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Pointwise Gap Metrics on Transfer Matrices*

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Abstract

A new family of metrics, called pointwise gap metrics, in the space of real rational matrices of fixed size is developed in this paper. These metrics are then used to study open loop and closed loop stability robustness of lumped linear time-invariant finite dimensional continuous time systems. It is shown that pointwise gap metrics have the desired qualitative properties for the study of stability robustness. Necessary and sufficient conditions on the open and closed loop stability robustness are obtained in terms of the radii of the pointwise gap metric balls centered at the nominal plant and/or the nominal controller. Comparison of the new metric with the available metrics, e.g. the gap metric and the graph metric, is made. All these metrics induce the same topology. Surprisingly, it is shown that many of the quantitative properties of pointwise gap metrics are the same as those of the gap metric, although they differ in value. A notable distinct property of pointwise gap metrics is that in the scalar case they have a very simple expression which is potentially useful to access the relationship between the uncertainty of physical parameters and uncertainty measured by pointwise gap metrics.

1 Introduction

The study of the stability robustness problem concerns the stability of uncertain systems. If such a study is to be carried out quantitatively, it is necessary to have a mechanism to measure the size of the uncertainty. A metric in the space of the systems under consideration can provide such a mechanism. In this paper, we consider the stability robustness of real rational matrices and assume that these matrices are transfer matrices of lumped linear time-invariant finite-dimensional continuous time systems. Hence, a real rational matrix is said to be stable if it is bounded in $\mathbb{C}^+$, where $\mathbb{C}^+ = \{s \in \mathbb{C} : \Re(s) > 0\}$. The purpose of this paper is to develop a metric in the space of all real rational matrices of certain size which facilitates the study of stability robustness. Before proceeding, we have to know what are the requirements for such a metric.

Let $\mathcal{P}$ be the field of all real rational functions and let $\mathcal{S}$ be the ring of all stable real rational functions. Then $\mathcal{P}$ can be considered as the quotient field of $\mathcal{S}$. We denote naturally by $\mathcal{P}^{p\times m}$ and $\mathcal{S}^{p\times m}$ the sets of all $p \times m$ matrices over (field) $\mathcal{P}$ and (ring) $\mathcal{S}$ respectively.

Each element of $\mathcal{P}^{p\times m}$ corresponds to an open loop system. If $F_0 \in \mathcal{S}^{p\times m}$, i.e. F_0 is stable, a desired property of the metric to be developed is that F_0 has certain stability robustness under the metric, which means that any uncertain system F which is close to F_0 should be stable. Formally, this requires that $\mathcal{S}^{p\times m}$ be an open subset of $\mathcal{P}^{p\times m}$.

To study closed loop stability robustness, consider the standard feedback configuration shown in Figure 1.

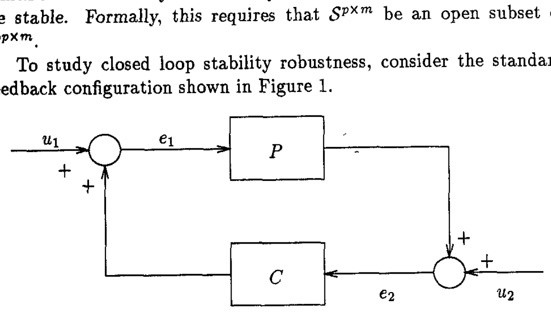

Figure 1: The Standard Feedback System

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In Figure 1, $P \in \mathcal{P}^{p\times m}$ represents the plant and $C \in \mathcal{P}^{m\times p}$ the controller. By the closed loop transfer matrix of the feedback configuration, we mean the transfer matrix from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ which is given by

$$\begin{aligned} \mathbf{H}(P,C) &= \left(I - \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}\right)^{-1} \\ &= \begin{bmatrix} (I-CP)^{-1} & C(I-PC)^{-1} \\ P(I-CP)^{-1} & (I-PC)^{-1} \end{bmatrix}. \end{aligned} \quad (1)$$

In order for $\mathbf{H}(P,C)$ to exist, $(I-CP)$ has to be invertible. If this is the case, we say that the closed loop system, or simply the pair (P,C), is well-posed. If we denote the set of all well-posed pairs, which is a subset of $\mathcal{P}^{p\times m} \times \mathcal{P}^{m\times p}$, by $\mathcal{W}(p,m)$, then $\mathbf{H}$ defines a function from $\mathcal{W}(p,m)$ to $\mathcal{P}^{(p+m)\times(p+m)}$. This function is injective since

$$\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}[I - (\mathbf{H}(P,C))^{-1}].$$

