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Performance Limitations of Non-minimum Phase Systems in the Servomechanism Problem*†

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A fundamental limitation exists in the achievable transient performance which is possible to be obtained in the tracking and disturbance rejection of a non-minimum phase system, and this limitation can be characterized completely by the number and locations of the right-half plane zeros.

Key Words—Linear optimal regulator; servomechanisms; non-minimum phase systems; transmission zeros; cheap control.

Abstract—This paper studies the cheap regulator problem and the cheap servomechanism problem for systems which may be non-minimum phase. The study extends some well-known properties of "perfect regulation" and the "perfect tracking and disturbance rejection" of minimum phase systems to non-minimum phase systems. It is shown that perfect rejection to disturbances applied to the plant input can be achieved no matter whether the system is minimum phase or non-minimum phase, whereas a fundamental limitation exists in the achievable transient performance of tracking and rejection to disturbances applied to the plant output for a non-minimum phase system, and that this limitation can be simply and completely characterized by the number and locations of those zeros of the system which lie in the right half of the complex plane. Furthermore, this limitation provides a quantitative measure of the "degree of difficulty" which is inherent in the control of such non-minimum phase systems.

1. INTRODUCTION

It has long been realized that minimum phase systems have certain advantages over non-minimum phase systems; for example, right-invertible minimum phase systems can achieve perfect regulation (Kwakernaak and Sivan, 1972b; Francis, 1979; Scherzinger and Davison, 1985) and perfect tracking/disturbance rejection (Davison and Chow, 1977; Davison and Scherzinger, 1987). These properties, however, are not possessed by non-minimum phase systems. In addition, a non-minimum phase system, unlike minimum phase systems, has various fundamental limitations associated with the achievable closed loop transfer matrix (Cheng and Desoer, 1980) the achievable closed loop gain margin (Tannenbaum, 1980) LQG loop transfer recovery (Stein and Athans, 1987; Zhang and Freudenberg, 1990) sensitivity or complementary sensitivity minimization (Freudenberg and Looze, 1985; Francis, 1987) model reference adaptive control (Miller and Davison, 1989) etc. On the other hand, it has been recognized that not all non-minimum phase systems behave in the same way; for example, some non-minimum phase systems produce results which are "almost as good" as minimum phase systems, whereas other non-minimum phase systems are indeed "almost impossible" to control. It is the purpose of this paper to study the quality of non-minimum phase systems with respect to tracking and disturbance rejection. We will show that for each non-minimum phase system there exists a fundamental limitation on the achievable transient response of the system. This limitation has a simple characterization in terms of the number and the locations of those zeros of the system which lie in the open-right complex plane, and provides a quantitative measure of the "degree of difficulty" which is inherent in the control of a non-minimum phase system.

In this paper, we first study the cheap linear quadratic regulator (LQR) problem for non-minimum phase systems. Consider a linear

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time-invariant system described by

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where u, x, y are finite dimensional vectors depending on the time t. Assume that (A, B, C, D) is stabilizable and detectable, and consider also the associated optimal cost functional

$$J_\epsilon = \min_u \int_0^\infty (y'y + \epsilon^2 u'u)\,\mathrm{d}t. \tag{2}$$

The cheap LQR problem concerns the limit $J_0 := \lim_{\epsilon \to 0} J_\epsilon$. It is shown in this paper that J_0 has a very simple expression when (A, B, C, D) has a particular special structure. This expression is not very important *per se* since very few systems have such a special structure. However, the significance of the result lies in the fact that for every transfer matrix, there always exists a realization which has such a special structure. Therefore the result has important applications in control problems which depend only on the system transfer matrix rather than on the internal realization of the system. The cheap optimal servomechanism problem, which is the focus of this paper, belongs to this class of problems. An expression for J_0 involving the concept of "zero directions" was obtained by Shaked (Grimble and Johnson, 1988) for the case when D is a zero matrix and A, B, C are general matrices; this expression, although interesting, is quite complicated and is not convenient to use in our application.

Consider now a system with noise corrupted input and output:

$$\begin{aligned} \dot{x} &= Ax + B(u + \xi), \quad x(0) = 0, \\ y &= Cx + D(u + \xi) + \eta, \end{aligned} \tag{3}$$

where ξ is the input disturbance, η is the output disturbance. We assume again that (A, B, C, D) is stabilizable and detectable. A control problem which often arises is to design a controller for system (3) such that the overall system is internally stable and such that the output y asymptotically tracks a reference signal y_{ref} for arbitrary ξ, η and y_{ref} contained in a certain class of signals. This problem is called a *servomechanism problem*. It is well-known that a general servomechanism problem can be treated as one with the reference signal being equal to zero since the distinguishing role between y_{ref} and η disappears if the tracking error $y - y_{\mathrm{ref}}$ is taken to be the output under consideration. Therefore, we will assume that $y_{\mathrm{ref}} = 0$ throughout the paper. In practice, a controller which solves the servomechanism problem is also required to have a good transient response, i.e. it is desired that the closed loop system should have a "fast speed of response" without "excessive peaking/oscillation" occurring in the output y and other system variables, as they approach their steady state values. To achieve such a response, we seek a controller which generates an input to achieve the following optimal quadratic cost functional

$$J_\epsilon = \min_{\tilde{u}} \int_0^\infty (y'y + \epsilon^2 \tilde{u}'\tilde{u})\,\mathrm{d}t, \tag{4}$$

where $\tilde{u}$ is a variable associated with the transient behaviour of the input u. The problem concerning the limit $J_0 := \lim_{\epsilon \to 0} J_\epsilon$ is called the *cheap servomechanism problem*. It is clear that J_ϵ and J_0 depend on the controller structure used since the controller structure will determine the physical meaning of $\tilde{u}$. In this paper, we first look at the ideal case: we assume that the system parameters are exactly known and that the disturbances are measurable. In this case, a feedforward controller structure can be used (Davison, 1973). It is shown in this case that J_0 is a quadratic form on the output disturbance only, and that a norm of this quadratic form can be characterized explicitly by the locations of the zeros of system (3) which lie in the open right hand side of the complex plane. In practice, the system parameters are always somewhat uncertain and it is often impossible to measure the disturbances; in this case, the servomechanism problem is still solvable but the controller has to contain an internal model of the disturbances, see e.g. Francis and Wonham (1976) and Davison (1976). Such a controller is called a robust servomechanism controller. We will show in this case that J_0 has the same characteristics as in the feedforward controller case, i.e. J_0 is a quadratic form on the output disturbance only and a norm of this quadratic form can be characterized explicitly in the same way by the locations of those zeros of sytem (3) which lie in the open right hand side of the complex plane. The significance of the results for the feedforward controller lies in that it tells us what is the best possible result that can be achieved in the ideal case, while the results for the robust servomechanism controller show that even though the controller does not have as much information, the limiting performance is identical to the ideal case.

The structure of this paper is as follows: Section 2 gives some preliminary material on the factorization of a system into the product of an inner system and a minimum phase systems.

Section 3 gives the main result on the cheap LQR problem for a special system resulting from the factorization given in Section 2. Section 4 studies the servomechanism problem with cheap quadratic cost using feedforward controllers. Section 5 studies the same problem using robust controllers. It is assumed in Sections 4 and 5 that the disturbances are constant signals. Section 6 extends the result obtained in Sections 4 and 5 to sinusoidal disturbances. Section 7 contains an example. Section 8 contains conclusions.

