Performance Limitations of Discrete-Time Systems in Tracking Sinusoidal Signals[1]

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Abstract

This paper studies the performance limitation of an LTI multivariable discrete-time system in tracking a reference signal which is a linear combination of a step signal and several sinusoids with different frequencies. It attempts to extend some recent results for continuous-time systems on the same issue. The tracking performance is measured by the energy of the error between the output of the plant and the reference signal. Our purpose is to find the fundamental limitation for the attainable tracking performance, under any control structure and parameters, in terms of the characteristics and structural parameters of the given plant as well as the reference signal under consideration. It is shown that this fundamental limitation depends on the interaction between the reference signal and the nonminimum phase zeros of the plant and their frequency dependent directional information.

Keywords: Linear system structure, Performance limitation, Optimal Tracking, Nonminimum phase.

1 Introduction

Tracking a given signal is a common task in feedback control systems. This paper will consider the best achievable performance, often called *performance limit* in the literature, of LTI discrete-time systems in tracking given reference signals. The signals under consideration are a linear combination of a step and several sinusoids in various frequencies. In general, these signals can be modelled as outputs of a signal generator. The state of this generator contributes further information to the feedback controller in addition to the reference signal itself. In light of this, we first consider the case where the full state information of the reference signal is available for the controller. Then the case where only the reference signal is available will be considered.

The setup where full information is available is depicted in Figure 1. Here λ is a unit delay operator; $P(\lambda)$ is the given plant transfer matrix; $K(\lambda)$ is the controller transfer matrix which is to be designed; $S(\lambda)$ is the exosystem which is excited by a unit impulse $\delta(k)$ and generates the reference. The controller is assumed to have full information of the reference in the sense that it takes the state $v(k)$ of the exosystem $S(\lambda)$, in addition to the measurement $y(k)$ of the plant, as its inputs. Whether or not the measurement $y(k)$ contains the full information of the plant, i.e., the state of the plant, is not important. The tracking problem is to design a controller $K(\lambda)$ so that the closed loop system is internally stabilized and the plant output $z(k)$ asymptotically tracks a reference signal $r(k)$ of the form:

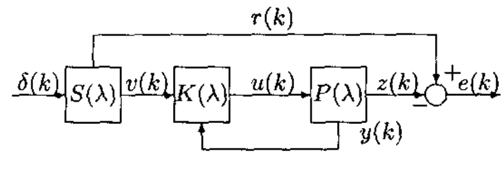

Figure 1: A general two-parameter control structure

$$r(k) = \sum_{l=-n}^{n} v_l e^{j\omega_l k} \qquad (1)$$

where ω_l, $l = 0, \pm1, \ldots, \pm n$, are distinct real frequencies satisfying $\omega_{-l} = -\omega_l$ and v_l, $l = 0, \pm1, \ldots, \pm n$, are complex vectors satisfying $v_{-l} = \bar{v}_l$. Implicitly, we have $\omega_0 = 0$ and v_0 is real. The reference defined in such a way is always a real valued signal. We use the vector $v = [v_{-n}^* \quad \cdots \quad v_{-1}^* \quad v_0^* \quad v_1^* \cdots \quad v_n^*]^*$ to capture the magnitude and phase information of all frequency components of the reference. The tracking performance is usually measured by the energy of transient tracking error:

$$J(v) = \sum_{k=0}^{\infty} \|r(k) - z(k)\|^2 = \sum_{k=0}^{\infty} \|e(k)\|^2. \qquad (2)$$

The problem considered in this paper is to find an explicit expression of the smallest tracking error, i.e., the *performance limit* of the system,

$$J_{opt}(v) = \inf_K J(v) \qquad (3)$$

when the controller K is chosen among all possible stabilizing 2DOF controllers. In this paper, we achieve this understanding in the form of an explicit, simple, and informative relationship between this performance limit and the plant characteristics as well as the parameter vector v of the reference.

If we are interested in an overall performance measure of the feedback system in tracking all references of the type (1), then we normally turn our attention to an averaged version of the tracking error and we normally take the average over all possible v whose entries have zero mean, are mutually uncorrelated, and have a

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unit variance. Such an averaged performance measure is given by

$$J = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\}. \tag{4}$$

In this case, the performance limitation becomes

$$J_{opt} = \inf_K J. \tag{5}$$

It turns out that the solution to this problem is simple enough to be stated as follows: Under some minor assumptions,

$$J_{opt} = \sum_{i=1}^{m} \sum_{l=-n}^{n} \frac{1 + e^{j\omega_l} q_i}{1 - e^{j\omega_l} q_i} \tag{6}$$

where $q_i, i = 1, 2, \ldots, m$, are the nonminimum phase zeros, i.e., those zeros inside the unit circle, of the transfer function from $u(k)$ to $z(k)$.