Let us denote the range of $\mathbf{H}(P,C)$, which is a subset of $\mathcal{P}^{(p+m)\times(p+m)}$, by $\mathcal{C}(p,m)$. Then $\mathbf{H}$ is a bijective function from $\mathcal{W}(p,m)$ to $\mathcal{C}(p,m)$. A desired property of the metric to be developed for closed loop stability robustness analysis is that if $(P_0,C_0) \in \mathcal{W}(p,m)$ and (P,C) is close to (P_0,C_0), then $(P,C) \in \mathcal{W}(p,m)$ and $\mathbf{H}(P,C)$ is close to $\mathbf{H}(P_0,C_0)$. Formally, this is equivalent to the requirement that $\mathcal{W}(p,m)$ be an open subset of $\mathcal{P}^{p\times m} \times \mathcal{P}^{m\times p}$ and that the function $\mathbf{H}$ from $\mathcal{W}(p,m)$ to $\mathcal{C}(p,m)$ be continuous.

The closed loop system given by the feedback configuration is said to be stable if the closed loop transfer matrix $\mathbf{H}(P,C)$ is stable. In this case, we say that C stabilizes P. The open loop and closed loop requirements together now imply that if (P_0,C_0) is well-posed and $\mathbf{H}(P_0,C_0)$ is stable, then (P,C) is well-posed and $\mathbf{H}(P,C)$ is stable for (P,C) close to (P_0,C_0).

The requirements given above are qualitative in nature since only the existence of stability robustness is asked; these requirements only depend on the topology induced by the metric, but not on the quantitative values of the metric. However, such existence arguments are usually not enough. We also want to use the metric to determine how much uncertainty can be tolerated on a stable transfer matrix F in order to maintain stability, and how much uncertainty can be tolerated on a pair $(P,C) \in \mathcal{W}(r,m)$ with stable $\mathbf{H}(P,C)$ such that the closed loop system is still stable. This requires that the metric developed should have certain desirable quantitative properties so that the questions above can be easily answered in terms of the size of uncertainty measured by the metric.

Two metrics serving our purpose have been developed in the last decade. One is the gap metric introduced by Zames and El-Sakkary [20] and the second one is the graph metric defined by Vidyasagar [19]. Both metrics induce the same topology in $\mathcal{P}^{p\times m}$ and this topology satisfies our requirements. However, they possess different quantitative properties. A computable formula of the gap metric is given in [7]; simple necessary and sufficient conditions are derived in [8] for the open loop and closed loop stability robustness in terms of the radii of the gap metric balls centered at the nominal plant and/or the nominal controller, and a procedure is given in [8] to design a robust controller which maximizes the closed loop stability robustness with respect to plant uncertainty. In contrast, the graph metric is difficult to compute, the quantitative conditions for the open loop and closed loop stability robustness in terms of the graph metric are conservative, and it is not clear how to incorporate it in the design of robust control systems.

For the stability robustness analysis of SISO systems, El-Sakkary [5] defines a metric in $\mathcal{P}$ which is based on the chord distance of the values of the transfer functions on the Riemann sphere. This metric induces the same topology in $\mathcal{P}$ as the gap metric and the graph metric. The advantage of this metric is that it has a simple expression in terms of the transfer functions which may provide an access to the relationship between the uncertainty of physical parameters and the uncertainty measured by the metric.

In this paper, we develop a new family of metrics, called pointwise gap metrics. It is shown that pointwise gap metrics induce the same topology as the gap metric and the graph metric, but differ from them in value. Pointwise gap metrics are in principle computable directly from its definition. In the scalar case, this family of metrics degenerates to a single metric and this metric turns out to be exactly the same as the metric defined in [5]. Necessary and sufficient conditions on the open loop and closed loop stability robustness are obtained in terms of the radii of the pointwise gap metric balls centered at the nominal plant and/or the nominal controller. Surprisingly, the conditions are nearly identical to the ones obtained in [8], in which the gap metric is used. From this we can conclude that a closed loop system is robust with respect to the gap metric if and only if it is robust with respect to any of the pointwise gap metrics. The same procedure in [8] can be used to design a controller with optimal closed loop stability robustness measured by pointwise gap metrics.

