeros of G as $z_1, z_2, \dots, z_\nu$, then $\tilde{M}$ and $\tilde{N}$ can be factorized as

$$\begin{aligned} \tilde{M} &= \tilde{M}_0 \tilde{M}_\mu \cdots \tilde{M}_1 \\ \tilde{N} &= \tilde{N}_0 \tilde{N}_\nu \cdots \tilde{N}_1 \end{aligned}$$

with

$$\tilde{M}_i(s) = I - \frac{2\operatorname{Re} p_i}{s + p_i^*}\zeta_i\zeta_i^*, \quad i = 1, 2, \dots, \mu \qquad (4)$$

or

$$\tilde{M}_i(s) = I - \frac{2\operatorname{Re} p_i}{p_i}\frac{s}{s + p_i^*}\zeta_i\zeta_i^*, \quad i = 1, 2, \dots, \mu \qquad (5)$$

and

$$\tilde{N}_i(s) = I - \frac{2\operatorname{Re} z_i}{s + z_i^*}\eta_i\eta_i^*, \quad i = 1, 2, \dots, \nu \qquad (6)$$

or

$$\tilde{N}_i(s) = I - \frac{2\operatorname{Re} z_i}{z_i}\frac{s}{s + z_i^*}\eta_i\eta_i^*, \quad i = 1, 2, \dots, \nu \qquad (7)$$

where $\zeta_1, \zeta_2, \dots, \zeta_\mu$ are zero Blaschke vectors (of type I or II) of $\tilde{M}$ and $\eta_1, \eta_2, \dots, \eta_\nu$ are those of $\tilde{N}$. Here also $\tilde{N}_0$ and $\tilde{M}_0$ have no nonminimum phase zeros.

Consequently, for any real rational matrix G with nonminimum phase zeros $z_1, z_2, \dots, z_\nu$ and antistable poles $p_1, p_2, \dots, p_\mu$, it can always be factorized to

$$G = G_p^{-1} G_0 G_z, \qquad (8)$$

as shown in Fig. 2, where

$$\begin{aligned} G_p(s) &= \prod_{i=1}^{\mu}\left[I - \frac{2\operatorname{Re} p_i}{s + p_i^*}\zeta_i\zeta_i^*\right] \quad \text{or} \\ &\quad \prod_{i=1}^{\mu}\left[I - \frac{2\operatorname{Re} p_i}{p_i}\frac{s}{s + p_i^*}\zeta_i\zeta_i^*\right] \\ G_z(s) &= \prod_{i=1}^{\nu}\left[I - \frac{2\operatorname{Re} z_i}{s + z_i^*}\eta_i\eta_i^*\right] \quad \text{or} \\ &\quad \prod_{i=1}^{\nu}\left[I - \frac{2\operatorname{Re} z_i}{z_i}\frac{s}{s + z_i^*}\eta_i\eta_i^*\right] \end{aligned}$$

and G_0 is a real rational matrix with neither nonminimum phase zero nor antistable pole. Although coprime factorizations of G are not unique, this does not affect factorization (8). $\eta_1, \eta_2, \dots, \eta_\nu$ are called zero Blaschke vectors (of type I or II) and $\zeta_1, \zeta_2, \dots, \zeta_\nu$ pole Blaschke vectors (of type I or II) of G.

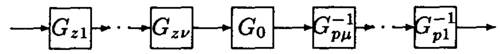

Figure 2: Cascade factorization

3 Estimation under White Measurement Noise

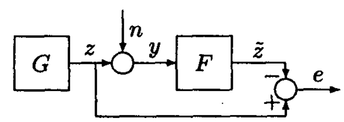

Figure 3: Estimation under measurement noise

Consider the estimation problem shown in Fig. 3. Here G is a given FDLTI plant, and n is a standard white noise. The purpose is to design a stable LTI filter F such that it generates an estimate $\tilde{z}$ of the true output z using the corrupted output y. This problem is clearly a special case of the general estimation problem stated in Sect. 1 with $P = \begin{bmatrix} G \\ G \end{bmatrix}$ and $u = 0$. The error of estimation is given by Fn. Since n is a standard white noise, the steady state variance of the error is given by

