PERFORMANCE LIMITATIONS OF NON-MINIMUM PHASE SYSTEMS*

L. Qiu and E. J. Davison

Department of Electrical Engineering, University of Toronto, Toronto, Ontario M5S 1A4, Canada

Abstract. This paper studies the cheap regulator problem and the cheap servomechanism problem for non-minimum phase systems. Some well-known properties of the "perfect regulation" (Francis, 1979; Kwakernaak and Sivan, 1972; Scherzinger and Davison, 1985) and the "perfect tracking" (Davison and Scherzinger, 1987) problem of minimum phase systems are generalized to systems which may be non-minimum phase. It is shown that a fundamental limitation exists re the speed of tracking and disturbance rejection for a non-minimum phase system, and that this limitation is completely characterized by the location of the unstable transmission zeros of the system. Furthermore, this limitation provides a quantitative measure of the "degree of difficulty" which is inherent in the control of such non-minimum phase systems.

Key Words. Linear optimal regulator; Servomechanisms; Non-minimum phase systems; Transmission zeros; Cheap control; Inner-outer factorization.

1 INTRODUCTION

It has long been realized that minimum phase systems have certain advantages over non-minimum phase systems; in particular, minimum phase systems have the desirable property that their response can be made arbitrarily fast with no "peaking" occurring in the output. This, however, is not the case for non-minimum phase systems. See, e.g. (Davison and Scherzinger, 1987; Francis, 1979; Kwakernaak and Sivan, 1972a; Scherzinger and Davison, 1985). In particular, in studying the "perfect control problem" for a "high gain servo controller" and the "robust servo controller" (Davison and Scherzinger, 1987), it is shown that a necessary condition which must be satisfied in order to obtain "perfect control" is that the system must be minimum phase. On the other hand, it is to be recognized that not all non-minimum phase systems behave the same; for example, some non-minimum phase systems produce results which are "almost as good" as minimum phase systems, whereas other non-minimum phase systems are indeed "almost impossible" to control.

In this paper, we first study the cheap quadratic regulator problem for non-minimum phase systems, which is the limiting case of the optimal quadratic regulator problem when the weight on the input energy of the performance index goes to zero. When a special state space realization is adopted, it is shown that a simple expression for the limiting performance can be obtained. This result is then applied to the study of the cheap optimal servomechanism problem, in which the original servomechanism problem is transformed to a linear quadratic regulator problem and the cheap control of this regulator problem is then studied. Two control schemes are considered. One is the high gain servomechanism controller (Davison and Scherzinger, 1987); the other is the robust servomechanism controller (Davison, 1976b). It is shown that for any non-minimum phase system there exists a fundamental limitation on the resultant optimal cost which characterizes the transient behavior of the closed loop system. This limitation can be completely characterized by the number and the locations of the transmission zeros of the system contained in the open right half of the complex plane, and it provides a quantitative measure of the "degree of difficulty" which is inherent in the control of a non-minimum phase system.

*This work has been supported by the Natural Sciences and Engineering Research Council of Canada under grant no. A4396.

2 PRELIMINARIES

Throughout this paper, a transfer matrix means a proper real rational matrix. Let $F(s)$ be a $r \times m$ transfer matrix. $F(s)$ is said to be tall (or wide, square) if $r \geq m$ (or $r \leq m$, $r = m$). By a realization of $F(s)$, we mean a 4-tuple of matrices over the real numbers (A, B, C, D) such that $D + C(sI - A)^{-1}B = F(s)$. The following notation is used to divide the complex plane into three parts: $\mathbf{C}^+ = \{s \in \mathbf{C} : \Re(s) > 0\}$, $\mathbf{C}^0 = \{s \in \mathbf{C} : \Re(s) = 0\}$, $\mathbf{C}^- = \{s \in \mathbf{C} : \Re(s) < 0\}$. A transfer matrix is said to be stable if all of its poles are contained in $\mathbf{C}^-$, and a square real matrix is said to be stable if all of its eigenvalues are contained in $\mathbf{C}^-$.

The transmission zeros of a realization (A, B, C, D) are defined to be the complex numbers λ which make the matrix

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

reduce rank. The transmission zeros of a transfer matrix $F(s)$ are defined to be the transmission zeros of its minimal realization. It can be shown that the transmission zeros of $F(s)$ are also the roots of the numerator polynomials of the diagonal elements of the Smith-McMillan form of $F(s)$. $F(s)$ is said to be minimum phase if it has no transmission zeros in $\mathbf{C}^+$; otherwise it is said to be non-minimum phase. For more details about transmission zeros, see (Davison and Wong, 1974).

Associated with any realization (A, B, C, D) in which A is stable, there are two Lyapunov equations:

$$\begin{aligned} AP + PA' &= -BB' \qquad (1) \\ A'Q + QA &= -C'C. \qquad (2) \end{aligned}$$

The solutions P, Q to equation (1)-(2) are called the controllability gramian and the observability gramian of (A, B, C, D) respectively. A minimal realization (A, B, C, D) of a stable transfer matrix $F(s)$ is called a balanced realization if the solution P, Q to equations (1)-(2) are diagonal and equal. The diagonal elements of such P or Q are called the Hankel singular values of $F(s)$. It is shown in (Moore, 1981) that every transfer matrix has a balanced realization. A procedure to find a balanced realization from any minimal realization of a transfer matrix is also given in (Moore, 1981).

A stable transfer matrix $F(s)$ is called inner if $F'(-s)F(s) = I$. An inner transfer matrix must be tall. All the transmission zeros of an inner transfer matrix must be located in $\mathbf{C}^+$.