2. PRELIMINARIES

The transfer matrix F of the system (1) or (3) is given by

$$F(s) = D + C(sI - A)^{-1}B. \qquad (5)$$

Conversely, a four-tuple of real matrices (A, B, C, D) is said to be a realization of a proper real rational matrix (transfer matrix in short) F if (5) is satisfied. Throughout this paper, the following notation is used to divide the complex plane into three parts: $\mathbb{C}^+ = \{s \in \mathbb{C} : \mathrm{Re}\,(s) > 0\}$, $\mathbb{C}^0 = \{s \in \mathbb{C} : \mathrm{Re}\,(s) = 0\}$, $\mathbb{C}^- = \{s \in \mathbb{C} : \mathrm{Re}\,(s) < 0\}$. A transfer matrix is said to be stable if all of its poles are contained in $\mathbb{C}^-$, and a square constant real matrix is said to be stable if all of its eigenvalues are contained in $\mathbb{C}^-$.

The zeros of a transfer matrix are defined to be the roots of the numerator polynomials of the nonzero elements of its Smith–McMillan form. A transfer matrix is said to be *minimum phase* if it has no zeros in $\mathbb{C}^+$; otherwise it is said to be *non-minimum phase*. The zeros of system (1), system (3) or simply a realization (A, B, C, D) are defined to be the roots of the invariant polynomials of the matrix $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$. Similarly, we can define the concepts of minimum phase and non-minimum phase for system (1), system (3) and a four-tuple (A, B, C, D), as was done for a transfer matrix. Although the zeros of a real rational matrix and those of its realization may be different, their minimum phase or non-minimum phase property is always the same as long as the realization is stabilizable and detectable.

Associated with any realization (A, B, C, D) in which A is stable, there are two Lyapunov equations:

$$AP + PA' = -BB', \qquad (6)$$

$$A'Q + QA = -C'C. \qquad (7)$$

The solutions P, Q to equations (6)–(7) are called the controllability grammian and the observability grammian of (A, B, C, D), respectively. A minimal realization (A, B, C, D) of a stable transfer matrix F is called a *balanced realization* if the solutions P, Q to equations (6)–(7) are diagonal and equal. It is shown in Moore (1981) that every stable transfer matrix has a balanced realization. Procedures to find a balanced realization from any minimal realization of a stable transfer matrix are given in Moore (1981) and Laub *et al.* (1987).

A stable transfer matrix F is called *inner* if $F'(-s)F(s) = I$. All the zeros of an inner matrix must be located in $\mathbb{C}^+$.

Lemma 1 (Glover, 1984). Let (A, B, C, D) be a balanced realization of an inner matrix F and let P, Q be the solutions to equations (6)–(7). Then

(a) $P = Q = I$,

(b) $D'D = I$,

(c) $D'C + B' = 0$, $DB' + C = 0$.

A transfer matrix F is said to be *right-invertible* if F has full row rank for at least one $s \in \mathbb{C}$. If (A, B, C, D) is any realization of F, then the right-invertibility of F is equivalent to the fact that $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$ has full row rank for at least one $\lambda \in \mathbb{C}$. Therefore, no confusion will be caused when we talk about the right-invertibility of system (1), system (3) or realization (A, B, C, D).

The following factorization result serves as a foundation for our development. It is noted here that when the poles and/or zeros of two transfer matrices are compared in the following, we consider not only their values but also their multiplicities in the Smith–McMillan sense.

Lemma 2. A transfer matrix F can always be factorized as $F = F_1F_2$ such that F_1 is inner, F_2 is minimum phase and right-invertible, and the unstable poles of F_2 are equal to the unstable poles of F.

The result has been known for a long time but its original proof is hard to trace. Readers are referred to Qiu and Davison (1990) for a proof and Zhang and Freudenberg (1990) for a proof of its dual version.

Given a transfer matrix F, let $F = F_1F_2$ be the factorization described in Lemma 2. Let (A_1, B_1, C_1, D_1) be a balanced realization of F_1 and let (A_2, B_2, C_2, D_2) be any stabilizable and detectable realization of F_2. Then a stabilizable and detectable realization of F is given by

$$A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \qquad (8)$$
$$C = [C_1 \quad D_1C_2], \quad D = D_1D_2.$$

This realization is called a *factorized realization* of F and plays an important role in the development.

The following lemma gives some useful properties of right-invertible transfer matrices with respect to the factorization described in Lemma 2.

Lemma 3. Let F be a right-invertible transfer matrix and let $F = F_1 F_2$ be the factorization described in Lemma 2. Then F_1 is square, the zeros of F_1 are equal to those zeros of F contained in $\mathbb{C}^+$, and the poles of F_1 are equal to the negatives of the zeros of F_1.

Proof. Let F be $r \times m$. Then F_1 has r rows and at most r columns. If F_1 has less than r columns, then the rank of F is less than r for all $s \in \mathbb{C}$, which contradicts the assumption that F is right-invertible. Therefore F_1 must have r columns, i.e. it must be square. It follows directly from the identity $F_1^{-1}(s) = F_1'(-s)$ that the poles of F_1 are equal to the negatives of the zeros of F_1.

Let (A, B, C, D) be a factorized realization of F. Those zeros of F which are contained in $\mathbb{C}^+$ are the complex numbers $\lambda \in \mathbb{C}^+$ which make the matrix

$$\begin{bmatrix} A_1 - \lambda I & B_1 C_2 & B_1 D_2 \\ 0 & A_2 - \lambda I & B_2 \\ C_1 & D_1 C_2 & D_1 D_2 \end{bmatrix},$$

reduce rank. Adding $-B_1 D_1^{-1}$ times of the third row to the first row and multiplying the third row by D_1^{-1}, we transform the above matrix into

$$\begin{bmatrix} A_1 - B_1 D_1^{-1} C_1 - \lambda I & 0 & 0 \\ 0 & A_2 - \lambda I & B_2 \\ D_1^{-1} C_1 & C_2 & D_2 \end{bmatrix}.$$

Hence the zeros of F in $\mathbb{C}^+$ are the eigenvalues of $A_1 - B_1 D_1^{-1} C_1$. Notice from Lemma 1 that

$$\begin{aligned} A_1 - B_1 D_1^{-1} C_1 &= A_1 + B_1 D_1^{-1} D_1 B_1' \\ &= A_1 + B_1 B_1' \\ &= A_1 - (A_1 + A_1') = -A_1'. \end{aligned}$$

This completes the proof. □

We end this section with a few words about the norm of quadratic forms. A (real) quadratic form f on $v \in \mathbb{R}^p$ is a function of the form $f(v) = v'Qv$ for some symmetric matrix $Q \in \mathbb{R}^{p \times p}$. Therefore a norm of Q gives a norm of the quadratic form f. The trace norm of Q, i.e. the sum of the singular values of Q, is of particular interest. If Q is positive semi-definite, then its trace norm is equal to its trace. The following lemma gives a physical meaning to the trace norm. Let $\mathscr{E}(\cdot)$ denote the expectation operator.

Lemma 4 (Levine and Athans, 1970). Let $Q \in \mathbb{R}^{p \times p}$ be a positive semi-definite matrix and let v be a random vector in $\mathbb{R}^p$ with $\mathscr{E}(v) = 0$ and $\mathscr{E}(vv') = I$. Then $\mathscr{E}(v'Qv) = \operatorname{tr} Q$.