$$V = \|F\|_2^2$$

where $\|\cdot\|_2$ is the $\mathcal{H}_2$ norm. If we want V to be finite, we need to have $F(\infty) = 0$, in addition to $F, G - FG \in \mathcal{RH}_\infty$. Therefore

$$V^* = \inf_{F, G - FG \in \mathcal{RH}_\infty, F(\infty) = 0} \|F\|_2^2.$$

Let $G = \tilde{M}^{-1}\tilde{N}$ be a left coprime factorization of G. Then $F \in \mathcal{RH}_\infty$ and $G - FG = (I - F)G = (I - F)\tilde{M}^{-1}\tilde{N} \in \mathcal{RH}_\infty$ if and only if $I - F = Q\tilde{M}$ for some $Q \in \mathcal{RH}_\infty$. Therefore

$$V^* = \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}(\infty) = I} \|I - Q\tilde{M}\|_2^2.$$

Now assume that G has antistable poles $p_1, p_2, \dots, p_\mu$ with $\zeta_1, \zeta_2, \dots, \zeta_\mu$ being the corresponding pole Blaschke vectors of type I. Then $\tilde{M}$ has the factorization

$$\tilde{M} = \tilde{M}_0 \tilde{M}_\mu \cdots \tilde{M}_1$$

where

$$\tilde{M}_i(s) = I - \frac{2\operatorname{Re} p_i}{s + p_i^*}\zeta_i\zeta_i^*.$$

Since $\tilde{M}_i(\infty) = I$, $i = 1, 2, \ldots, \mu$, it follows that $Q(\infty)\tilde{M}(\infty) = I$ is equivalent to $Q(\infty)\tilde{M}_0(\infty) = I$. Hence, by using the facts that $\tilde{M}_i$, $i = 1, 2, \ldots, \mu$, are unitary operators in $\mathcal{L}_2$ and that $\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I \in \mathcal{H}_2^\perp$ and $I - Q\tilde{M}_0 \in \mathcal{H}_2$, we obtain

$$\begin{aligned} V^* &= \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty) = I} \|I - Q\tilde{M}_0\tilde{M}_\mu \cdots \tilde{M}_1\|_2^2 \\ &= \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty) = I} \|\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I + \\ &\qquad I - Q\tilde{M}_0\|_2^2 \\ &= \|\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I\|_2^2 + \\ &\qquad \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty) = I} \|I - Q\tilde{M}_0\|_2^2. \end{aligned}$$

Since $\tilde{M}_0$ is co-inner with invertible $\tilde{M}_0(\infty)$, there exists a sequence $\{Q_k\} \in \mathcal{RH}_\infty$ with $Q_k(\infty)\tilde{M}_0(\infty) = I$ such that $\lim_{k\to\infty} \|I - Q_k\tilde{M}_0\|_2 = 0$. This shows

$$\begin{aligned} V^* &= \|\tilde{M}_1^{-1} \cdots \tilde{M}_\mu^{-1} - I\|_2^2 \\ &= \|\tilde{M}_2^{-1} \cdots \tilde{M}_\mu^{-1} - I + I - \tilde{M}_1\|_2^2 \\ &= \|\tilde{M}_2^{-1} \cdots \tilde{M}_\mu^{-1} - I\|_2^2 + \|I - \tilde{M}_1\|_2^2 \\ &= \sum_{i=1}^{\mu} \|I - \tilde{M}_i\|_2^2 = 2\sum_{i=1}^{\mu} p_i. \end{aligned}$$

Here the first equality follows from the fact that $\tilde{M}_1$ is a unitary operator in $\mathcal{L}_2$, the second from that $\tilde{M}_2^{-1} \cdots \tilde{M}_\mu^{-1} - I \in \mathcal{H}_2^\perp$ and $I - \tilde{M}_1 \in \mathcal{H}_2$, the third by repeating the procedure in the first and second equalities, and the last from straightforward computation. The derivation shows that an arbitrarily near-optimal Q can be chosen from the sequence $\{Q_k\}$. Therefore

$$V^*(\xi) = \lim_{k\to\infty} \|\xi'(I - Q_k\tilde{M}_0\tilde{M}_\mu \cdots \tilde{M}_1)\|_2^2.$$