Throughout this paper, we always assume that the singular values of a matrix $A \in \mathbb{C}^{p\times m}$ are ordered nonincreasingly and denote the i-th singular value of A by $\sigma_i(A)$, $i = 1, 2, \ldots, \min\{p, m\}$. The matrix norms used in this paper, denoted by $\|\cdot\|$, belong to the family of unitarily invariant matrix norms, i.e. matrix norms satisfying

(a) $\|UAV\| = \|A\|$ for all $A \in \mathbb{C}^{p\times m}$ and all compatible unitary matrices U, V, and

(b) $\|uv^*\| = \|u\|_2\|v\|_2$ for all $u \in \mathbb{C}^p$ and $v \in \mathbb{C}^m$.

For an introduction to unitarily invariant matrix norms, see [14]. Frequently used examples of unitarily invariant matrix norms are the spectral norm $\|A\|_s = \sigma_1(A)$, Frobenius norm $\|A\|_F = \left[\sum_{i=1}^{\min\{p,m\}} \sigma_i^2(A)\right]^{\frac{1}{2}}$ and the trace norm $\|A\|_t = \sum_{i=1}^{\min\{p,m\}} \sigma_i(A)$. A clear distinction has to be made in this paper between a real rational matrix and its value at a certain point in the complex plane. If F denotes a $p \times m$ real rational matrix, then $F(s)$, where $s \in \mathbb{C}$ is not a pole of F, is considered to be a matrix in $\mathbb{C}^{p\times m}$ and is just equal to the value of F at s. Thus $F^*(s)$ means a matrix in $\mathbb{C}^{m\times p}$ which is the conjugate transpose of $F(s)$ and is equal to $F'(s^*)$. By a norm on real rational matrices, we always mean in this paper a norm belonging to the following class of norms: the norm on stable real rational matrices corresponding to an arbitrary unitarily invariant norm $\|\cdot\|$, also denoted by $\|\cdot\|$, is defined by

$$\|F\| = \sup_{s\in\mathbb{C}^+} \|F(s)\|.$$

In particular, the norm of F corresponding to the spectral norm is denoted by $\|F\|_s$. Since all unitarily invariant norms are equivalent, so are all norms on real rational matrices. A useful fact, as a consequence of the maximum modulus principle, is that $\sup_{s\in\mathbb{C}^+} \|F(s)\| = \sup_{\omega\in\mathbb{R}} \|F(j\omega)\|$.

2 Canonical Angles and Gaps

We introduce, in this section, the concepts of canonical angles and gaps between subspaces of the unitary space $\mathbb{C}^n$. The origins of these concepts are hard to trace, but a good set of references is given in [2]. See also [1] for the computation of canonical angles. A detailed treatment of these concepts is also given in [16] with emphasis in their application in the study of pointwise gap metrics.

Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathbb{C}^n$ with $\dim(\mathcal{X}) = \dim(\mathcal{Y}) = l$ and let X_1 and Y_1 be matrices whose columns form orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Since $\|Y_1^*X_1\|_s \le 1$, all singular values of $Y_1^*X_1$ are bounded by 1. Hence there exists an angle θ_i in $[0, \frac{\pi}{2}]$ such that $\cos\theta_i = \sigma_{l-i+1}(Y_1^*X_1)$ for $i = 1, 2, \ldots, l$. Furthermore, the singular values of $Y_1^*X_1$ are invariant to the choice of X_1 and Y_1 and are determined by $\mathcal{X}$ and $\mathcal{Y}$ completely. This leads to our definition: the angles $\{\theta_i, i = 1, 2, \ldots, l\}$ are called the *canonical angles* between $\mathcal{X}$ and $\mathcal{Y}$.

Let $\|\cdot\|$ be any unitarily invariant norm. Let θ_i, $i = 1, 2, \ldots, l$, be the i-th canonical angles between $\mathcal{X}$ and $\mathcal{Y}$. The *gap* between subspaces $\mathcal{X}$ and $\mathcal{Y}$ corresponding to norm $\|\cdot\|$ is defined as

$$\gamma(\mathcal{X}, \mathcal{Y}) = \|\text{diag}(\sin\theta_1, \sin\theta_2, \ldots, \sin\theta_l)\|.$$

Apparently, a gap γ depends on the underlying norm $\|\cdot\|$. If $\|\cdot\|$ is the spectral norm $\|\cdot\|_s$, then the corresponding gap is denoted by γ_s. The value of γ_s is always in $[0, 1]$. Denote the range of some matrix by $\mathcal{R}(\cdot)$. The following result concerns the computation of gaps.