Lemma 1 (Glover, 1984) *Let (A,B,C,D) be a minimal realization of an inner transfer matrix $F(s)$ and let P, Q be the solutions to equations (1)-(2). Then*

(a) $PQ = I$,

(b) $D'D = I$,

(c) $D'C + B'Q = 0$, $DB' + CP = 0$.

Corollary 1 *Let (A,B,C,D) be a balanced realization of an inner transfer matrix $F(s)$ and let P, Q be the solutions to equations (1)-(2). Then*

(a) $P = Q = I$,

(b) $D'D = I$,

(c) $D'C + B' = 0$, $DB' + C = 0$.

A stable transfer matrix $F(s)$ is outer if $F(s)$ has full row rank for every $s \in \mathbf{C}^+$. The concept of outer matrices is not as important as that of inner matrices in our development. Instead, we are more interested in the class of transfer matrices which are minimum phase and wide. Clearly, outer matrices belong to this class, but a matrix in this class does not have to be stable.

The following factorization result, which serves as a foundation for our development, is obtained. It is remarked here that when the poles and/or zeros of two transfer matrices are compared in the following, we consider not only their values but also their multiplicities in the Smith-McMillan sense.

Lemma 2 *A transfer matrix $F(s)$ can always be factorized as $F(s) = F_1(s)F_2(s)$ such that $F_1(s)$ is inner, $F_2(s)$ is minimum phase and wide, and the unstable poles of $F_2(s)$ are equal to the unstable poles of $F(s)$.*

Proof: In this proof, we will use the notation $\mathbf{RH}_\infty$ to denote the set of all stable transfer matrices and will use some knowledge of the factorization theory in $\mathbf{RH}_\infty$ (Francis, 1987). It is known that $F(s)$ can be decomposed as $F(s) = F_s(s) + F_a(s)$ where $F_s(s) \in \mathbf{RH}_\infty$ and $F_a(s)$ is strictly unstable, i.e. all its poles are contained in $\mathbf{C}^0 \cup \mathbf{C}^+$. The first factorization result we have to use is the coprime factorization result which says that any transfer matrix $G(s)$ can be factorized as $G(s) = N(s)M^{-1}(s)$ where $N(s), M(s) \in \mathbf{RH}_\infty$ and there exist $X(s), Y(s) \in \mathbf{RH}_\infty$ such that

$$X(s)N(s) + Y(s)M(s) = I$$

Moreover, $M(s)$ can be chosen so that $M^{-1}(s)$ is proper (Francis, 1987). Let $N(s)M^{-1}(s)$ be such a coprime factorization of $F_a(s)$. Then the transmission zeros of $M(s)$ are the poles of $F_a(s)$. Let $L(s)$ be any matrix (compatible in size) in $\mathbf{RH}_\infty$ with the property that $L^{-1}(s) \in \mathbf{RH}_\infty$ and $M(s)L^{-1}(s), N(s)L^{-1}$ are proper. The existence of such $L(s)$ can be verified simply by choosing $L(s) = I$. Then

$$\begin{aligned} F(s) &= F_s(s) + N(s)L^{-1}(s)L(s)M^{-1}(s) \\ &= [F_s(s)M(s)L^{-1}(s) + N(s)L^{-1}(s)]L(s)M^{-1}(s) \end{aligned}$$

in which $F_s(s)M(s)L^{-1}(s) + N(s)L^{-1}(s)$ belongs to $\mathbf{RH}_\infty$, $L(s)M^{-1}(s)$ is minimum phase, square and its unstable poles are the unstable poles of $F(s)$.

The second factorization result we have to use is the inner-outer factorization result which says that any $G(s) \in \mathbf{RH}_\infty$ can be factorized as $G(s) = G_i(s)G_o(s)$ where $G_1(s)$ is inner and $G_o(s)$ is outer (Chen, 1987). Let the inner outer factorization of $F_s(s)M(s)L^{-1}(s) + N(s)L^{-1}(s)$ be $F_i(s)F_o(s)$ and let $F_1(s) = F_i(s)$ and $F_2(s) = F_o(s)L(s)M^{-1}(s)$. Then we immediately have that $F(s) = F_1(s)F_2(s)$ and $F_1(s)$ is inner. For all $s \in \mathbf{C}^+$, $F_o(s)$ has full row rank and $L(s)M^{-1}(S)$ is a nonsingular square matrix. Therefore $F_2(s)$ must have full row rank for all $s \in \mathbf{C}^+$, which means that $F_2(s)$ must be minimum phase and wide. □

Given a transfer matrix $F(s)$, let $F(s) = F_1(s)F_2(s)$ be the factorization described in Lemma 2. Let (A_1, B_1, C_1, D_1) be a balanced realization of $F_1(s)$ and (A_2, B_2, C_2, D_2) be any stabilizable and detectable realization of $F_2(s)$. Then a stabilizable and detectable realization of $F(s)$ is given by

$$\begin{aligned} A &= \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} \qquad B = \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} \\ C &= [C_1 \ \ D_1C_2] \qquad D = D_1D_2. \end{aligned} \tag{3}$$

This realization is called a factorized realization of $F(s)$ and plays an important role in the development.

A class of transfer matrices called right-invertible transfer matrices deserves special treatment. A transfer matrix $F(s)$ is said to be right-invertible if it has full row rank for at least one $s \in \mathbf{C}$. If (A,B,C,D) is a realization of $F(s)$, then the right-invertibility of $F(s)$ is equivalent to the fact that

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

has full row rank for at least one $\lambda \in \mathbf{C}$. The following lemma gives some useful properties of right-invertible transfer matrices with respect to the factorization described by Lemma 2.