Results of this sort first appeared for continuous time systems [3], [6]. The discrete time version only started to appear in recent years [5], [12]. For the case when $r(k)$ is a step signal of the form $r(k) = v, k \geq 0$, it is shown in [5]

$$J_{opt}(v) = \|v\|_2^2 \sum_{i=1}^{m} \frac{1 + q_i}{1 - q_i} \cos^2 \angle(v, \eta_i)$$

where η_i is a vector associated with the zero q_i and

$$J_{opt} = \sum_{i=1}^{m} \frac{1 + q_i}{1 - q_i}.$$

Further study for other types of reference signals, including a sinusoid with a single frequency, was carried out in [12]

In formula (6), the assumption that the state of the exosystem is available to the controller is crucial. This simply means that not only the reference but also the magnitude and phase information of all of its frequency components is known to the controller. This may look unrealistic in the first glance, but this gives a limitation more fundamental to any other one assuming a partial information structure. Note that when the reference only contains a constant term, the value of the reference already gives its full information.

This paper gives a rather complete picture for the tracking performance limitation problem for general reference containing several frequency components. We first give some new insight on linear system structure. We show that each nonminimum phase zero has associated frequency dependent directions. A key technical result in this paper is a relation among directions at different frequencies. By using this technical result, we derive an expression for $J_{opt}(v)$ which elegantly exhibits the effect of the plant nonminimum phase zeros and the interaction between each frequency component and the directions mentioned above.

Figure 2 shows the structure of an LTI discrete-time system in which only the reference is available for the controller $K(\lambda)$. If the parameter vector v of the reference is available for controller design, then we will obtain a v dependent controller $K(\lambda)$ and the same performance limit $J_{opt}(v)$ as that of the systems shown in Figure 1. However, this is an idea case. A more practical case is that the parameter vector v is not available

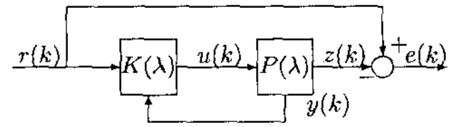

Figure 2: A two-parameter control structure with only reference information

for the controller K. In this case, we turn our attention to an averaged version of the tracking error J_{opt} defined in (4) and (5). It turns out that deriving a simple expression for J_{opt} is hard for the general reference of the form (1). We will consider instead a special case when $r(k)$ is a scalar signal containing a single sinusoid

$$v(k) = v_{-1} e^{-j\omega k} + v_1 e^{j\omega k}.$$

Under some mild assumptions, we are able to find that

$$\begin{aligned} J_{opt} &= \sum_{i=1}^{m} \left(\frac{1 + e^{-j\omega} q_i}{1 - e^{-j\omega} q_i} + \frac{1 + e^{j\omega} q_i}{1 - e^{j\omega} q_i} \right) \\ &+ \frac{2}{\sin \omega} \sin^2 2 \sum_{i=1}^{m} \left[\angle(1 - e^{-j\omega} q_i) + \frac{\omega}{2} \right] \end{aligned}$$

where $\angle(1 - e^{-j\omega} q_i)$ stands for the phase or argument of the complex number $1 - e^{-j\omega} q_i$. Comparing this with the performance limit in the full reference information case, which is

$$J_{opt} = \sum_{i=1}^{m} \left(\frac{1 + e^{-j\omega} q_i}{1 - e^{-j\omega} q_i} + \frac{1 + e^{j\omega} q_i}{1 - e^{j\omega} q_i} \right),$$

we are able to pinpoint exactly the performance deterioration due to the limited information.

The organization of this paper is as follows. In Section 2, preliminary materials on linear system factorizations are presented. It is shown that a right-invertible system can be factorized as a cascade connection of series of first order inner factors and a minimum phase factor. The factorization is frequency dependent. The inner factors then contain all the information associated to the nonminimum phase zeros. In Section 3, we formally formulate the problems studied and then state and discuss the main result and some of its consequences for the case where the full state information is available. Section 4 gives result for the case where only reference signal is available. Section 5 is the conclusion. The proofs of the main results in Section 3 and 4 are given in Appendix.