3. CHEAP LQR PROBLEM

Consider the cheap LQR problem defined by (1)–(2). It is known that $J_\epsilon = x_0' P_\epsilon x_0$ and that the optimal control which stabilizes the system and which achieves the optimal cost is given by $u = -(\epsilon^2 I + D'D)^{-1}(B'P_\epsilon + D'C)x$, where P_ϵ is the unique positive semi-definite solution to the following algebraic Riccati equation (ARE)

$$\begin{aligned} &[A - B(\epsilon^2 I + D'D)^{-1} D'C]' P_\epsilon \\ &\quad + P_\epsilon [A - B(\epsilon^2 I + D'D)^{-1} D'C] \\ &\quad + C'[I - D(\epsilon^2 I + D'D)^{-1} D']C \\ &\quad - P_\epsilon B(\epsilon^2 I + D'D)^{-1} B' P_\epsilon = 0. \qquad (9) \end{aligned}$$

It is easy to show (Kwakernaak and Sivan (1972b)) that P_ϵ monotonically decreases as ϵ goes to zero, and so the limit $P_0 := \lim_{\epsilon \to 0} P_\epsilon$ exists. The following result was proved in Scherzinger and Davison (1985); the same result for the case when $D = 0$ was obtained in Kwakernaak and Sivan (1972b) and Francis (1979).

Lemma 5. $P_0 = 0$ if and only if (A, B, C, D) is minimum phase and right-invertible.

For systems which do not satisfy the conditions given in Lemma 5, (Francis, 1979) characterized the null space of P_0, which is simply the set of all x_0 with $J_0 = 0$, and (Saberi and Sannuti, 1987) gave a complete decomposition of the state space in terms of the transient speed of the state trajectories. In the following, we will show that P_0 takes on a very simple form if (A, B, C, D) is a factorized realization of an arbitrary transfer matrix.

Lemma 6. Let (A, B, C, D) be a factorized realization of a transfer matrix. Then $P_\epsilon = \begin{bmatrix} I & 0 \\ 0 & P_{\epsilon 2} \end{bmatrix}$, where $P_{\epsilon 2}$ is the unique positive semi-definite solution to the ARE

$$\begin{aligned} &[A_2 - B_2(\epsilon^2 I + D_2' D_2)^{-1} D_2' C_2]' P_{\epsilon 2} \\ &\quad + P_{\epsilon 2}[A_2 - B_2(\epsilon^2 I + D_2' D_2)^{-1} D_2' C_2] \\ &\quad + C_2'[I - D_2(\epsilon^2 I + D_2' D_2)^{-1} D_2] C_2 \\ &\quad - P_{\epsilon 2} B_2 (\epsilon^2 I + D_2' D_2)^{-1} B_2 P_{\epsilon 2} = 0. \end{aligned}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be partitioned accordingly with the partition of A. Then the optimal control is given by $u = -(\epsilon^2 I + D_2'D_2)^{-1}(B_2'P_{\epsilon 2} + D_2'C_2)x_2$.

The proof of the first statement of this lemma is obtained simply by verifying that the given solution indeed satisfies ARE (9). The second statement follows by using Lemma 1. The details of the proof are dry algebra and are omitted.

A direct application of Lemma 6 leads to the following corollary.

Corollary 1. If (A, B, C, D) is a balanced realization of an inner transfer matrix, then $P_\epsilon = I$ and the optimal control is $u = 0$.

Since any minimal realization of a transfer matrix is similar to a balanced realization, this corollary implies that if (A, B, C, D) is a minimal realization of an inner matrix, then the optimal control of the system (1) under cost (2) is always zero, and thus the optimal performance is independent of ϵ. This is consistent with the well-known fact that cheap control asymptotically puts all the poles of the closed loop system to the mirror points of the system's zeros in $\mathbb{C}^+$ with respect to the imaginary axes. Since an inner matrix already has this property, no control is therefore needed to make it optimal.

Since (A_2, B_2, C_2, D_2) is a stabilizable and detectable realization of a minimum phase and right-invertible transfer matrix, the following theorem is obtained immediately from Lemmas 5 and 6. (In the statement of the theorem, we assume that P_ϵ is partitioned accordingly with the partition of A given in (8).)

Theorem 1. Let (A, B, C, D) be a factorized realization of a transfer matrix. Then

$$P_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

4. FEEDFORWARD SERVOMECHANISM CONTROLLER

In the rest of this paper, we apply Theorem 1 to study various optimal servomechanism problems with transient performance measured by cheap quadratic functionals. Consider the servomechanism problem for system (3). Denote the zeros of (A, B, C, D) which are contained in $\mathbb{C}^+$ (if any) by $\lambda_1, \lambda_2, \ldots, \lambda_l$. To achieve clarity in the presentation, we assume in this and the next section that the disturbances are constant signals. The results will be extended to the sinusoidal signals in Section 6.

In order for the servomechanism problem to be solvable, it is necessary and sufficient to have the following assumption.

Assumption 1. Assume that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ has full row rank.

Assumption 1 implies that system (3) is right-invertible and has no zero at the origin. Assume that disturbances ξ and η are measurable. Then under Assumption 1, the following feedforward controller solves the servomechanism problem:

$$u = Kx - \xi - [D - (C + DK)(A + BK)^{-1}B]^{\dagger}\eta, \tag{10}$$

where K is any matrix which makes $A + BK$ stable and $[\cdot]^{\dagger} = [\cdot]'([\cdot][\cdot]')^{-1}$ (Davison, 1973). The inverse involved exists due to Assumption 1.

Assume controller (10) is applied to the system (3). The closed loop stability implies that the input and the state of the system will approach constant values in the steady-state. Denote the steady-state values of the input and the state by $\bar{u}$ and $\bar{x}$, respectively. Then $\bar{u}$ and $\bar{x}$ must satisfy equations

$$\begin{aligned} 0 &= A\bar{x} + B(\bar{u} + \xi), \\ 0 &= C\bar{x} + D(\bar{u} + \xi) + \eta, \\ \bar{u} &= K\bar{x} - \xi \\ &\quad - [D - (C + DK)(A + BK)^{-1}B]^{\dagger}\eta. \end{aligned}$$

Let the transient part of the input and the state be denoted by $\tilde{u} := u - \bar{u}$ and $\tilde{x} := x - \bar{x}$, respectively. Then these values are governed by the following equations:

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u}, \quad \tilde{x}(0) = -\bar{x}, \\ y &= C\tilde{x} + D\tilde{u}, \\ \tilde{u} &= K\tilde{x}. \end{aligned}$$

These equations suggest that in order to have a good transient response, we can choose K to achieve the following optimal quadratic cost functional:

$$J_\epsilon = \min_{\tilde{u}} \int_0^\infty (y'y + \epsilon^2 \tilde{u}'\tilde{u})\, dt. \tag{11}$$

From the knowledge of the LQR problem, J_ϵ is a positive semi-definite quadratic form on $\tilde{x}(0) = -\bar{x}$, which in turn is linear in ξ and η. Hence, J_ϵ is a positive semi-definite quadratic form on ξ and η. Since J_ϵ monotonically decreases as ϵ^2 goes to zero, it follows that $J_0 := \lim_{\epsilon \to 0} J_\epsilon$ exists. The following theorem says that J_0 is a positive semi-definite quadratic form on η only, i.e. J_0 is independent of the input disturbance, and that a norm of J_0 can be simply given by the locations of the zeros of system (3) in $\mathbb{C}^+$.

Theorem 2. $J_0 = \eta' H \eta$ for some positive semi-definite H and $\operatorname{tr} H = 2 \sum_{i=1}^{l} \frac{1}{\lambda_i}$.

Proof. The nature of the problem setup indicates that J_ϵ depends solely on the transfer matrix F of system (3). Therefore, we can assume that (A, B, C, D) is a factorized realization which is of the form of (8). Let $F_1(s) = D_1 + C_1(sI - A_1)^{-1}B_1$ and $F_2(s) = D_2 + C_2(sI - A_2)^{-1}B_2$. Since F is right-invertible by Assumption 1, it follows from Lemma 3 that F_1 must be square and that the poles of F_1 are $-\lambda_1, -\lambda_2, \ldots, -\lambda_l$.