Lemma 3 *Let $F(s)$ be a right-invertible transfer matrix and let $F(s) = F_1(s)F_2(s)$ be the factorization described in Lemma 2. Then $F_1(s)$ is square, the transmission zeros of $F_1(s)$ are equal to those transmission zeros of $F(s)$ contained in $\mathbf{C}^+$, and the poles of $F_1(s)$ are equal to the negative of the transmission zeros of $F_1(s)$.*

Proof: Let $F(s)$ be $r \times m$. Then $F_1(s)$ has r rows and at most r columns. If $F_1(s)$ has less than r columns, then the rank of $F_1(s)$ is less than r for all $s \in \mathbf{C}$, which implies that the rank of $F(s)$ is less than r for all $s \in \mathbf{C}$. This contradicts with the fact that $F(s)$ is right-invertible. Therefore $F_1(s)$ must have r columns, i.e. it must be square.

The transmission zeros of $F_1(s)$ are the poles of $F_1^{-1}(s)$. A minimal realization of $F_1^{-1}(s)$ is given by

$$(A_1 - B_1D_1^{-1}C_1, B_1D_1^{-1}, -D_1^{-1}C_1, D_1^{-1}),$$

where D_1^{-1} exists, so that the transmission zeros of $F_1(s)$ are equal to the eigenvalues of $A_1 - B_1D_1^{-1}C_1$. For the proof of the remaining part of the lemma, it is enough to show that

(a) the eigenvalues of $A_1 - B_1D_1^{-1}C_1$ are equal to the negative of the eigenvalues of A_1;

(b) the eigenvalues of $A_1 - B_1D_1^{-1}C_1$ are equal to those transmission zeros of $F(s)$ contained in $\mathbf{C}^+$.

By Corollary 1,

$$\begin{aligned} A_1 - B_1D_1^{-1}C_1 &= A_1 + B_1D_1^{-1}D_1B_1' \\ &= A_1 + B_1B_1' \\ &= A_1 - (A_1 + A_1') \\ &= -A_1'. \end{aligned}$$

This proves (a).

Since the factorized realization is stabilizable and detectable, those transmission zeros of $F(s)$ which are contained in $\mathbf{C}^+$ are the complex numbers $\lambda \in \mathbf{C}^+$ which make the matrix

$$\begin{bmatrix} A_1 - \lambda I & B_1C_2 & B_1D_2 \\ 0 & A_2 - \lambda I & B_2 \\ C_1 & D_1C_2 & D_1D_2 \end{bmatrix}$$

reduce rank. This matrix can be shown to be similar to the matrix

$$\begin{bmatrix} A_1 - B_1D_1^{-1}C_1 - \lambda I & 0 & 0 \\ 0 & A_2 - \lambda I & B_2 \\ D_1^{-1}C_1 & C_2 & D_2 \end{bmatrix}.$$

This proves (b). □

3 CHEAP LQR PROBLEM

Let $F(s)$ be a transfer matrix and let (A, B, C, D) be a stabilizable and detectable realization of $F(s)$. Consider the linear time-invariant system defined by (A, B, C, D):

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) = x_0 \\ y &= Cx + Du \end{aligned} \tag{4}$$

and consider the optimal linear quadratic regulator problem of this system with respect to the cost

$$J_\epsilon = \int_0^\infty (y'y + \epsilon x'x)dt, \qquad \epsilon > 0. \tag{5}$$

The problem concerning the limit of J_ϵ as $\epsilon \searrow 0$ is called the cheap LQR problem. It is known that the optimal control which minimizes J_ϵ is given by $u = -(\epsilon I + D'D)^{-1}(B'P_\epsilon + D'C)x$, the optimal value of J_ϵ is given by $x_0'P_\epsilon x_0$, where P_ϵ is the unique positive semi-definite solution to the algebraic Riccati equation (ARE)

$$\begin{aligned} &[A - B(\epsilon I + D'D)^{-1}D'C]'P_\epsilon + P_\epsilon[A - B(\epsilon I + D'D)^{-1}D'C] \\ &+C'[I - D(\epsilon I + D'D)^{-1}D']C - P_\epsilon B(\epsilon I + D'D)^{-1}B'P_\epsilon = 0 \end{aligned} \tag{6}$$

It is easy to show that P_ϵ is a monotonically increasing function of $\epsilon > 0$, and so the limit of P_ϵ as $\epsilon \searrow 0$ exists.

Lemma 4 (Scherzinger and Davison, 1985) *Let (A, B, C, D) be a stabilizable and detectable realization of a transfer matrix $F(s)$, and let P_ϵ be the unique positive semi-definite solution to ARE (6). Then $P_\epsilon \to 0$ as $\epsilon \searrow 0$ if $F(s)$ is minimum phase and wide.*

The condition in Lemma 4 is also necessary if matrices B and C are assumed to have full column rank and full row rank respectively. The same result for the case when $D = 0$ is obtained in (Francis, 1979; Kwakernaak and Sivan, 1972a). The main purpose of this section is to give a result on the limit of P_ϵ as $\epsilon \searrow 0$ for systems which do not satisfy the condition given in Lemma 4.

Lemma 5 *Let (A, B, C, D) be a factorized realization of any transfer matrix $F(s)$. Then the unique positive semi-definite solution P_ϵ to ARE (6) is given by $\begin{bmatrix} I & 0 \\ 0 & P_{\epsilon 2} \end{bmatrix}$, where $P_{\epsilon 2}$ is the unique positive semi-definite solution to the ARE*

$$\begin{aligned} &[A_2 - B_2(\epsilon I + D_2'D_2)^{-1}D_2'C_2]'P_{\epsilon 2} \\ &+ P_{\epsilon 2}[A_2 - B_2(\epsilon I + D_2'D_2)^{-1}D_2'C_2] \\ &+ C_2'[I - D_2(\epsilon I + D_2'D_2)^{-1}D_2]C_2 \\ &- P_{\epsilon 2}B_2(\epsilon I + D_2'D_2)^{-1}B_2P_{\epsilon 2} = 0. \end{aligned}$$

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be partitioned accordingly with the partition of A given by (3). Then the optimal control of system (4) under cost (5) is given by $u = -(\epsilon I + D_2'D_2)^{-1}(B_2'P_{\epsilon 2} + D_2'C_2)x_2$.