The notation used throughout this paper is fairly standard. For any complex number, vector and matrix, denote their conjugate, transpose, conjugate transpose, real and imaginary part by $\overline{(\cdot)}$, $(\cdot)^T$, $(\cdot)^*$, $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$, respectively. Denote the expectation of a random variable by $\boldsymbol{E}\{\cdot\}$. Let the open unit disk, the unit circle, and the part of the complex plane outside of the unit disk together with the infinity be denoted by $\mathbb{D}$, $\mathbb{T}$, and $\bar{\mathbb{D}}^c$ respectively. The usual Lebesgue space of all possibly vector valued square integral functions on $\mathbb{T}$ is denoted by $\mathcal{L}_2$. The set of those functions in $\mathcal{L}_2$ which are analytic on $\mathbb{D}$ is denoted by $\mathcal{H}_2$ and the set of those functions in $\mathcal{L}_2$ which are analytic on $\bar{\mathbb{D}}^c$ and vanish at the infinity is denoted by $\mathcal{H}_2^\perp$. It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are indeed orthogonal complement to each other as subspaces of $\mathcal{L}_2$. The Euclidean vector norm and the norm in the space $\mathcal{L}_2$ are both denoted by $\|\cdot\|_2$. $\mathcal{RH}_\infty$ is the set of all stable, rational transfer matrices (see [14]). Finally, the inner product between two complex vectors u, v is defined by $\langle u, v\rangle := u^* v$.

2 Preliminaries

Let $G(\lambda)$ be a real rational matrix representing the transfer function of a discrete time finite-dimensional, linear time invariant (FDLTI) system. Let us assume that $G(\lambda)$ is right invertible. Its poles and zeros (including multiplicity) are defined in the usual way according to its Smith-MacMillan form. $G(\lambda)$ is said to be minimum phase if it is free of zeros in $\mathbb{D}$; otherwise it is said to be nonminimum phase.

It is worth noticing that a particular class of nonminimum phase zeros are those at the origin. Such nonminimum phase zeros are possibly caused by pure delays in the output channels.

Let $G(\lambda) = N(\lambda)M^{-1}(\lambda)$, where $M(\lambda), N(\lambda) \in \mathcal{RH}_\infty$, be a right coprime factorization of $G(\lambda)$. Let $q \in \mathbb{C}$ be a nonminimum phase zero of $G(\lambda)$. Then q is also a nonminimum phase zero of $N(\lambda)$ and there exists a unit vector η such that $\eta^* N(q) = 0$. We call the vector η a (left or output) zero vector of $G(\lambda)$ corresponding to the nonminimum phase zero q.

Let us now order the nonminimum phase zeros of $G(\lambda)$ (or $N(\lambda)$ equivalently) as $q_1, q_2, \ldots, q_m$. Assume that each pair of complex conjugate zeros are ordered in adjacent order. Let us also fix a frequency $\omega_l \in \mathbb{R}$. We first find a unit vector $\eta_{\omega_l 1}$ of $G(\lambda)$ corresponding to q_1 and define

$$G_{\omega_l 1}(\lambda) = U_{\omega_l 1} \begin{bmatrix} \frac{1 - e^{j\omega_l} q_1^*}{e^{j\omega_l} - q_1} \frac{\lambda - q_1}{1 - \lambda q_1^*} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 1}^* \quad (7)$$

where $U_{\omega_{l1}}$ is a unitary matrix with the first column equal to $\eta_{\omega_l 1}$. Here $G_{\omega_l 1}(\lambda)$ is so constructed that it is inner, has the only zero at q_1 with $\eta_{\omega_l 1}$ as a zero vector corresponding to q_1, and $G_{\omega_l 1}(j\omega_l) = I$. Since it is a generalization of the standard scalar Blaschke factor, we call it a matrix Blaschke factor at the frequency w_l and call $\eta_{\omega_l 1}$ a corresponding Blaschke vector. Also notice that the choice of other columns in $U_{\omega_l 1}$ is immaterial. Now $G_{\omega_l 1}^{-1}(\lambda)G(\lambda)$ has zeros $q_2, q_3, \ldots, q_m$. Find a zero vector $\eta_{\omega_l 2}$ of $G_{\omega_l 1}^{-1}(\lambda)G(\lambda)$ corresponding to q_2 and define

$$G_{\omega_l 2}(\lambda) = U_{\omega_l 2} \begin{bmatrix} \frac{1 - e^{j\omega_l} q_2^*}{e^{j\omega_l} - q_2} \frac{\lambda - q_2}{1 - \lambda q_2^*} & 0 \\ 0 & I \end{bmatrix} U_{\omega_l 2}^*$$

where $U_{\omega_l 2}$ is a unitary matrix with the first column equal to $\eta_{\omega_l 2}$. Then $G_{\omega_l 2}^{-1}(\lambda)G_{\omega_l 1}^{-1}(\lambda)G(\lambda)$ has zeros $q_3, q_4, \ldots, q_m$. Continue this process until Blaschke vectors $\eta_{\omega_l 1}, \ldots, \eta_{\omega_l m}$ and Blaschke factors $G_{\omega_l 1}(\lambda), \ldots, G_{\omega_l m}(\lambda)$ are obtained. This procedure shows that $G(\lambda)$ can be factorized as