It is known that $J_\epsilon = \bar{x}'(0)P_\epsilon \bar{x}(0) = \bar{x}'P_\epsilon \bar{x}$, where P_ϵ is the unique positive semi-definite solution of ARE (9). By Theorem 1, $P_0 := \lim_{\epsilon \to 0} P_\epsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Let $\bar{x}$ be partitioned as $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ according to the partition of A as given by (8); then $J_0 = \bar{x}_1'\bar{x}_1$.

Now assume that the closed loop system is at steady-state. The output of F_1 must be $-\eta$ and the output of F_2 is therefore $-F_1^{-1}(0)\eta$. It then follows that $\bar{x}_1 = A_1^{-1}B_1F_1^{-1}(0)\eta$. Let $H = F_1^{-1}(0)B_1'A_1'^{-1}A_1^{-1}B_1F_1^{-1}(0)$. Then $J_0 = \eta' H \eta$. Since the matrix $F_1(0)$ is unitary, it follows that

$$\operatorname{tr} H = \operatorname{tr}(B_1'A_1'^{-1}A_1^{-1}B_1) = \operatorname{tr}(A_1^{-1}B_1B_1'A_1'^{-1}).$$

By using Lemmas 1 and 3, we obtain

$$\operatorname{tr} H = -\operatorname{tr}[A_1^{-1}(A_1 + A_1')A_1'^{-1}]$$
$$= -2 \operatorname{tr}(A_1^{-1}) = 2 \sum_{i=1}^{l} \frac{1}{\lambda_i}. \qquad \square$$

It is not a surprise that the feedforward controller (10) produces perfect control, i.e. $J_0 = 0$, for the case when only the input disturbance is present, even if the system (3) is non-minimum phase. In fact, the feedforward controller generates a signal which completely cancels the input disturbance. However for the case when the output disturbance is present, perfect control cannot be obtained for non-minimum phase systems, and a norm (or an averaging effect) of the optimal performance J_ϵ is now bounded from below by $2 \sum_{i=1}^{l} \frac{1}{\lambda_i}$. This result shows that $2 \sum_{i=1}^{l} \frac{1}{\lambda_i}$ can be considered as a quantitative measure of the degree of difficulty in solving the servomechanism problem for non-minimum phase systems with constant disturbances. This result also emphasizes the fact that not all non-minimum phase systems behave the same. A system with a small positive zero is more difficult to control than a system whose zeros in $\mathbb{C}^+$ are far away from the origin. On the other hand, a conjugate pair of complex zeros $\alpha \pm j\beta$ in $\mathbb{C}^+$ with $\alpha \ll |\beta|$ will not cause significant difficulty in control since its contribution to the limiting performance J_0 is $\frac{4\alpha}{\alpha^2 + \beta^2}$. This later phenomenon has been observed in Davison and Gesing (1985) in the control design for a large flexible space structure. A similar result to Theorem 2 was given in Morari and Zafiriou (1989) for SISO systems by using frequency domain techniques.

5. ROBUST SERVOMECHANISM CONTROLLER

In practical applications, the parameters of system (3) are always somewhat uncertain and we often do not have access to the disturbances. In either case, controller (10) cannot generally be used. To overcome these difficulties, a robust servomechanism controller has been proposed to solve the servomechanism problem (Davison, 1976). The robust servomechanism controller does not require the disturbances to be measured, but must contain a servocompensator which contains the modes of the disturbances. A significant advantage of the robust servomechanism controller over the feedforward controller is that tracking and disturbance rejection occur for all perturbations in the system provided only that the perturbed closed loop system remains stable.

Assume again that Assumption 1 holds. The robust servomechanism controller for system (3) with constant disturbances can take the following form

$$\dot{z} = y, \quad z(0) = 0,$$
$$u = K_0 x + K z, \qquad (12)$$

where $[K_0 \quad K]$ is chosen to stabilize matrix $\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}[K_0 \quad K]$.

On combining the original system and the servo-compensator (the integrator), the augmented system with input u and output z is then described by the state-space equation:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}(u + \xi) + \begin{bmatrix} 0 \\ I \end{bmatrix}\eta,$$

$$z = [0 \quad I]\begin{bmatrix} x \\ z \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The action of the controller becomes that of a state feedback, i.e. $u = [K_0 \quad K]\begin{bmatrix} x \\ z \end{bmatrix}$. Define new variables: $\tilde{x} := \dot{x}$, $\tilde{z} := \dot{z}$, $\tilde{u} := \dot{u}$. On noticing that $\dot{z} = y$, the augmented system then becomes

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}\tilde{u},$$

$$\begin{bmatrix} \tilde{x}(0) \\ \tilde{z}(0) \end{bmatrix} = \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad (13)$$

$$y = [0 \quad I]\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix},$$

and the controller becomes $\tilde{u} = [K_0 \quad K]\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix}$. It can be easily shown that system

$$\left(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} B \\ D \end{bmatrix}, [0 \quad I], 0\right),$$

is always stabilizable and detectable under Assumption 1 and the assumption that (A, B, C, D) is stabilizable and detectable.

This suggests that in order to have a good transient response, we can choose $[K_0 \quad K]$ to achieve the following optimal quadratic cost functional

$$J_\epsilon = \min_{\tilde{u}} \int_0^\infty (y'y + \epsilon^2 \tilde{u}'\tilde{u})\, dt. \qquad (14)$$

By the same argument as made in the last section, we see that J_ϵ is a positive semi-definite quadratic form on ξ and η, and that $J_0 := \lim_{\epsilon \to 0} J_\epsilon$ exists. The statement of the following theorem is exactly the same as in Theorem 2.

Theorem 3. $J_0 = \eta' H \eta$ for some real positive semi-definite H and $\operatorname{tr} H = 2 \sum_{i=1}^{l} \frac{1}{\lambda_i}$.

Proof. Again J_ϵ depends solely on the transfer matrix F of system (3). Therefore (A, B, C, D) can be assumed to be a factorized realization of the form of (8). Consequently (13) can be rewritten as

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 C_2 & 0 \\ 0 & A_2 & 0 \\ C_1 & D_1 C_2 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} B_1 D_2 \\ B_2 \\ D_1 D_2 \end{bmatrix}\tilde{u}, \qquad (15)$$

$$y = [0 \quad 0 \quad I]\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix},$$

with initial condition

$$\begin{bmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \\ \tilde{z}(0) \end{bmatrix} = \begin{bmatrix} B_1 D_2 & 0 \\ B_2 & 0 \\ D_1 D_2 & I \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Let

$$T = \begin{bmatrix} A_1 & B_1 & 0 \\ 0 & 0 & I \\ C_1 & D_1 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} = T\begin{bmatrix} \hat{x}_1 \\ \hat{z} \\ \hat{x}_2 \end{bmatrix}.$$

Assumption 1 guarantees that T is invertible. System (15) can then be transformed to

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{z}} \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 \\ 0 & 0 & C_2 \\ 0 & 0 & A_2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{z} \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ D_2 \\ B_2 \end{bmatrix}\tilde{u}, \qquad (16)$$

$$y = [C_1 \quad D_1 \quad 0]\begin{bmatrix} \hat{x}_1 \\ \hat{z} \\ \hat{x}_2 \end{bmatrix},$$

and the initial condition then becomes

$$\begin{bmatrix} \hat{x}_1(0) \\ \hat{z}(0) \\ \hat{x}_2(0) \end{bmatrix} = T^{-1}\begin{bmatrix} B_1 D_2 & 0 \\ B_2 & 0 \\ D_1 D_2 & I \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

Note that (16) is a factorized realization with (A_1, B_1, C_1, D_1) being inner and $\left(\begin{bmatrix} 0 & C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} D_2 \\ B_2 \end{bmatrix}, [I \quad 0], 0\right)$ being minimum phase and right-invertible. Let P_ϵ be the unique positive semi-definite solution to the ARE associated with system (16) and optimal cost functional (14). Theorem 1 leads to

$$\lim_{\epsilon \to 0} P_\epsilon = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $J_0 = \hat{x}_1'(0)\hat{x}_1(0)$. Direct calculation shows that

$$\hat{x}_1(0) = [I \quad 0 \quad 0]T^{-1}\begin{bmatrix} B_1 D_2 & 0 \\ B_2 & 0 \\ D_1 D_2 & I \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$$
$$= -A_1^{-1} B_1 F_1^{-1}(0)\eta.$$

where $F_1(0) = D_1 - C_1 A_1^{-1} B_1$. Let $H = F_1'^{-1}(0) B_1' A_1'^{-1} A_1^{-1} B_1 F_1^{-1}(0)$. Then $J_0 = \eta' H \eta$.