The proof of the first statement of this lemma is obtained simply by checking that the given solution indeed satisfies ARE (6). The second statement follows by using Corollary 1. The details of the proof are omitted.

Direct application of Lemma 5 and Corollary 1 leads to the following corollary.

Corollary 2 *If (A, B, C, D) is a balanced realization of an inner transfer matrix, then the unique positive semi-definite solution P_ϵ to the ARE (6) is $P_\epsilon = I$ and the optimal control of system (4) under cost (5) is $u = 0$.*

Since any minimal realization of a transfer matrix is similar to a balanced realization, the corollary implies that if (A, B, C, D) is a minimal realization of an inner transfer matrix, then the optimal control of the system (4) under cost (5) is always zero and thus the optimal performance is independent of ϵ. This is consistent with the well-known fact that cheap control asymptotically puts all the poles of the closed loop system to the mirror points of the system's unstable zeros with respect to the imaginary axes. Since an inner matrix already has this property, no control is needed to make it optimal.

Since (A_2, B_2, C_2, D_2) is a stabilizable and detectable realization of a minimum phase and wide system, the following theorem is obtained immediately from Lemma 4-5. (In the statement of the theorem, we assume that P_ϵ is partitioned accordingly with the partition of A given in (3).)

Theorem 1 *Let (A, B, C, D) be a factorized realization of a transfer matrix $F(s)$, and let P_ϵ be the unique positive semi-definite solution to ARE (6). Then $P_\epsilon \to \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ as $\epsilon \to 0$.*

4 HIGH GAIN SERVOMECHANISM CONTROLLER

Consider a plant described by the state space equation:

$$\begin{aligned} \dot{x} &= Ax + Bu + B\omega, \qquad x(0) = 0 \\ y &= Cx + Du + D\omega + \eta \\ e &= y_{\text{ref}} - y. \end{aligned} \tag{7}$$

where ω is the input disturbance, η is the output disturbance and y_{ref} is the output reference. Assume that ω, η and y_{ref} are constant signals. A control problem which often arises is to design a controller for system (7) such that the closed loop system is stable, and such that the tracking error $e \to 0$ as $t \to \infty$ for arbitrary ω, η and y_{ref}. In order for this to be possible, the following assumption is necessary (Davison, 1976a).

Assumption 1 *Assume that system (7) satisfies the following condition:*

(a) *(A, B, C, D) is stabilizable and detectable;*

(b) $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *has full row rank.*

Assumption 1(b) implies that $F(s) = D + C(sI - A)^{-1}B$ is right-invertible and has no transmission zeros at the origin.

Under Assumption 1, it can be shown that the following controller can accomplish the required task:

$$\begin{aligned} u &= [D - (C + DK)(A + BK)^{-1}B]^\dagger e \\ &+ \{K - [D - (C + DK)(A + BK)^{-1}B]^\dagger(C + DK)\}x \end{aligned} \tag{8}$$

where K is any matrix which makes $A + BK$ stable and $M^\dagger$ denotes the Moore-Penrose pseudo-inverse of matrix M. Assumption 1 implies that $M = D - (C + DK)(A + BK)^{-1}B$ has full row rank so that $M^\dagger = M'(MM')^{-1}$. It is easy to check that the closed loop state matrix of (7)-(8) is given by $A + BK$ which is stable. The fact that the error e goes to zero is proved in (Davison and Scherzinger, 1987).

Now assume controller (8) is applied to system (7), and assume that the steady-state values of the input and the state are given by $\bar{u}$ and $\bar{x}$ respectively. Let $v := u - \bar{u}$ and $z := x - \bar{x}$. Then $\bar{u}$, $\bar{x}$ must satisfy equations

$$\begin{aligned} 0 &= A\bar{x} + B\bar{u} + B\omega \\ y_{\text{ref}} &= C\bar{x} + D\bar{u} + D\omega + \eta, \end{aligned}$$

and as a result, system (7) can be written as

$$\begin{aligned} \dot{z} &= Az + Bv \\ e &= Cz + Du. \end{aligned} \tag{9}$$

Equation (9) suggests that

$$J_\epsilon = \int_0^\infty (e'e + \epsilon v'v)dt \tag{10}$$

can be used as a performance index for the closed loop system, where the stabilizing gain matrix K is chosen to minimize the performance index J_ϵ. The purpose of this section is to investigate the behavior of J_ϵ as $\epsilon \searrow 0$. It is obvious that J_ϵ in general depends on y_{ref}, ω and η. It is shown in (Davison and Scherzinger, 1987) that $\lim_{\epsilon\searrow 0} J_\epsilon = 0$ for all $y_{\rm ref}$, ω and η, if (A,B,C,D) is minimum phase. We would like to determine what the limit of $\lim_{\epsilon\searrow 0} J_\epsilon$ is when (A,B,C,D) is non-minimum phase. Before stating our main result, we review some results on quadratic forms.