$$G(\lambda) = G_{\omega_l 1}(\lambda) \cdots G_{\omega_l m}(\lambda) G_{\omega_l 0}(\lambda) \quad (8)$$

where $G_{\omega_l i}(\lambda)$ is in the same form as that of $G_{\omega_l 1}(\lambda)$ in (7). and $G_{\omega_l 0}(\lambda)$ has no nonminimum phase zero. Let us call such a factorization a cascade factorization at frequency ω_l, which is shown schematically in Figure 3. In this factorization, each Blaschke vector and Blaschke factor correspond to one nonminimum phase zero. Keep in mind that these vectors and factors depend on the order of the nonminimum zeros, as well as on the frequency ω_l. The product

$$G_{\omega_l 1}(\lambda) \cdots G_{\omega_l m}(\lambda)$$

is called a matrix Blaschke product.

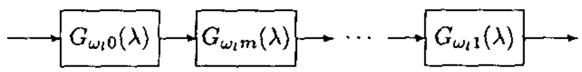

Figure 3: Cascade factorization

One should note that even though the order of $q_1, q_2, \ldots, q_m$ is fixed, the factorization at the frequency ω_l is not unique since $\eta_{\omega_l i}$ is not uniquely determined. Furthermore, if we have $2n+1$ different frequencies $\omega_l, l = 0, \pm 1, \ldots, \pm n$, then the factorizations at different frequencies are in general different. Nevertheless, they can be intimately related if we make the choices properly. For example, it is easy to see from the above construction that $\eta_{\omega_l 1}$, the first Blaschke vector, can be chosen independent of ω_l. The following lemma provides such relations and is the key technical result that leads to the main result of this paper.

Lemma 2.1 *Suppose that the order of $q_1, q_2, \ldots, q_m$ is fixed. Also suppose that we are given $2n+1$ different frequencies $\omega_l, l = 0, \pm 1, \ldots, \pm n$. Then the $2n+1$ cascade factorizations (8) can be chosen so that for all $l, l' = 0, \pm 1, \cdots, \pm n$ and $i = 1, 2, \ldots, m$,*

$$\eta_{\omega_l i} = G_{\omega_{l'} 1}(e^{j\omega_l}) G_{\omega_{l'} 2}(e^{j\omega_l}) \cdots G_{\omega_{l'} i-1}(e^{j\omega_l}) \eta_{\omega_{l'} i}. \quad (9)$$

Proof: See [11].

For SISO LTI discrete-time systems, proper choices of Blaschke vectors are given as below:

$$\eta_{\omega_k i} = \frac{1 - q_1^*}{1 - q_1} \frac{e^{j\omega} - q_1}{1 - e^{j\omega} q_1^*} \cdots \frac{1 - q_{i-1}^*}{1 - q_{i-1}} \frac{e^{j\omega} - q_{i-1}}{1 - e^{j\omega} q_{i-1}^*}$$

for $k = 0, \pm 1, \cdots, \pm n$.

3 The Main Result

Let us go back to the setup shown in Figure 1. The measurement output $y(k)$ of the plant is assumed to be different from the tracking output $z(t)$. We denote the transfer function from $u(k)$ to $z(k)$ by $G(\lambda)$ and that from $u(k)$ to $y(k)$ by $H(\lambda)$, i.e., $P(\lambda) = \begin{bmatrix} G(\lambda) \\ H(\lambda) \end{bmatrix}$. In order for the tracking problem to be meaningful and solvable, we make the following assumptions throughout the paper.

Assumption 3.1

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1. $P(\lambda)$, $G(\lambda)$ and $H(\lambda)$ have the same unstable poles.

2. $G(\lambda)$ has no zero at $e^{j\omega_l}, l = 0, \pm 1, \cdots, \pm n$.

The first item in the assumption means that the measurement can be used to stabilize the system and at the same time does not introduce any additional unstable modes. A more precise way of stating this is that if $P(\lambda) = \begin{bmatrix} N(\lambda) \\ L(\lambda) \end{bmatrix} M^{-1}(\lambda)$ is a coprime factorization, then we assume that $N(\lambda)M^{-1}(\lambda)$ and $L(\lambda)M^{-1}(\lambda)$ are also coprime factorizations. The second item is of course necessary for the solvability of the tracking problem.

Now it is ready to state our main result.