The rest of the proof now proceeds in exactly the same way as done in the last part of the proof of Theorem 2. □

Two interesting points can be observed on comparing Theorem 3 with Theorem 2. Firstly, although we can no longer completely cancel the input disturbance when the robust servomechanism controller is used (as was done for the feedforward controller) perfect control still occurs for the case when only the input disturbance is present even if the system is non-minimum phase. In other words, no matter whether the system is minimum phase or not, the robust servomechanism controller's reaction to the input disturbance can be made arbitrarily fast. This result is perhaps somewhat surprising. Secondly, we are concerned if the use of the robust servomechanism controller sacrifices the potential performance of the controlled system, in comparison to the use of the feedforward controller. Since the variable $\tilde{u}$ in (11) and (14) have different physical meanings, it may appear that the norms of J_0 given in Theorem 2 and Theorem 3 are incomparable. However, it has been shown in Kwakernaak and Sivan (1972b) that if we denote the outputs of the optimal system in both cases by y_ϵ, then $J_0 = \lim_{\epsilon \to 0} \int_0^\infty y_\epsilon' y_\epsilon \, dt$, i.e. when ϵ is small, J_ϵ essentially contains only the output term. Therefore, in both cases, the limiting transient speed of the output, measured by a norm of J_0, are the same. In other words, the use of the robust servomechanism controller will not lead to a significant loss in the potential limiting performance of the system.

6. SINUSOIDAL DISTURBANCES

In this section, we extend the results obtained in the last two sections to the case when the disturbances are sinusoidal signals. We will obtain results which are in the same spirit as in Theorems 2–3.

Assume that the disturbances in system (3) are now of the following form

$$\xi(t) = \xi_{e1} \sin \omega t + \xi_{e2} \cos \omega t, \qquad (17)$$

$$\eta(t) = \eta_{e1} \sin \omega t + \eta_{e2} \cos \omega t, \qquad (18)$$

where $\xi_e := \begin{bmatrix} \xi_{e1} \\ \xi_{e2} \end{bmatrix}$ and $\eta_e := \begin{bmatrix} \eta_{e1} \\ \eta_{e2} \end{bmatrix}$ are real constant vectors. In order for the servomechanism problem to be solvable for system (3) with disturbances of the form (17)–(18), it is necessary and sufficient to have the following assumption.

Assumption 2. Assume that $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full row rank.

Assumption 2 implies that system (3) is right-invertible and has no zero at $j\omega$.

First, we consider the ideal case when the system parameters (A, B, C, D) are exactly known, and the disturbances as well as their derivatives are available for measurement. In this case, the following feedforward controller solves the servomechanism problem:

$$\begin{aligned} u = Kx - \xi - \operatorname{Re}\{&[D + (C + DK) \\ &\times (j\omega I - A - BK)^{-1}B]^\dagger\}\eta \\ &- \frac{1}{\omega} \operatorname{Im}\{[D + (C + DK) \\ &\times (j\omega I - A - BK)^{-1}B]^\dagger\}\dot{\eta}, \end{aligned} \qquad (19)$$

where K is any matrix which makes $A + BK$ stable and $[\cdot]^\dagger = [\cdot]'([\cdot][\cdot]')^{-1}$ (Davison, 1973). The inverse involved exists due to Assumption 2.

Assume controller (19) is applied to system (3), and assume that the steady-state signals of the input and the state are given by $\bar{u}$ and $\bar{x}$, respectively. Then $\bar{u}$ and $\bar{x}$ must satisfy equations:

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} + B\xi, \\ 0 &= C\bar{x} + D\bar{u} + D\xi + \eta, \\ \bar{u} &= K\bar{x} - \xi - \operatorname{Re}\{[D + (C + DK) \\ &\quad \times (j\omega I - A - BK)^{-1}B]^\dagger\}\eta \\ &\quad - \frac{1}{\omega} \operatorname{Im}\{[D + (C + DK) \\ &\quad \times (j\omega I - A - BK)^{-1}B]^\dagger\}\dot{\eta}. \end{aligned}$$

Let the transient part of the input and the state be denoted by $\tilde{u} := u - \bar{u}$ and $\tilde{x} := x - \bar{x}$, respectively. Then these values are governed by the following equations:

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{u}, \quad \tilde{x}(0) = -\bar{x}(0), \\ y &= C\tilde{x} + D\tilde{u}, \\ \tilde{u} &= K\tilde{x}. \end{aligned}$$

These equations suggest that in order to have a good transient response, we can choose K to achieve the following optimal quadratic cost functional

$$J_\epsilon = \min_{\tilde{u}} \int_0^\infty (y'y + \epsilon^2 \tilde{u}'\tilde{u}) \, dt. \qquad (20)$$

From the knowledge of the LQR problem, J_ϵ is a positive semi-definite quadratic form on $\tilde{x}(0) = -\bar{x}(0)$, which in turn depends linearly on ξ_e and η_e. Hence J_ϵ is a positive semidefinite quadratic form on ξ_e and η_e. Since J_ϵ decreases monotonically as ϵ^2 goes to zero, it follows that $J_0 := \lim_{\epsilon \to 0} J_\epsilon$ exists. The following theorem says that J_0 is a positive semidefinite quadratic form on η_e only and a norm of J_0 is given by a simple expression involving only the zeros of the system (3) in $\mathbb{C}^+$.

Theorem 4. $J_0=\eta_e'M\eta_e$ for some positive semi-definite M and

$$\operatorname{tr} M=\sum_{i=1}^{l}\left(\frac{1}{\lambda_i-j\omega}+\frac{1}{\lambda_i+j\omega}\right).$$

Proof. Similar to the constant disturbance case, we can assume that (A, B, C, D) is a factorized realization of the form of (8). Let $F_1(s)=D_1+C_1(sI-A_1)^{-1}B_1$ and $F_2(s)=D_2+C_2(sI-A_2)^{-1}B_2$. Since system (3) is assumed to be right-invertible, it follows from Lemma 3 that F_1 must be square and that the poles of F_1 are given by $-\lambda_1, -\lambda_2, \ldots, -\lambda_l$.

It is known that $J_\epsilon=\bar{x}'(0)P_\epsilon\bar{x}(0)=\bar{x}'(0)P_\epsilon\bar{x}(0)$ where P_ϵ is the unique positive semidefinite solution of ARE (9). By Theorem 1, $P_0:=\lim_{\epsilon\to 0}P_\epsilon=\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}$. Let $\bar{x}$ be partitioned as $\bar{x}=\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}$ according to the partition of A given by (8); then $J_0=\bar{x}_1'(0)\bar{x}_1(0)$.