The same reasoning as in the above derivation gives

$$V^*(\xi) = \sum_{i=1}^{\mu} \|\xi(I - \tilde{M}_i)\|_2^2 = 2\sum_{i=1}^{\mu} p_i \cos^2 \angle(\xi, \zeta_i).$$

The directional steady state error variance is

$$V_\xi = \|\xi' F\|_2^2.$$

The optimal directional steady state error variance is

$$\begin{aligned} V_\xi^* &= \inf_{F, G - FG \in \mathcal{RH}_\infty} \|\xi' F\|_2^2 \\ &= \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{M}_0(\infty) = I} \|\xi'(I - Q\tilde{M}_0\tilde{M}_\mu \cdots \tilde{M}_1)\|_2^2. \end{aligned}$$

By following an almost identical derivation as in the non-directional case, we can show that the same sequence $\{Q_k\}$ giving the near-optimal solutions there also gives near-optimal solutions herein, for every $\xi \in \mathbb{R}^m$. Hence,

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\mu} p_i \cos^2 \angle(\xi, \zeta_i).$$

We have thus established the following theorem.

Theorem 1 *Let G's antistable poles be $p_1, p_2, \ldots, p_\mu$ with $\zeta_1, \zeta_2, \ldots, \zeta_\mu$ being the corresponding pole Blaschke vectors of type I. Then*

$$V^* = 2\sum_{i=1}^{\mu} p_i$$

and

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\mu} p_i \cos^2 \angle(\xi, \zeta_i).$$

4 Estimation of Brownian Motion

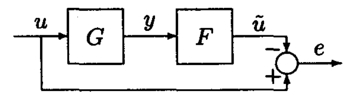

Figure 4: Estimation of a stochastic process

Consider the estimation problem shown in Fig. 4. Here G is a given FDLTI plant, u is the input to the plant which is assumed to be a Brownian motion process, i.e., the integral of a standard white noise, which can be used to model a slowly varying "constant". Assume that $G(0)$ is left invertible. The objective is to design an LTI filter F such that it measures the output of G and generates an estimate $\tilde{u}$ of u. This problem is clearly a special case of the general estimation problem stated in Sect. 1 with $P = \begin{bmatrix} G \\ I \end{bmatrix}$ and $n = 0$. The error of estimation is given by $(I - FG)u$. Since u is a Brownian process, the variance of the error is given by

$$V = \|(I - FG)U\|_2^2,$$

where $U(s) = \frac{1}{s}I$ is the transfer matrix of m channels of integrators. If we want V to be finite, we need to have $I - F(0)G(0) = 0$, in addition to $F, I - FG \in \mathcal{RH}_\infty$. This requires $G(0)$, the DC gain of of G, to be left invertible, which will be assumed. Equivalently, we need to have $F, FG \in \mathcal{H}_\infty$ and $F(0)G(0) = I$. Therefore,

$$V^* = \inf_{F, FG \in \mathcal{RH}_\infty, F(0)G(0) = I} \|(I - FG)U\|_2^2.$$

Let $G = \tilde{M}^{-1}\tilde{N}$ be a left coprime factorization of G. Then it is easy to see that $F, FG \in \mathcal{H}_\infty$ is equivalent to $F = Q\tilde{M}$ for some $Q \in \mathcal{H}_\infty$. Hence

$$V^* = \inf_{Q \in \mathcal{RH}_\infty, Q(0)\tilde{N}(0) = I} \|(I - Q\tilde{N})U\|_2^2.$$

Let G have nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding input Blaschke vectors of type II. Then $\tilde{N}$ has the factorization

$$\tilde{N} = \tilde{N}_0\tilde{N}_\nu \cdots \tilde{N}_1$$

where

$$\tilde{N}_i = I - \frac{2\,\mathrm{Re}\, z_i}{z_i}\frac{s}{s+z_i^*}\eta_i\eta_i^*.$$