Let $\mathcal{W}$ be an Euclidean space with inner product $\langle\cdot,\cdot\rangle$. A quadratic form Q on $\mathcal{W}$ is a function $\mathcal{W} \to \mathbf{R}$ mapping w to $Q(w) = \langle w, Hw\rangle$ where H is a given (real) Hermitian operator on $\mathcal{W}$. The quadratic form Q uniquely determine the matrix H and is completely characterized by H. The collection of all quadratic forms on $\mathcal{W}$ is a linear space and clearly is isomorphic to the linear space of all Hermitian operators on $\mathcal{W}$. A quadratic form Q is said to be positive semi-definite if $Q(w) \geq 0$ for all $w \in \mathcal{W}$. A quadratic form is positive semi-definite if and only if its corresponding Hermitian operator is positive semi-definite. A partial ordering can then be defined in the space of all quadratic forms on $\mathcal{W}$: $Q_1 \geq Q_2$ if $Q_1 - Q_2$ is positive semi-definite. Define the norm of a quadratic form to be the norm of its corresponding Hermitian operator. In the following, we always use the operator norm which is given by the sum of all of the singular values of the operator. For positive semi-definite quadratic forms, which are the type we are interested in, this norm is equal to the trace of the corresponding Hermitian operator and it has following two properties:

Lemma 6 (Levine and Athans, 1970) *Let Q be a positive semi-definite quadratic form on Euclidean space $\mathbf{R}^p$ with the usual inner product $\langle w_1, w_2\rangle = w_1'w_2$, and let w be a random vector in $\mathbf{R}^p$ with $E(w) = 0$ and $E\{ww'\} = I$. Then $E\{Q(w)\} = \|Q\|$.*

The notation $E(\cdot)$ in Lemma 6 denotes the expectation operator. Lemma 6 can be interpreted that the norm of Q is the "average" value of $Q(w)$ over all w on a sphere with radius $\sqrt{p}$.

Let $\mathcal{W}_1$ be a subspace of $\mathcal{W}$, and let Q be a quadratic form on $\mathcal{W}$. Then the restriction of Q on $\mathcal{W}_1$, denoted by $Q|\mathcal{W}_1$, is a quadratic form on $\mathcal{W}_1$.

Lemma 7 *Let $\mathcal{W}_1$, $\mathcal{W}_2$ be mutually orthogonal subspaces of an Euclidean space $\mathcal{W}$, and let Q be a positive semi-definite quadratic form on $\mathcal{W}$. Then $\|Q|\mathcal{W}_1\| + \|Q|\mathcal{W}_2\| = \|Q|(\mathcal{W}_1 + \mathcal{W}_2)\|$.*

Proof: Let H be the corresponding operator of Q and let $\mathcal{W}_3$ be the orthogonal complement of $\mathcal{W}_1 + \mathcal{W}_2$. Then H can be represented by the following 3×3 matrix with operator entries:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix},$$

where $H_{ij} = P_i H|\mathcal{W}_j$ and P_i, $i = 1,2,3$ is the projection of $\mathcal{W}$ onto $\mathcal{W}_i$. In this case, H_{11} and H_{22} are just the corresponding operators of $Q|\mathcal{W}_1$ and $Q|\mathcal{W}_2$, and $\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ is the corresponding operator of $Q|(\mathcal{W}_1 + \mathcal{W}_2)$. The positive semi-definiteness of H implies the positive semi-definiteness of H_{11}, H_{22} and $\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$. The lemma then directly follows from the fact that $\mathrm{tr}H_{11} + \mathrm{tr}H_{22} = \mathrm{tr}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$. □

If $\mathcal{W}$ has an orthogonal decomposition $\{\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_q\}$, i.e. $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_q$ are mutually orthogonal and their sum is $\mathcal{W}$, then $\|Q\|$, as well as any $\|Q|\sum_{i\in\mathcal{I}} \mathcal{W}_i\|$ where $\mathcal{I} \subset \{1,2,\ldots,q\}$, is completely determined by all $\|Q|\mathcal{W}_i\|$, $i = 1,2,\ldots,q$.

Now let $\mathcal{W}$ be the space of all vectors of the form $\begin{bmatrix} \omega \\ \eta \\ y_{\rm ref} \end{bmatrix}$. The inner product on $\mathcal{W}$ is defined as usual, e.g. $\langle w_1, w_2\rangle = w_1'w_2$. Define subspaces

$$\mathcal{W}_\omega = \{\begin{bmatrix} \omega \\ \eta \\ y_{\rm ref} \end{bmatrix} \in \mathcal{W} : y_{\rm ref} = 0,\ \eta = 0\}$$

$$\mathcal{W}_\eta = \{\begin{bmatrix} \omega \\ \eta \\ y_{\rm ref} \end{bmatrix} \in \mathcal{W} : y_{\rm ref} = 0,\ \omega = 0\}$$

$$\mathcal{W}_y = \{\begin{bmatrix} \omega \\ \eta \\ y_{\rm ref} \end{bmatrix} \in \mathcal{W} : \omega = 0,\ \eta = 0\}.$$

Then $\mathcal{W}_y$, $\mathcal{W}_\omega$ and $\mathcal{W}_\eta$ form an orthogonal decomposition of $\mathcal{W}$.

Theorem 2 *Given the system (7), assume that Assumption 1 holds; then the performance index J_ϵ given by (10) is a positive semi-definite quadratic form on $\mathcal{W}$, and J_ϵ is a monotonically increasing function of ϵ. Let $\lim_{\epsilon\searrow 0} J_\epsilon = J_0$, and let the transmission zeros of (A,B,C,D) which are contained in $\mathbf{C}^+$ be given $\lambda_1, \lambda_2, \ldots, \lambda_l$; then*

$$\begin{aligned} \|J_0|\mathcal{W}_\omega\| &= 0 \\ \|J_0|\mathcal{W}_\eta\| &= 2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_l}) \\ \|J_0|\mathcal{W}_y\| &= 2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_l}). \end{aligned}$$

This result states that the high gain servomechanism controller (8) produces perfect control, i.e. the optimal performance goes to zero as $\epsilon \searrow 0$, for the case when input disturbances are only present, even if the system (7) is non-minimum phase. However for the case when output disturbances or/and nonzero reference are present, perfect control cannot be obtained for non-minimum phase systems, and the optimal performance J_ϵ is now bounded from below by $2\sum_{i=1}^{l}\frac{1}{\lambda_i}$. This result shows that $\sum_{i=1}^{l}\frac{1}{\lambda_i}$ can be considered as a quantitative measure of the degree of difficulty in the control of non-minimum phase systems. This result also emphasizes the fact that not all non-minimum phase systems behave the same. For example, a plant with one unstable transmission zero at θ has a degree of difficulty equal to $\frac{1}{\theta}$ which is large if θ is small, whereas a plant with two unstable transmission zeros at $\theta \pm j\sigma$ has a degree of difficulty equal to $\frac{2\theta}{\theta^2+\sigma^2}$ which is small if $\sigma \gg \theta$.