Theorem 3.1 *Let $G(\lambda)$ have nonminimum phase zeros $q_1, q_2, \ldots, q_m$ with associated Blaschke vectors $\eta_{\omega_l 1}, \ldots, \eta_{\omega_l m}$, $l = 0, \ldots, n$, satisfying Lemma 2.1. Then*

$$\begin{aligned} J_{opt}(v) &= \sum_{i=1}^{m}(1 - q_i q_i^*)\left|\sum_{l=-n}^{n} \frac{\langle \eta_{-\omega_l i}, v_l\rangle}{1 - e^{j\omega_l} q_i}\right|^2 \\ &= \sum_{i=1}^{m}\sum_{l=-n}^{n}\sum_{l'=-n}^{n} (1 - q_i q_i^*) \frac{\langle v_l, \eta_{-\omega_l i}\rangle \langle \eta_{-\omega_{l'} i}, v_{l'}\rangle}{(1 - e^{-j\omega_l} q_i^*)(1 - e^{j\omega_{l'}} q_i)}. \end{aligned} \tag{10}$$

Proof: See Appendix A.

This formula for the performance limit clearly shows that it depends on the nonminimum phase zeros in an additive way and the contribution of each nonminimum phase zero to the performance limitation depends in a quadratic way on the frequency components of the reference. It also clearly shows how various frequency components enter the performance limitation through the inner products with the Blaschke vectors at the corresponding frequencies.

In the case when $n = 0$, i.e., the reference only has a step component, we get

$$J_{opt}(v) = \sum_{i=1}^{m} \frac{(1 - q_i q_i^*)}{|1 - q_i|^2} |\langle \eta_{0i}, v\rangle|^2 = \sum_{i=1}^{m} \frac{1 + q_i}{1 - q_i} |\langle \eta_{0i}, v\rangle|^2.$$

This is exactly the formula given in [5], [12].

The proof of Theorem 3.1 also shows that a controller, or a sequence of controllers, independent of v can be found to attain the performance limitation. Therefore

$$\begin{aligned} &J_{opt} \\ &= \inf_K \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \qquad (11) \\ &= \boldsymbol{E}\{\inf_K J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \qquad (12) \\ &= \sum_{i=1}^{m}\sum_{l=-n}^{n}\sum_{l'=-n}^{n} (1 - q_i q_i^*) \frac{\eta^*_{-\omega_{l'} i} \boldsymbol{E}(v_{l'} v_l^*) \eta_{-\omega_l i}}{(1 - e^{-j\omega_l} q_i^*)(1 - e^{j\omega_{l'}} q_i)} \\ &= \sum_{i=1}^{m}\sum_{l=-n}^{n} (1 - q_i q_i^*) \frac{\eta^*_{-\omega_l i} \eta_{-\omega_l i}}{(1 - e^{-j\omega_l} q_i^*)(1 - e^{j\omega_l} q_i)} \\ &= \sum_{i=1}^{m}\sum_{l=-n}^{n} \frac{1 - q_i q_i^*}{|1 - e^{j\omega_l} q_i|^2} = \sum_{i=1}^{m}\sum_{l=-n}^{n} \frac{1 + e^{j\omega_l} q_i}{1 - e^{j\omega_l} q_i}. \end{aligned}$$

This immediately leads to the following theorem.

Theorem 3.2 *Let $G(s)$ have nonminimum phase zeros $q_1, q_2, \ldots, q_m$. Then*

$$J_{opt} = \sum_{i=1}^{m}\sum_{l=-n}^{n} \frac{1 - q_i q_i^*}{|1 - e^{j\omega_l} q_i|^2} = \sum_{i=1}^{m}\sum_{l=-n}^{n} \frac{1 + e^{j\omega_l} q_i}{1 - e^{j\omega_l} q_i}. \tag{13}$$

4 Design Limitation from Partial Information of the Reference

In this section, the performance limit of optimal tracking problem is discussed for the system shown in Fig. 2. Denote $P(\lambda) = \begin{bmatrix} G(\lambda) \\ H(\lambda) \end{bmatrix}$ and assume that Assumption 3.1 holds. Moreover, it is assumed that only reference signal of the system is available for a controller. To solve this tracking problem, we will start at the same setup as that in the Section 3. If the magnitude and phase information v of all frequency components of the reference is still available for controller design, a v-dependent controller and performance limit $J_{opt}(v)$ as that in last section can be obtained. It must be noted that this is only an idea case. Here we will consider a more practical case where the parameter vector v is unknown for controller designing. To find meaningful solution for the tracking problem, we will consider the average tracking performance J defined in (4). But, in this case, the exchange of the infimum and expectation in the the step from (11) to (12) is no longer valid. Furthermore, the result in Theorem 3.2 is no longer true. Essentially, this constraint is caused by the partial information of the reference signal. It will be shown that more performance limitation is imposed on the optimal tracking problem by this constraint.

In general, without full information of a reference signal, it is very complicated to find the performance limit for a tracking problem. In this paper, we only discuss the performance limit of a SISO linear system in tracking a single frequency sinusoidal signal as follows:

$$r(k) = v_{-1} e^{-j\omega k} + v_1 e^{j\omega k}. \tag{14}$$

The magnitude and phase information of the reference is denoted by $v' = [v_{-1}\ v_1]$.