Now assume that the closed loop system is at steady-state. The output of F_1 must be $-\eta$. To obtain the state $\bar{x}_1$ of F_1, we use $\mathscr{L}(\cdot)$ to denote the Laplace transform operator. Then $\bar{x}_1$ is related to η by

$$\mathscr{L}(\bar{x}_1)=-(sI-A_1)^{-1}B_1F_1^{-1}(s)\mathscr{L}(\eta).$$

Let $L(s)=(sI-A_1)^{-1}B_1F_1^{-1}(s)$. Steady state sinusoidal analysis tells us that

$$\bar{x}_1=-\operatorname{Re} L(j\omega)\eta-\frac{1}{\omega}\operatorname{Im} L(j\omega)\dot{\eta}.$$

Hence,

$$\bar{x}_1(0)=-\operatorname{Re} L(j\omega)\eta_{e2}-\operatorname{Im} L(j\omega)\eta_{e1}.$$

Let

$$M=\begin{bmatrix}\operatorname{Im} L(j\omega)\\ \operatorname{Re} L(j\omega)\end{bmatrix}'[\operatorname{Im} L(j\omega)\ \operatorname{Re} L(j\omega)].$$

Then $J_0=\eta_e'M\eta_e$ and

$$\begin{aligned}\operatorname{tr} M&=\operatorname{tr}[L^*(j\omega)L(j\omega)]\\ &=\operatorname{tr}[F_1^{*-1}(j\omega)B_1'(j\omega I-A_1)^{*-1}\\ &\quad\times(j\omega I-A_1)^{-1}B_1F_1^{-1}(j\omega)]\\ &=\operatorname{tr}[B_1'(j\omega I-A_1)^{*-1}(j\omega I-A_1)^{-1}B_1]\\ &=\operatorname{tr}[(j\omega I-A_1)^{-1}B_1B_1'(-j\omega I-A_1')^{-1}].\end{aligned}$$

By using Lemmas 1 and 3, we obtain

$$\begin{aligned}\operatorname{tr} M&=\operatorname{tr}[(j\omega I-A_1)^{-1}\\ &\quad\times(j\omega I-A_1-j\omega I-A_1')(-j\omega I-A_1')^{-1}]\\ &=\operatorname{tr}[(-j\omega I-A_1')^{-1}+(j\omega I-A_1)^{-1}]\\ &=\sum_{i=1}^{l}\left(\frac{1}{\lambda_i-j\omega}+\frac{1}{\lambda_i-j\omega}\right).\end{aligned}$$

□

In this case, we see that a system with zeros in $\mathbb{C}^+$ close to $j\omega$, where ω is the frequency of the disturbances, is more difficult to control than a system whose zeros in $\mathbb{C}^+$ are far away from $j\omega$.

We can see, as in the constant disturbance case, that the feedforward controller (19) cannot be used either when the system parameters (A, B, C, D) are uncertain or when the disturbances are not available for measurement. The robust servomechanism controller does not have these disadvantages. The robust servomechanism controller for disturbance signals of the form (17)–(18) can take the following form (Davison, 1976)

$$\begin{gathered}\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}0 & -\omega^2I\\ I & 0\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\begin{bmatrix}I\\ 0\end{bmatrix}y,\\ \begin{bmatrix}z_1(0)\\ z_2(0)\end{bmatrix}=0,\\ u=K_0x+K_1z_1+K_2z_2,\end{gathered}\tag{21}$$

where $[K_0\ K_1\ K_2]$ is chosen to stabilize the following matrix

$$\begin{bmatrix}A & 0 & 0\\ C & 0 & -\omega^2I\\ 0 & I & 0\end{bmatrix}+\begin{bmatrix}B\\ D\\ 0\end{bmatrix}[K_0\quad K_1\quad K_2].$$

The augmented system (with input u and output z_2) is then described by the state-space equation

$$\begin{bmatrix}\dot{x}\\ \dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}A & 0 & 0\\ C & 0 & -\omega^2I\\ 0 & I & 0\end{bmatrix}\begin{bmatrix}x\\ z_1\\ z_1\end{bmatrix}+\begin{bmatrix}B\\ D\\ 0\end{bmatrix}(u+\xi)+\begin{bmatrix}0\\ I\\ 0\end{bmatrix}\eta,\quad\begin{bmatrix}x(0)\\ z_1(0)\\ z_2(0)\end{bmatrix}=0,$$

$$z_2=[0\quad 0\quad I]\begin{bmatrix}x\\ z_1\\ z_2\end{bmatrix}.$$

Define new variables $\bar{x}:=\ddot{x}+\omega^2x$, $\bar{z}_1:=\ddot{z}_1+\omega^2z_1$, $\bar{z}_2:=\ddot{z}_2+\omega^2z_2$, and $\bar{u}:=\ddot{u}+\omega^2u$. On noting that $\ddot{z}_2+\omega^2z_2=y$, the augmented system then becomes

$$\begin{gathered}\begin{bmatrix}\dot{\bar{x}}\\ \dot{\bar{z}}_1\\ \dot{\bar{z}}_2\end{bmatrix}=\begin{bmatrix}A & 0 & 0\\ C & 0 & -\omega^2I\\ 0 & I & 0\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{z}_1\\ \bar{z}_2\end{bmatrix}+\begin{bmatrix}B\\ D\\ 0\end{bmatrix}\bar{u},\\ y=[0\quad 0\quad I]\begin{bmatrix}\bar{x}\\ \bar{z}_1\\ \bar{z}_2\end{bmatrix},\end{gathered}\tag{22}$$

where $\left(\begin{bmatrix}A & 0 & 0\\ C & 0 & -\omega^2I\\ 0 & I & 0\end{bmatrix},\begin{bmatrix}B\\ D\\ 0\end{bmatrix},[0\quad 0\quad I],0\right)$

is always stabilizable and detectable under Assumption 2 and the assumption that (A, B, C, D) is stabilizable and detectable.

The initial condition of system (22) is given by

$$\begin{bmatrix} \ddot{x}(0) \\ \ddot{z}_1(0) \\ \ddot{z}_2(0) \end{bmatrix} = \begin{bmatrix} \ddot{x}(0) \\ \ddot{z}_1(0) \\ \ddot{z}_2(0) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ C & 0 & -\omega^2 I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(0) \\ \dot{z}_1(0) \\ \dot{z}_2(0) \end{bmatrix} + \begin{bmatrix} B \\ D \\ 0 \end{bmatrix} [\dot{u}(0) + \dot{\xi}(0)] + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \dot{\eta}(0)$$

$$= \begin{bmatrix} A & 0 & 0 \\ C & 0 & -\omega^2 I \\ 0 & I & 0 \end{bmatrix} \times \left\{ \begin{bmatrix} B \\ D \\ 0 \end{bmatrix} [u(0) + \xi(0)] + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \eta(0) \right\} + \begin{bmatrix} B \\ D \\ 0 \end{bmatrix} [\dot{u}(0) + \dot{\xi}(0)] + \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \dot{\eta}(0).$$

These equations suggest that in order to have a good transient response, we can choose $[K_0\ K_1\ K_2]$ to achieve the following optimal quadratic cost functional

$$J_\epsilon = \min_{\bar{u}} \int_0^\infty (y'y + \epsilon^2 \bar{u}'\bar{u})\, dt. \tag{23}$$

Again the optimal cost J_ϵ is a positive semi-definite quadratic form on ξ_e and η_e, and $J_0 = \lim_{\epsilon \to 0} J_\epsilon$ exists, The statement of the next theorem is exactly the same as that of Theorem 4.