The proof of Theorem 2: Since the initial condition of system (7) is assumed to be zero, the input-output relation of the system (7) is determined solely by its transfer matrix $F(s) = D + C(sI - A)^{-1}B$. Thus (A,B,C,D) can be assumed to be the factorized realization of $F(s)$ which is of the form of (3). Let $F_1(s) = D_1 + C_1(sI - A_1)^{-1}B_1$ and $F_2(s) = D_2 + C_2(sI - A_2)^{-1}B_2$. Since $F(s)$ is assumed to be right-invertible, it follows from Lemma 4 that $F_1(s)$ must be square and the poles of $F_1(s)$ are $-\lambda_1, -\lambda_2, \ldots, -\lambda_l$.

It is known that $J_\epsilon = z'(0)P_\epsilon z(0)$ where P_ϵ is the unique positive semi-definite solution of ARE (6) and

$$z(0) = x(0) - \bar{x} = -\bar{x}.$$

Since $x(0)$ is zero, the closed loop system can be considered as a linear system with input $\begin{bmatrix} \omega \\ \eta \\ y_{\rm ref} \end{bmatrix}$ and output x. Hence there exist a linear transformation T such that $\bar{x} = T\begin{bmatrix} \omega \\ \eta \\ y_{\rm ref} \end{bmatrix}$. It therefore follows that $J_\epsilon = \bar{x}'P_\epsilon\bar{x}$ is a positive semi-definite quadratic form on $\mathcal{W}$ for all $\epsilon > 0$. The fact that J_ϵ is monotonically increasing with respect ϵ follows directly from the monotonicity of P_ϵ.

By Theorem 1, $P_\epsilon \to \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ as $\epsilon \searrow 0$. Let $\bar{x}$ be partitioned as $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$ according to the partition of A given by (3). Then

$$J_0 = \lim_{\epsilon\searrow 0} J_\epsilon = \bar{x}'\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\bar{x} = \bar{x}_1'\bar{x}_1.$$

Now assume that the system (7), under control (8), is at steady-state. The following relation exists

$$y_{\text{ref}} - \eta = F(0)(\bar{u} + \omega) = F_1(0)F_2(0)(\bar{u} + \omega).$$

Since

$$\bar{x}_1 = -A_1^{-1}B_1F_2(0)(\bar{u} + \omega),$$

where A_1^{-1} is well defined, it follows that

$$\bar{x}_1 = -A_1^{-1}B_1F_1^{-1}(0)(y_{\text{ref}} - \eta).$$

Therefore, the following expression is obtained for J_0:

$$\begin{aligned} J_0 &= (y_{\text{ref}} - \eta)'F_1(0)'B_1'A_1'^{-1}A_1^{-1}B_1F_1(0)(y_{\text{ref}} - \eta) \\ &= \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & H & -H \\ 0 & -H & H \end{bmatrix} \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}, \end{aligned}$$

where H is used to denote $F_1(0)'B_1'A_1'^{-1}A_1^{-1}B_1F_1(0)$. It becomes clear that $\|J_0|\mathcal{W}_\omega\| = 0$ and

$$\|J_0|\mathcal{W}_\eta\| = \|J_0|\mathcal{W}_y\| = \|F_1(0)'B_1'A_1'^{-1}A_1^{-1}B_1F_1(0)\|.$$

By using the fact that (A_1, B_1, C_1, D_1) is a balanced realization of square inner function $F_1(s)$, we obtain:

$$\begin{aligned} \|J_0|\mathcal{W}_\eta\| &= \|J_0|\mathcal{W}_y\| \\ &= \operatorname{tr}[F_1(0)'B_1'A_1'^{-1}A_1^{-1}B_1F_1(0)] \\ &= \operatorname{tr}[F_1(0)F_1(0)'B_1'A_1'^{-1}A_1^{-1}B_1] \\ &= \operatorname{tr}[A_1^{-1}B_1B_1'A_1'^{-1}] \qquad \text{(since } F_1(s) \text{ is square)} \\ &= -\operatorname{tr}[A_1^{-1}(A_1 + A_1')A_1'^{-1}] \qquad \text{(by Corollary 1)} \\ &= -\operatorname{tr}(A_1'^{-1} + A_1^{-1}) \\ &= -2\operatorname{tr}(A_1^{-1}) \\ &= 2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_l}) \qquad \text{(by Lemma 3).} \end{aligned}$$

□

5 ROBUST SERVOMECHANISM CONTROLLER

Consider system (7), where ω, η and y_{ref} are constant signals. It is now desired to apply a controller to solve the robust servomechanism problem (Davison, 1976b) for (7). Assume that Assumption 1 holds. In this case, a controller which solves the problem must include a servo-compensator, and so consider now the following controller for (7):

$$\begin{aligned} \dot{z} &= e, \qquad z(0) = 0 \qquad (11) \\ u &= K_0x + Kz \end{aligned}$$

where $[K_0\ K]$ is chosen to stabilize the following matrix

$$\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}[K_0\ K]. \qquad (12)$$

Controller (11) has the significant advantage over controller (8) in that tracking and disturbance rejection occur for all perturbations of the system parameters $(A, B, C, D)$and controller parameters $[K_0\ K]$, provided only that the perturbed closed loop system remains stable.