Theorem 4.1 *Let $G(\lambda)$ have nonminimum phase zeros $q_1, \cdots, q_m$. Then,*

$$\begin{aligned} J_{opt} &= \sum_{i=1}^{m}\left(\frac{1 + e^{-j\omega} q_i}{1 - e^{-j\omega} q_i} + \frac{1 + e^{j\omega} q_i}{1 - e^{j\omega} q_i}\right) \\ &\quad + \frac{2}{\sin\omega} \sin^2 2 \sum_{i=1}^{m}\left[\angle(1 - e^{-j\omega} q_i) + \frac{\omega}{2}\right]. \end{aligned} \tag{15}$$

Proof: See [11].

In Theorem 3.2, it is shown that, if the full information of the reference (14) is available, the tracking performance limit of the linear system $P(\lambda)$ is given by

$$J_{opt} = \sum_{i=1}^{m}\left(\frac{1 + e^{-j\omega} q_i}{1 - e^{-j\omega} q_i} + \frac{1 + e^{j\omega} q_i}{1 - e^{j\omega} q_i}\right).$$

Theorem 4.1 gives explicit expression of the tracking performance limit for the case where the full state information of the reference is unavailable. This expression reveals that an extra limitation caused by this partial information is imposed on the tracking performance. Indeed, in this case, the controller has to estimate the state of reference first and then track the reference based on the estimated information. The extra limitation on the tracking performance is from the cost of estimation.

5 Conclusions

In this paper, a formula is obtained for the best tracking performance for discrete-time LTI multivariable systems when the reference is a given linear combination of step and sinusoidal signals. This formula clearly reveals the role that each nonminimum phase zero, as well as its associated frequency dependent directions, plays towards the performance limitation. A formula is also obtained for the average tracking performance for all references with the same frequency components.

APPENDIX A

Proof of the Theorem 3.1

Now let us consider the tracking problem as shown in Figure 1. Let $G(\lambda) = N(\lambda)M^{-1}(\lambda)$ be a coprime factorization. By using the parameterization of all stabilizing two degree of freedom controllers [13], we can see that, under Assumption 3.1, all possible transfer functions from $v(k)$ to $z(k)$ are given by $N(\lambda)Q(\lambda)$ where $Q(\lambda)$ is an arbitrary $\mathcal{H}_\infty$ transfer matrix which can be designed. Let us denote the λ-transform of $r(k)$ by $R(\lambda)$ and that of $v(k)$ by $V(\lambda)$. Then the integral square error (2) becomes

$$J(v) = \|R(\lambda) - N(\lambda)Q(\lambda)V(\lambda)\|_2^2.$$

Notice that $N(\lambda)$ is stable and its nonminimum phase zeros are the same as those of $G(\lambda)$. If $G(\lambda)$ is factored as

$$G(\lambda) = G_{\omega_0 1}(\lambda)\cdots G_{\omega_0 m}(\lambda)G_{\omega_0 0}(\lambda) \qquad \text{(A-1)}$$

where $G_{\omega_0 i}(\lambda)$ is a Blaschke factor of the form of (7) and $G_{\omega_0 0}(\lambda)$ is minimum phase. Then $N(\lambda)$ has inner-outer factorization

$$\begin{aligned} N(\lambda) &= G_{in}(\lambda)N_{out}(\lambda) \\ &= [G_{\omega_0 1}(\lambda)\cdots G_{\omega_0 m}(\lambda)][G_{\omega_0 0}(\lambda)M(\lambda)]. \end{aligned}$$

The tracking performance $J(v)$ can be rewritten as

$$\begin{aligned} &J(v) \\ &= \left\|\sum_{l=-n}^{n} \frac{v_l}{1-\lambda e^{j\omega_l}} - G_{in}(\lambda)N_{out}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2 \\ &= \left\|\left[G_{in}^{-1}(\lambda)\sum_{l=-n}^{n} \frac{v_l}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}}\right] \right. \\ &\quad \left. + \left[\sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{out}(\lambda)Q(\lambda)V(\lambda)\right]\right\|_2^2. \end{aligned}$$

It is easy to see that

$$G_{in}^{-1}(\lambda)\sum_{l=-n}^{n} \frac{v_l}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} \in \mathcal{H}_2^\perp.$$

and

$$\sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{out}(z)Q(\lambda)V(\lambda)$$

can be made to belong to $\mathcal{H}_2$ by properly choosing $Q(z)$. It then follows that

$$\begin{aligned} J(v) &= \left\|G_{in}^{-1}(\lambda)\sum_{l=-n}^{n} \frac{v_l}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}}\right\|_2^2 \\ &\quad + \left\|\sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{out}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2 \quad \text{(A-2)} \end{aligned}$$