Since $\tilde{N}_i(0) = I$, $i = 1, 2, \ldots, \nu$, it follows that $Q(0)\tilde{N}(0) = I$ is equivalent to $Q(0)\tilde{N}_0(0) = I$. Hence, by using the facts that $\tilde{N}_i$, $i = 1, 2, \ldots, \nu$, are unitary operators in $\mathcal{L}_2$ and that $\tilde{N}_1^{-1}\cdots\tilde{N}_\nu^{-1} - I \in \mathcal{H}_2^\perp$ and $I - Q\tilde{N}_0 \in \mathcal{H}_2$, we obtain

$$\begin{aligned} V^* &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I} \|(I - Q\tilde{N}_0\tilde{N}_\nu\cdots\tilde{N}_1)U\|_2^2 \\ &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I} \|(\tilde{N}_1^{-1}\tilde{N}_2^{-1}\cdots\tilde{N}_\nu - I)U \\ &\qquad + (I - Q\tilde{N}_0)U\|_2^2 \\ &= \|(\tilde{N}_1^{-1}\tilde{N}_2^{-1}\cdots\tilde{N}_\nu - I)U\|_2^2 + \\ &\qquad \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I} \|(I - Q\tilde{N}_0)U\|_2^2. \end{aligned}$$

Since $\tilde{N}_0$ is co-inner with invertible $\tilde{N}(0)$, there exists a sequence $\{Q_k\} \in \mathcal{RH}_\infty$ with $Q_k(0)\tilde{N}_0(0) = I$ such that $\lim_{k\to\infty} \|(I - Q_k\tilde{N}_0)U\|_2 = 0$. This shows

$$\begin{aligned} V^* &= \|(\tilde{N}_1^{-1}\cdots\tilde{N}_\nu^{-1} - I)U\|_2^2 \\ &= \|(\tilde{N}_2^{-1}\cdots\tilde{N}_\nu^{-1} - I + I - \tilde{N}_1)U\|_2^2 \\ &= \|(\tilde{N}_2^{-1}\cdots\tilde{N}_\nu^{-1} - I)U\|_2^2 + \|(I - \tilde{M}_1)U\|_2^2 \\ &= \sum_{i=1}^{\nu} \|(I - \tilde{N}_i)U\|_2^2 = 2\sum_{i=1}^{\nu}\frac{1}{z_i}. \end{aligned}$$

The first equality follows from the fact that $\tilde{N}_1$ is a unitary operator in $\mathcal{L}_2$, the second from that $(\tilde{N}_2^{-1}\cdots\tilde{N}_\nu^{-1} - I)U \in \mathcal{H}_2^\perp$ and $(I - \tilde{N}_1)U \in \mathcal{H}_2$, the third from repeating the underlying procedure in the first and second equalities, and the last from straightforward computation.

The above derivation shows that an arbitrarily near-optimal Q can be chosen from the sequence $\{Q_k\}$. Thus

$$V^*(\xi) = \lim_{k\to\infty} \|\xi'(I - Q_k\tilde{N}_0\tilde{N}_\nu\cdots\tilde{N}_1)U\|_2^2.$$

The same reasoning gives

$$V^*(\xi) = \sum_{i=1}^{\mu} \|\xi(I - \tilde{N}_i)U\|_2^2 = 2\sum_{i=1}^{\mu}\frac{1}{z_i}\cos^2\angle(\xi, \eta_i).$$

The directional steady state error variance is

$$V_\xi = \|\xi'(I - FG)U\|_2^2.$$

The optimal directional steady state error variance is

$$\begin{aligned} V_\xi^* &= \inf_{F, G-FG\in\mathcal{RH}_\infty} \|\xi'(I - FG)U\|_2^2 \\ &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{N}_0(0)=I} \|\xi'(I - Q\tilde{N}_0\tilde{N}_\nu\cdots\tilde{N}_1)U\|_2^2. \end{aligned}$$

By following an almost identical derivation as in the non-directional case, we can show that the same sequence $\{Q_k\}$ giving near-optimal solutions there also gives near-optimal solutions here for every $\xi \in \mathbb{R}^m$. Hence,

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\nu}\frac{1}{z_i}\cos^2\angle(\xi, \eta_i).$$

We have thus established the following theorem.