The augmented system (with input u and output z), on combining the original system and the integrator, is then described by the state-space equation:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}(u + \omega) + \begin{bmatrix} 0 \\ I \end{bmatrix}(\eta - y_{\text{ref}})$$

$$\begin{bmatrix} x(0) \\ z(0) \end{bmatrix} = 0$$

$$z = [0\ \ I]\begin{bmatrix} x \\ z \end{bmatrix}.$$

Differentiate all variables once, and define new variables:

$$\tilde{x} := \dot{x}, \quad \tilde{z} := \dot{z}, \quad \tilde{u} := \dot{u}.$$

On noticing that $\dot{z} = e$, the augmented system then becomes

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{z}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix}\tilde{u} \qquad (13)$$

$$\begin{bmatrix} \tilde{x}(0) \\ \tilde{z}(0) \end{bmatrix} = \begin{bmatrix} B & 0 & 0 \\ D & I & I \end{bmatrix}\begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}$$

$$e = [0\ \ I]\begin{bmatrix} \tilde{x} \\ \tilde{z} \end{bmatrix}.$$

where $(\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \begin{bmatrix} B \\ D \end{bmatrix}, [0\ \ I], 0)$ is stabilizable and detectable if Assumption 1 holds.

This suggests that

$$J_\epsilon = \int_0^\infty (e'e + \epsilon\tilde{u}'\tilde{u})dt \qquad (14)$$

can be used as a performance index of the closed loop system, where the matrix $[K_0\ K]$ is chosen to minimize the performance index J_ϵ. The same question to the one studied in last section arises: what is the limit of J_ϵ as ϵ goes to zero? It is known that if (A, B, C, D) is minimum phase, then $\lim_{\epsilon\searrow 0} J_\epsilon = 0$ (Davison and Scherzinger, 1987). In this section, we will obtain a result similar to Theorem 2 for the case of non-minimum phase systems.

Assume that $\mathcal{W}$, $\mathcal{W}_\omega$, $\mathcal{W}_\eta$ and $\mathcal{W}_y$ have the same meaning as used in the last section, and that the same norm of quadratic forms is used.

Theorem 3 *Given the system (7), assume that Assumption 1 holds; then the performance index J_ϵ given by (14) is a positive semi-definite quadratic form on $\mathcal{W}$, and J_ϵ is a monotonically increasing function of ϵ. Let $\lim_{\epsilon\searrow 0} J_\epsilon = J_0$, and let the transmission zeros of (A, B, C, D) which are contained in $\mathbb{C}^+$ be given by $\lambda_1, \lambda_2, \ldots, \lambda_l$; then*

$$\begin{aligned} \|J_0|\mathcal{W}_\omega\| &= 0 \\ \|J_0|\mathcal{W}_\eta\| &= 2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_l}) \\ \|J_0|\mathcal{W}_y\| &= 2(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_l}). \end{aligned}$$

Since the same type of result, as obtained for the high gain controller case, is also obtained in this case, the discussion following Theorem 2 equally applies to this case.

The proof of Theorem 3: Since the initial condition of system (7) is assumed to be zero, the input-output relation of the system (7) is determined solely by its transfer matrix $F(s) = D+C(sI-A)^{-1}B$. Then (A, B, C, D) can be assumed to be the factorized realization of $F(s)$ which is of the form of (3). Consequently (13) can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ \dot{\tilde{z}} \end{bmatrix} &= \begin{bmatrix} A_1 & B_1C_2 & 0 \\ 0 & A_2 & 0 \\ C_1 & D_1C_2 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} B_1D_2 \\ B_2 \\ D_1D_2 \end{bmatrix}\tilde{u} \qquad (15) \\ e &= [0\ 0\ I]\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix}. \end{aligned}$$

Let

$$T := \begin{bmatrix} A_1 & B_1 & 0 \\ 0 & 0 & I \\ C_1 & D_1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{z} \end{bmatrix} := T\begin{bmatrix} \hat{x}_1 \\ \hat{z} \\ \hat{x}_2 \end{bmatrix}.$$

Then (15) becomes

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{z}} \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & B_1 & 0 \\ 0 & 0 & C_2 \\ 0 & 0 & A_2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{z} \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ D_2 \\ B_2 \end{bmatrix}\tilde{u} \qquad (16) \\ e &= [0\ 0\ I]\begin{bmatrix} \hat{x}_1 \\ \hat{z} \\ \hat{x}_2 \end{bmatrix}. \end{aligned}$$

and the initial condition becomes

$$\begin{bmatrix} \hat{x}_1(0) \\ \hat{z}(0) \\ \hat{x}_2(0) \end{bmatrix} = T^{-1} \begin{bmatrix} \bar{x}_1(0) \\ \bar{x}_2(0) \\ \bar{z}(0) \end{bmatrix} = T^{-1} \begin{bmatrix} B_1 D_2 & 0 & 0 \\ B_1 & 0 & 0 \\ D-1D_2 & I & I \end{bmatrix} \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}. \tag{17}$$

Let P_ϵ be the unique semi-definite solution to the following ARE

$$\begin{bmatrix} A_1' & 0 & 0 \\ B_1' & 0 & 0 \\ 0 & C_2' & A_2' \end{bmatrix} P_\epsilon + P_\epsilon \begin{bmatrix} A_1 & B_1 & 0 \\ 0 & 0 & C_2 \\ 0 & 0 & A_2 \end{bmatrix} + \begin{bmatrix} C_1' \\ D_1' \\ 0 \end{bmatrix} [C_1 \ D_1 \ 0]$$

$$-\frac{1}{\epsilon} P_\epsilon \begin{bmatrix} 0 \\ D_2 \\ B_2 \end{bmatrix} [0 \ \ D_2' \ \ B_2'] P_\epsilon = 0.$$

Then $J_\epsilon = [\hat{x}_1'(0) \ \hat{z}'(0) \hat{x}_2'(0)] P_\epsilon \begin{bmatrix} \hat{x}_1(0) \\ \hat{z}(0) \\ \hat{x}_2(0) \end{bmatrix}$. It follows from (17) and the monotonicity of P_ϵ that J_ϵ is a quadratic form on $\mathcal{W}$ for all $\epsilon > 0$ and is monotonically increasing with respect to ϵ.