Without loss of generality, we can assume

$$V(\lambda) = \left[\frac{v_{-n}^T}{1-\lambda e^{j\omega_{-n}}} \quad \cdots \quad \frac{v_0^T}{1-\lambda e^{j\omega_0}} \quad \cdots \quad \frac{v_n^T}{1-\lambda e^{j\omega_n}}\right]^T.$$

Partition $Q(\lambda)$ consistently as

$$Q(\lambda) = [Q_{-n}(\lambda) \quad \cdots \quad Q_0(\lambda) \quad \cdots \quad Q_n(\lambda)]$$

and let

$$Q_l(\lambda) = N_{out}^\dagger(e^{-j\omega_l})G_{in}^{-1}(e^{-j\omega_l}) + (1-\lambda e^{j\omega_l})\tilde{Q}_l(\lambda)$$

where $N_{out}^\dagger(e^{-j\omega_l})$ is a right inverse of $N_{out}(e^{-j\omega_l})$. Then we have

$$\begin{aligned} &\sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{out}(\lambda)Q(\lambda)V(\lambda) \\ &= \sum_{l=-n}^{n} \left\{[I - N_{out}(\lambda)N_{out}^\dagger(e^{-j\omega_l})]\frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} \right. \\ &\quad \left. - N_{out}(\lambda)\tilde{Q}_l(\lambda)v_l\right\} \in \mathcal{H}_2. \end{aligned}$$

Notice that

$$[I-N_{out}(\lambda)N_{out}^\dagger(e^{-j\omega_l})]\frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} \in \mathcal{H}_2, \;\; l = -n, \cdots, n$$

and $N_{out}(\lambda)$ is outer. Then we can always find $\tilde{Q}_l(\lambda)$, $l = 0, \pm 1, \ldots, \pm n$, such that

$$\left\|\sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} - N_{out}(\lambda)Q(\lambda)V(\lambda)\right\|_2^2 \longrightarrow 0.$$

This shows that the second term of (A-2) can be made arbitrarily small by choosing $Q(\lambda)$, independent of v. Consequently,

$$J_{opt}(v) = \left\|\sum_{l=-n}^{n} \frac{v_l}{1-\lambda e^{j\omega_l}} - G_{in}(\lambda)\sum_{l=-n}^{n} \frac{G_{in}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}}\right\|_2^2.$$

Following the definition of $G_{in}(\lambda)$ in (A-1) and selecting $\omega_0 = 0$, we have

$$\begin{aligned} J_{opt}(v) = &\left\|\sum_{l=-n}^{n} \left\{\left[\frac{G_{01}^{-1}(\lambda)}{1-\lambda e^{j\omega_l}} - \frac{G_{01}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}}\right]v_l \right.\right. \\ &\left.\left. + \left[\frac{G_{01}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} - \prod_{\kappa=2}^{m} G_{0i}(\lambda)\frac{\prod_{\kappa=m}^{1} G_{0\kappa}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}}\right]v_l\right\}\right\|_2^2 \end{aligned} \qquad \text{(A-3)}$$

It is easily be verified that

$$\frac{G_{01}^{-1}(\lambda)}{1-\lambda e^{j\omega_l}} - \frac{G_{01}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} \in \mathcal{H}_2^\perp$$

and

$$\frac{G_{01}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} - \prod_{\kappa=2}^{m} G_{0i}(\lambda) \frac{\prod_{\kappa=m}^{1} G_{0\kappa}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} \in \mathcal{H}_2.$$

Hence (A-3) is rewritten as below:

$$\begin{aligned} &J_{opt}(v) \\ &= \left\| \sum_{l=-n}^{n} \left[\frac{G_{01}^{-1}(\lambda)}{1-\lambda e^{j\omega_l}} - \frac{G_{01}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} \right] v_l \right\|_2^2 + \left\| \left[\frac{G_{01}^{-1}(e^{-j\omega_l})v_l}{1-\lambda e^{j\omega_l}} \right. \right. \\ &\left. \left. - \prod_{\kappa=2}^{m} G_{0i}(\lambda) \frac{\prod_{\kappa=m}^{2} G_{0\kappa}^{-1}(e^{-j\omega_l})[G_{01}^{-1}(e^{-j\omega_l})v_l]}{1-\lambda e^{j\omega_l}} \right] \right\|_2^2. \end{aligned}$$

Repeating this procedure for $m-1$ times leads to

$$\begin{aligned} J_{opt}(v) = \sum_{i=1}^{m} \Bigg\| &\left[\sum_{l=-n}^{n} \frac{G_{0i}^{-1}(\lambda)}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n} \frac{G_{0i}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} \right] \\ &\prod_{\kappa=i-1}^{1} G_{0\kappa}^{-1}(e^{-j\omega_l}) v_l \Bigg\|_2^2 \qquad \text{(A-4)} \end{aligned}$$

where $\prod_{\kappa=i-1}^{1} G_{0\kappa}^{-1}(e^{-j\omega_l}) = I$ for $i=1$.