Theorem 2 *Let G's nonminimum phase zeros be $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding Blaschke vectors of type II, then*

$$V^* = 2\sum_{i=1}^{\nu}\frac{1}{z_i}$$

and

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\nu}\frac{1}{z_i}\cos^2\angle(\xi, \eta_i).$$

5 Estimation under Brownian Noise

Consider the estimation problem considered in Sect. 3, but let us assume that the noise n is a Brownian motion process instead of a white noise. In this case, the steady state variance of the estimation error is given by

$$V = \|FN\|_2^2$$

where $N(s) = \frac{1}{s}I$. Following similar arguments as in Sect. 3, we get

$$\begin{aligned} V^* &= \inf_{F, G-FG\in\mathcal{RH}_\infty, F(0)=0} \|FN\|_2^2 \\ &= \inf_{Q\in\mathcal{RH}_\infty, Q(0)\tilde{M}(0)=I} \|(I - Q\tilde{M})N\|_2^2. \end{aligned}$$

Here we assume that $G = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorization of G. Now assme that G has antistable poles $p_1, p_2, \ldots, p_\mu$ with $\zeta_1, \zeta_2, \ldots, \zeta_\mu$ be the corresponding pole Blaschke vectors of type II. Then $\tilde{M}$ has the factorization

$$\tilde{M} = \tilde{M}_0\tilde{M}_\mu\cdots\tilde{M}_1$$

where

$$\tilde{M}_i(s) = I - \frac{2\,\mathrm{Re}\, p_i}{p_i}\frac{s}{s+p_i^*}\zeta_i\zeta_i^*.$$

Now the problem becomes similar to the one considered in Sect. 4 with $\tilde{N}_i$ replaced by $\tilde{M}_i$. This enables us to obtain the following theorem, based upon similar arguments.

Theorem 3 *Let G's antistable poles be $p_1, p_2, \cdots, p_\mu$ with $\zeta_1, \zeta_2, \cdots, \zeta_\mu$ being the corresponding pole Blaschke vectors of type II. Then*

$$V^* = 2\sum_{i=1}^{\mu}\frac{1}{p_i}$$

and

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\mu} \frac{1}{p_i} \cos^2 \angle(\xi, \zeta_i).$$

6 Estimation of White Noise

Consider the estimation problem as in Sect. 4, but now we assume that the signal to be estimated u is a white noise, instead of a Brownian motion. In this case, the variance of the estimation error is given by

$$V = \|I - FG\|_2^2.$$

If we want V to be finite, we need to have $I - F(\infty)G(\infty) = 0$, in addition to $F, I - FG \in \mathcal{RH}_\infty$. This requires $G(\infty)$, the direct feedthrough term of G, to be left invertible, which will be assumed. Equivalently, we need to have $F, FG \in \mathcal{H}_\infty$ and $F(\infty)G(\infty) = I$. Therefore

$$\begin{aligned} V^* &= \inf_{F, FG \in \mathcal{RH}_\infty, F(\infty)G(\infty)=I} \|I - FG\|_2^2 \\ &= \inf_{Q \in \mathcal{RH}_\infty, Q(\infty)\tilde{N}(\infty)=I} \|I - Q\tilde{N}\|_2^2. \end{aligned}$$

Here we assume that $G = \tilde{M}^{-1}\tilde{N}$ is a left coprime factorization of G. Now let G have nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding input Blaschke vectors of type I. Then $\tilde{N}$ has the factorization

$$\tilde{N} = \tilde{N}_0 \tilde{N}_\nu \cdots \tilde{N}_1$$

where

$$\tilde{N}_i = I - \frac{2\,\mathrm{Re}\, z_i}{s + z_i^*} \eta_i \eta_i^*.$$

The problem then becomes similar to the one considered in Sect. 3 with $\tilde{M}_i$ replaced by $\tilde{N}_i$. Thus, the following theorem is obtained.

Theorem 4 *Let G's nonminimum phase zeros be $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding Blaschke vectors of type I, then*

$$V^* = 2\sum_{i=1}^{\nu} z_i$$

and

$$V_\xi^* = V^*(\xi) = 2\sum_{i=1}^{\nu} z_i \cos^2 \angle(\xi, \eta_i).$$

7 Concluding Remarks

This paper relates the performance limitations in four typical estimation problems to simple characteristics of the plants involved. By estimation problems we mean actually filtering problems here. The general estimation problems can include prediction and smoothing problems. We are now trying to extend the results in this paper to smoothing and prediction problems.

We have considered two types of noises and signals: white noise and Brownian motion. We are trying to extending our results to possibly other types of noises and signals.

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