Theorem 5. $J_0 = \eta_e' M \eta_e$ for some positive semi-definite M and

$$\operatorname{tr} M = \sum_{i=1}^{l} \left(\frac{1}{\lambda_i - j\omega} + \frac{1}{\lambda_i + j\omega} \right).$$

Proof. Again we can assume that (A, B, C, D) is a factorized realization of F which is of the form of (8). Consequently (22) can be written as

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \\ \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 C_2 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ C_1 & D_1 C_2 & 0 & -\omega^2 I \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} + \begin{bmatrix} B_1 D_2 \\ B_2 \\ D_1 D_2 \\ 0 \end{bmatrix} \bar{u}, \tag{24}$$

$$y = [0\ \ 0\ \ 0\ \ I] \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix}.$$

Let

$$T = \begin{bmatrix} A_1^2 + \omega^2 I & A_1 B_1 & B_1 & 0 \\ 0 & 0 & 0 & I \\ C_1 A_1 & C_1 B_1 & D_1 & 0 \\ C_1 & D_1 & 0 & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{z}_1 \\ \bar{z}_2 \end{bmatrix} = T \begin{bmatrix} \hat{x}_1 \\ \hat{z}_1 \\ \hat{z}_2 \\ \hat{x}_2 \end{bmatrix}.$$

Assumption 2 guarantees that T is invertible. System (24) is then transformed to

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -\omega^2 I & 0 & C_2 \\ 0 & 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{z}_1 \\ \hat{z}_2 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D_2 \\ B_2 \end{bmatrix} \bar{u}, \tag{25}$$

$$y = [C_1\ \ D_1\ \ 0\ \ 0] \begin{bmatrix} \hat{x}_1 \\ \hat{z}_1 \\ \hat{z}_2 \\ \hat{x}_2 \end{bmatrix}.$$

System (25) is in the factorized form with (A_1, B_1, C_1, D_1) being inner and

$$\left(\begin{bmatrix} 0 & I & 0 \\ -\omega^2 I & 0 & C_2 \\ 0 & 0 & A_2 \end{bmatrix}, \begin{bmatrix} 0 \\ D_2 \\ B_2 \end{bmatrix}, [I\ \ 0\ \ 0], 0 \right),$$

being minimum phase and right-invertible. Let P_ϵ be the unique positive semi-definite solution of the ARE associated with system (25) and the optimal cost functional (23). Theorem 1 leads to

$$P_0 := \lim_{\epsilon \to 0} P_\epsilon = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $J_0 = \hat{x}_1'(0) \hat{x}_1(0)$.

The initial condition $\hat{x}_1(0)$ can be obtained as:

$$\hat{x}_1(0) = [I \quad 0 \quad 0 \quad 0]\begin{bmatrix}\hat{x}_1(0)\\ \hat{z}_1(0)\\ \hat{z}_2(0)\\ \hat{x}_2(0)\end{bmatrix}$$

$$= [I \quad 0 \quad 0 \quad 0]T^{-1}$$

$$\times\left\{\begin{bmatrix} A_1 & B_1C_2 & 0 & 0\\ 0 & A_2 & 0 & 0\\ C_1 & D_1C_2 & 0 & -\omega^2 I\\ 0 & 0 & I & 0\end{bmatrix}\right.$$

$$\times\left(\begin{bmatrix}B_1D_2\\ B_2\\ D_1D_2\\ 0\end{bmatrix}[u(0)+\xi(0)] + \begin{bmatrix}0\\0\\I\\0\end{bmatrix}\eta(0)\right)$$

$$\left. + \begin{bmatrix}B_1D_2\\ B_2\\ D_1D_2\\ 0\end{bmatrix}[\dot{u}(0)+\dot{\xi}(0)] + \begin{bmatrix}0\\0\\I\\0\end{bmatrix}\dot{\eta}(0)\right\}.$$

Straightforward computation leads to:

$$\hat{x}_1(0) = -[(A_1 - B_1D_1^{-1}C_1)^2 + \omega^2 I]^{-1} \times [(A_1 - B_1D_1^{-1}C_1)B_1D_1^{-1}\eta(0) + B_1D_1^{-1}\dot{\eta}(0)]$$
$$= -[(A_1 - B_1D_1^{-1}C_1)^2 + \omega^2 I]^{-1} \times [(A_1 - B_1D_1^{-1}C_1)B_1D_1^{-1}\eta_{e2} + \omega B_1D_1^{-1}\eta_{e1}].$$

Let $L(s) = (sI - A_1 + B_1D_1^{-1}C_1)^{-1}B_1D_1^{-1}$. Then simple algebra shows that

$$L(s) = (sI - A_1)^{-1}B_1 \times [D_1 + C_1(sI - A_1)^{-1}B_1]^{-1} = (sI - A_1)^{-1}B_1F_1^{-1}(s),$$

and

$$\hat{x}_1(0) = -\mathrm{Re}\,[L(j\omega)]\eta_{e2} - \mathrm{Im}\,[L(j\omega)]\eta_{e1}.$$

Let

$$M = \begin{bmatrix}\mathrm{Im}\,L(j\omega)\\ \mathrm{Re}\,L(j\omega)\end{bmatrix}'[\mathrm{Im}\,L(j\omega)\ \mathrm{Re}\,L(j\omega)].$$

Then $J_0 = \eta_e' M \eta_e$. The rest of this proof is the same as the last part of the proof of Theorem 4. □

Again we have two similar observations as in the constant disturbance case. Firstly, perfect control still occurs for the case when only the input disturbance is present, even if the system is non-minimum phase. Secondly, the use of the robust servomechanism controller does not lead to a significant loss in the potential limiting performance of the system, compared to the feedforward controller case.

7. AN EXAMPLE

An experimental flexible beam system is described by the following transfer function (MacLean, 1990)

$$F(s) = \frac{8.26s^4 - 1.66s^3 - 2878s^2 + 453s + 95400}{5s^6 + 4.83s^5 + 2312s^4 + 488s^3 + 60657s^2 + 40.5s},$$

which has a state space model of the form (3) given by

$$A = \begin{bmatrix} -0.996 & -463 & -97.8 & -12131 & -8.11 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix},$$

$$B = \begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix},$$

$$C = [0 \quad 1.65 \quad -0.331 \quad -576 \quad 90.6 \quad 19080], \qquad D = 0.$$

Note that this is not in the factorized form as in (8).

This system has two zeros in $\mathbb{C}^+$: $\lambda_1 = 6.18$ and $\lambda_2 = 17.7$. Our purpose is to solve the servomechanism problem for this system with respect to a constant reference signal y_{ref} (assuming that there are no external disturbances present). As we have discussed in Section 1, we can consider $y - y_{\mathrm{ref}}$ as the output and $-y_{\mathrm{ref}}$ as the output disturbance for this problem.