From the definition of $G_i(\lambda)$, it holds

$$\begin{aligned} &\sum_{l=-n}^{n} \frac{G_{0i}^{-1}(\lambda)}{1-\lambda e^{j\omega_l}} - \sum_{l=-n}^{n} \frac{G_{0i}^{-1}(e^{-j\omega_l})}{1-\lambda e^{j\omega_l}} \\ &= \frac{1-q_i}{1-q_i^*} \left[\sum_{l=-n}^{n} \frac{1-q_i q_i^*}{1-e^{j\omega_l} q_i} \eta_{0i} \eta_{0i}^* \right] \frac{1}{\lambda - q_i}. \qquad \text{(A-5)} \end{aligned}$$

Substituting (A-5) into (A-4) leads to

$$\begin{aligned} &J_{opt}(v) \\ &= \sum_{i=1}^{m} \left\| \left[\sum_{l=-n}^{n} \frac{1-q_i q_i^*}{1-e^{j\omega_l} q_i} \eta_{0i} \eta_{0i}^* \right] \frac{1}{\lambda - q_i} \prod_{\kappa=i-1}^{1} G_{0\kappa}^{-1}(e^{-j\omega_l}) v_l \right\|_2^2 \\ &= \sum_{i=1}^{m} (1-q_i q_i^*) \left| \sum_{l=-n}^{n} \frac{\langle \eta_{0i}, \prod_{\kappa=i-1}^{1} G_{0\kappa}^{-1}(e^{-j\omega_l}) v_l \rangle}{1-e^{j\omega_l} q_i} \right|^2. \end{aligned}$$

Finally, Lemma 2.1 immediately yields

$$J_{opt}(v) = \sum_{i=1}^{m} (1-q_i q_i^*) \left| \sum_{l=-n}^{n} \frac{\langle \eta_{-\omega_l i}, v_l \rangle}{1-e^{j\omega_l} q_i} \right|^2.$$

This completes the proof.

References

[1] J. Chen, L. Qiu and O. Toker, "Limitations on maximal tracking accuracy", *IEEE Trans. Automat. Contr.*, vol. AC-45, pp. 326-331.

[2] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

[3] M. Morari and E. Zafiriou, *Robust Process Control*, Prentice Hall, Englewood Cliffs, NJ, 1989.

[4] K. Poolla and T. Ting, "Nonlinear Time-Varying Controllers for Robust Stabilization". *IEEE Trans. Auto. Contro.*, **32**, pp. 195-200, 1987.

[5] L. Qiu and J. Chen, "Time Domain Characterizations of Performance Limitations of Feedback Control", In Y. Yamamoto and S. Hara, editors, *Learning, Control and Hybrid Systems*, Springer-Verlag, London, pp. 397-415, 1999.

[6] L. Qiu and E. J. Davison, "Performance limitations of nonminimum phase systems in the servomechanism problem", *Automatica*, vol. 29, pp. 337-349, 1993.

[7] M. M. Seron, J. H. Braslavsky and G. C. Goodwin, *Fundamental Limitations in Filtering and Control*, Springer-Verlag, London, 1997.

[8] M. M. Seron, J. H. Braslavsky, P. V. Kokotovic, and D. Q. Mayne, "Feedback limitations in nonlinear systems: from Bode integrals to cheap control", *IEEE Trans. on Automatic Control*, vol. 44, pp. 829–833, 1999.

[9] M. M. Seron, J. H. Braslavsky, D. G. Mayne, and P. V. Kokotovic, "Limiting performance of optimal linear filters", *Automatica*, vol 35, pp. 189–199, 1999.

[10] W. Su, L Qiu, and J. Chen, "Fundamental performance limitations in tracking sinusoidal signals", *IEEE Trans. on Automatic Control*, 2002, submitted.

[11] W. Su, L Qiu, and J. Chen, "Performance Limitations of Discrete-Time Systems in Tracking Sinusoidal Signals", Working paper 2003.

[12] O. Toker, J. Chen and L. Qiu, "Tracking performance limitations in LTI multivariable discrete-time systems", *IEEE Trans. Circuits and Systems, Part I*, vol. 49, 657-670, 2002.

[13] M. Vidyasagar, *Control System Synthesis: a Factorization Approach*, Cambridge, MA: MIT Press, 1985.

[14] K. Zhou, J. C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, NJ: Upper Saddle River, 1995.