Note that (16) is a factorized realization with (A_1, B_1, C_1, D_1) being inner and $\left(\begin{bmatrix} 0 & C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} D_2 \\ B_2 \end{bmatrix}, [0 \ \ I], 0 \right)$ being wide and minimum phase. By Theorem 1, $\lim_{\epsilon \searrow 0} P_\epsilon = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Hence

$$\begin{aligned} J_0 \ = \ & \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}' \begin{bmatrix} B_1 D_2 & 0 & 0 \\ B_1 & 0 & 0 \\ D_1 D_2 & I & I \end{bmatrix}' T'^{-1} \\ & \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} T^{-1} \begin{bmatrix} B_1 D_2 & 0 & 0 \\ B_1 & 0 & 0 \\ D_1 D_2 & I & I \end{bmatrix} \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}. \end{aligned}$$

Direct calculation shows that

$$J_0 = \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}' \begin{bmatrix} 0 & 0 & 0 \\ 0 & H & -H \\ 0 & -H & H \end{bmatrix} \begin{bmatrix} \omega \\ \eta \\ y_{\text{ref}} \end{bmatrix}$$

where $H = F_1(0)' B_1' A_1'^{-1} A_1^{-1} B_1 F_1(0)$ and $F_1(0) = D_1 - C_1 A_1^{-1} B_1$.

The rest of the proof now proceeds in exactly the same way as in the last part of the proof of Theorem 2. □

6 CONCLUSION

This paper considers the cheap regulator problem and the cheap optimal servomechanism problem for systems which may be non-minimum phase. The basic tool used is a factorization which factorizes an arbitrary system transfer matrix into the product of an inner transfer matrix and a wide minimum phase transfer matrix. Based on this factorization, the study of an arbitrary system can be decomposed into the study of a system with an inner transfer matrix and the study of a minimum phase and wide system. The cheap control problem of a system with an inner transfer matrix becomes easy to analyze by exploiting various properties of inner matrices, while the cheap control problem of a wide minimum phase system has been intensively studied.

A novel contribution of this paper is the establishment of the fact that the number and the locations of the system's transmission zeros in the open right half of the complex plane, are crucial factors of a system which determines the best attainable closed loop system performance. In another words, we have shown that the design limitations on the closed loop system performance, for the servomechanism problem, can be completely characterized by the number and the locations of the system's open loop transmission zeros in the open right half of the complex plane. This design limitation can be used to evaluate an open loop system, i.e. to determine whether the system is inherently hard to control, and to assess a given closed loop design, i.e. to determine how near the closed loop system's performance is from the best attainable.

The servomechanism problem considered in this paper is only for the case of constant reference and disturbances. The extension of the results obtained to more general reference and disturbance signals offers a direction for future research.

References

Chen, T. (1987). *Spectral and Inner-Outer Factorizations of Rational Matrices.* Master's thesis, Dept. of Electrical Eng., Univ. of Toronto, Toronto, Canada.

Davison, E. J. (1976a). The steady-state invertibility and feedforward control of linear time-invariant systems. *IEEE Trans. Automat. Contr.*, AC-21:529–534.

Davison, E. J. (1976b). The robust control of a servomechanism problem for linear time invariant multivariable systems. *IEEE Trans. Automat. Contr.*, AC-21:25–34.

Davison, E. J. and Scherzinger, B. M. (1987). Perfect control of the robust servomechanism problem. *IEEE Trans. Automat. Contr.*, AC-32:689–702.

Davison, E. J. and Wong, S. H. (1974). Properties and calculation of transmission zeros of linear multivariable systems. *Automatica*, 10:634–658.

Francis, B. A. (1979). The optimal linear-quadratic time-invariant regulator with cheap control. *IEEE Trans. Automat. Contr.*, AC-24:616–621.

Francis, B. A. (1987). *A Course in H_∞ Control Theory.* Springer-Verlag, Berlin.

Glover, K. (1984). All optimal hankel-norm approximations of linear multivariable systems and their l^∞-error bounds. *Int. J. Control*, 39:1115–1193.

Kwakernaak, H. and Sivan, R. (1972a). The maximal achievable accuracy of linear optimal regulators and linear optimal filters. *IEEE Trans. Automat. Contr.*, AC-17:79–86.

Kwakernaak, H. and Sivan, R. (1972b). *Linear Optimal Control Systems.* Wiley-Interscience, New York.

Levine, W. S. and Athans, M. (1970). On the determination of the optimal constant output feedback gains for the linear multivariable systems. *IEEE Trans. Automat. Contr.*, AC-15:44–48.

Moore, B. C. (1981). Principle component analysis of linear systems: controllability, observability, and model reduction. *IEEE Trans. Automat. Contr.*, AC-26:17–32.

Scherzinger, B. M. and Davison, E. J. (1985). The optimal LQ regulator with cheap control for not strictly proper systems. *Optimal Control Application & Method*, 6:291–303.