Let us first apply the feedforward controller of the form (10) (with $D = 0$):

$$u = K_\epsilon x + [-C(A + BK_\epsilon)^{-1}B]^{-1}y_{\mathrm{ref}},$$

where K_ϵ is chosen to achieve the optimal cost

$$J_\epsilon = \min_u \int_0^\infty [(y - y_{\mathrm{ref}})'(y - y_{\mathrm{ref}}) + \epsilon^2(u - \bar{u})'(u - \bar{u})]\,dt.$$

Assume that the system has a zero initial condition. Let $\bar{x}_\epsilon$ be the steady state value of the state variable; then $\bar{x}_\epsilon = -(A + BK_\epsilon)^{-1}B[-$

TABLE 1. COMPARISON OF OPTIMAL COSTS FOR FEEDFORWARD CONTROLLER CASE

ϵ	J_ϵ	$J_{y\epsilon}$	$J_{u\epsilon}$
1	$1.21y_{\rm ref}^2$	$0.957y_{\rm ref}^2$	$2.54\times 10^{-1}y_{\rm ref}^2$
10^{-1}	$0.640y_{\rm ref}^2$	$0.592y_{\rm ref}^2$	$4.86y_{\rm ref}^2$
10^{-2}	$0.513y_{\rm ref}^2$	$0.499y_{\rm ref}^2$	$1.39\times 10^{2}y_{\rm ref}^2$
10^{-3}	$0.468y_{\rm ref}^2$	$0.462y_{\rm ref}^2$	$6.70\times 10^{3}y_{\rm ref}^2$
10^{-4}	$0.448y_{\rm ref}^2$	$0.445y_{\rm ref}^2$	$2.62\times 10^{5}y_{\rm ref}^2$

TABLE 2. COMPARISON OF OPTIMAL COSTS FOR ROBUST CONTROLLER CASE

	J_ϵ	$J_{y\epsilon}$	$J_{u\epsilon}$
1	$1.78y_{\rm ref}^2$	$1.50y_{\rm ref}^2$	$2.76\times 10^{-1}y_{\rm ref}^2$
10^{-1}	$0.962y_{\rm ref}^2$	$0.867y_{\rm ref}^2$	$9.49y_{\rm ref}^2$
10^{-2}	$0.690y_{\rm ref}^2$	$0.650y_{\rm ref}^2$	$4.00\times 10^{2}y_{\rm ref}^2$
10^{-3}	$0.563y_{\rm ref}^2$	$0.547y_{\rm ref}^2$	$1.63\times 10^{4}y_{\rm ref}^2$
10^{-4}	$0.506y_{\rm ref}^2$	$0.497y_{\rm ref}^2$	$9.59\times 10^{5}y_{\rm ref}^2$
10^{-5}	$0.472y_{\rm ref}^2$	$0.447y_{\rm ref}^2$	$5.48\times 10^{7}y_{\rm ref}^2$

$C(A+BK_\epsilon)^{-1}B]^{-1}y_{\rm ref}$ and $J_\epsilon=(-\bar{x}_\epsilon)'P_\epsilon(-\bar{x}_\epsilon)$, where P_ϵ is the unique positive semi-definite solution to ARE (9) (with $D=0$). From Theorem 2, we obtain $J_\epsilon\to 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)y_{\rm ref}^2=0.437y_{\rm ref}^2$ as $\epsilon\to 0$. Table 1 gives the computed value of J_ϵ for several different values of ϵ. For the convenience of comparison, Table 1 also gives

$$J_{y\epsilon}:=\int_0^\infty (y_\epsilon-y_{\rm ref})'(y_\epsilon-y_{\rm ref})\,{\rm d}t,$$

and

$$J_{u\epsilon}:=\int_0^\infty (u_\epsilon-\bar{u})'(u_\epsilon-\bar{u})\,{\rm d}t,$$

where y_ϵ and u_ϵ are the output and control trajectories of the optimally controlled system. The quantities $J_{y\epsilon}$ and $J_{u\epsilon}$ can be obtained as ${\rm tr}\,(CL_\epsilon C')$ and ${\rm tr}\,(K_\epsilon L_\epsilon K_\epsilon')$, respectively, where L_ϵ is the solution of the following Lyapunov equation

$$(A+BK_\epsilon)L_\epsilon+L_\epsilon(A+BK_\epsilon)'+\bar{x}_\epsilon\bar{x}_\epsilon'=0.$$

It is seen that as $\epsilon\to 0$, the computed value of J_ϵ is approaching the limiting cost $J_0=0.437y_{\rm ref}^2$ obtained from Theorem 2.

Now let us apply the robust controller

$$\dot{z}=y-y_{\rm ref},$$
$$u=K_{0\epsilon}x+K_\epsilon z,$$

where $[K_{0\epsilon}\ \ K_\epsilon]$ is chosen to achieve the optimal cost

$$J_\epsilon=\min_u\int_0^\infty [(y-y_{\rm ref})'(y-y_{\rm ref})+\epsilon^2\dot{u}'\dot{u}]\,{\rm d}t.$$

We know from Section 5 that $J_\epsilon=[0\ \ -y_{\rm ref}]P_\epsilon\begin{bmatrix}0\\ -y_{\rm ref}\end{bmatrix}$ where P_ϵ is the unique positive semi-definite solution of the following ARE:

$$\begin{bmatrix}A' & C'\\ 0 & 0\end{bmatrix}P_\epsilon+P_\epsilon\begin{bmatrix}A & 0\\ C & 0\end{bmatrix}+\begin{bmatrix}0\\ I\end{bmatrix}[0\ \ I]-\frac{1}{\epsilon^2}P_\epsilon\begin{bmatrix}B\\ 0\end{bmatrix}[B'\ \ 0]P_\epsilon=0.$$

From Theorem 3, we again obtain that $J_\epsilon\to 0.437y_{\rm ref}^2$ as $\epsilon\to 0$. Table 2 gives the computed value of J_ϵ for several different values of ϵ. Table 2 also gives

$$J_{y\epsilon}:=\int_0^\infty (y_\epsilon-y_{\rm ref})'(y_\epsilon-y_{\rm ref})\,{\rm d}t,$$

and

$$J_{u\epsilon}:=\int_0^\infty \dot{u}_\epsilon'\dot{u}_\epsilon\,{\rm d}t,$$

where y_ϵ and u_ϵ are the output and control trajectories of the optimally controlled system. The quantities $J_{y\epsilon}$ and $J_{u\epsilon}$ can be obtained as ${\rm tr}\,([0\ \ I]L_\epsilon[0\ \ I]')$ and ${\rm tr}\,([K_{0\epsilon}\ \ K_\epsilon]L_\epsilon[K_{0\epsilon}\ \ K_\epsilon]')$, respectively, where L_ϵ is the solution of the following Lyapunov equation

$$\left(\begin{bmatrix}A & 0\\ C & 0\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}[K_{0\epsilon}\ \ K_\epsilon]\right)L_\epsilon+L_\epsilon\left(\begin{bmatrix}A & 0\\ C & 0\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}[K_{0\epsilon}\ \ K_\epsilon]\right)'+\begin{bmatrix}0\\ I\end{bmatrix}[0\ \ I]=0.$$

Again, it is seen that as $\epsilon\to 0$, the computed value of J_ϵ is approaching the limiting cost $J_0=0.437y_{\rm ref}^2$ obtained from Theorem 3.

8. CONCLUSION

This paper considers the cheap regulator problem and the cheap optimal servomechanism problems for systems which may be non-minimum phase. The basic tool used is a factorization which factorizes an arbitrary system into the product of an inner system and a right-invertible minimum phase system. Based on this factorization, the study of an arbitrary system can be decomposed into the study of an inner system and the study of a right-invertible minimum phase system. The cheap control problem of an inner system becomes easy to analyse by exploiting various properties of inner matrices, while the cheap control problem of a right-invertible minimum phase system has already been intensively studied.

A novel contribution of this paper is the establishment of the fact that the number and the locations of the zeros of a system in the open right half of the complex plane, are crucial factors which determine the best attainable closed loop performance of the system. In particular, it is shown that the fundamental design limitations on the closed loop performance of the servomechanism problem can be completely characterized by the number and the locations of the zeros of the open loop system which lie in the open right half of the complex plane. This design limitation can be used to evaluate an open loop system, i.e. to determine whether the system is "inherently hard to control", and to assess a given closed loop design, i.e. to determine how near the closed loop system's performance is from the best attainable.

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