Proposition 1 *Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathbb{C}^n$ with equal dimensions. Suppose $X = [X_1\ X_2]$ and $Y = [Y_1\ Y_2]$ are unitary matrices with $\mathcal{X} = \mathcal{R}(X_1)$ and $\mathcal{Y} = \mathcal{R}(Y_1)$. Then*

$$\gamma(\mathcal{X}, \mathcal{Y}) = \|Y_1^*X_2\| = \|Y_2^*X_1\|.$$

Let us denote by $\Gamma_l(\mathbb{C}^n)$ the set of all l-dimensional subspaces of $\mathbb{C}^n$. Then γ is a function from $\Gamma_l(\mathbb{C}^n) \times \Gamma_l(\mathbb{C}^n)$ to $[0, \infty)$.

Proposition 2 ([12]) *γ is a metric on $\Gamma_l(\mathbb{C}^n)$ and the metric space $(\Gamma_l(\mathbb{C}^n), \gamma)$ is compact.*

A different proof of Proposition 2 is given in [16].

3 Definition of Pointwise Gap Metrics

It is well-known that every real rational matrix has both a right-coprime factorization and a left-coprime factorization over the ring of all stable real rational functions. This means that for each real rational matrix F, there exist stable real rational matrices M, N, U, V, and $\tilde{M}$, $\tilde{N}$, $\tilde{U}$, $\tilde{V}$ such that $F = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $UM + VN = I$, $\tilde{M}\tilde{U} + \tilde{N}\tilde{V} = I$. Right-coprime factorizations of a real rational matrix are not unique, but the set of all right-coprime factorizations can be parameterized by a free unimodular matrix. A stable square real rational matrix is said to be *unimodular* if its inverse exists and is stable. Let NM^{-1} be a right-coprime factorization of $F \in \mathcal{P}^{p\times m}$. Then the set of all right-coprime factorizations of F is given by $\{(ND)(MD)^{-1} : D \in \mathcal{S}^{m\times m}$ is unimodular$\}$. A similar parameterization exists for left-coprime factorizations. We refer to [19] and [6] for detailed exposition of coprime factorizations.

Let NM^{-1} be a right-coprime factorization of $F \in \mathcal{P}^{p\times m}$. Although such a factorization is not unique, the parameterization of all right-coprime factorizations given in the last paragraph shows that the subspace $\mathcal{R}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right) \in \mathbb{C}^{p+m}$ at each $s \in \mathbb{C}^+$ is uniquely determined by F. Denote this subspace by $\mathcal{G}(F, s)$. The right-coprimeness of N and M implies that the dimension of $\mathcal{G}(F, s)$ is m for all $s \in \mathbb{C}^+$.

Let γ be the gap between the m-dimensional subspaces of $\mathbb{C}^{p+m}$ corresponding to a unitarily invariant norm $\|\cdot\|$. Define a function $\delta : \mathcal{P}^{p\times m} \times \mathcal{P}^{p\times m} \to [0, \infty)$ by

$$\delta(F_1, F_2) = \sup_{s\in\mathbb{C}^+} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)]. \tag{2}$$

Proposition 3 *δ is a metric on $\mathcal{P}^{p\times m}$.*

We call δ a *pointwise gap metric* on $\mathcal{P}^{p\times m}$ corresponding to the unitary invariant norm $\|\cdot\|$. If the underlying norm used to define the pointwise gap metric is $\|\cdot\|_s$, then the corresponding pointwise gap metric is written as δ_s. The space $\mathcal{P}^{p\times m}$ equipped with a pointwise gap metric forms a metric space. Since all unitarily invariant matrix norms are equivalent, all pointwise gap metrics are uniformly equivalent (— see [18] for the definitions).

Let NM^{-1} be a right-coprime factorization of F and $\tilde{M}^{-1}\tilde{N}$ be a left-coprime factorization of F. Then an orthonormal basis of $\mathcal{G}(F, s)$ is given by the columns of

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} [M^*(s)M(s) + N^*(s)N(s)]^{-\frac{1}{2}}.$$

Since $-\tilde{N}M + \tilde{M}N = 0$, an orthonormal basis of the orthogonal complement of $\mathcal{G}(F, s)$ is given by the columns of

$$\begin{bmatrix} -\tilde{N}^*(s) \\ \tilde{M}^*(s) \end{bmatrix} [\tilde{M}(s)\tilde{M}^*(s) + \tilde{N}(s)\tilde{N}^*(s)]^{-\frac{1}{2}}.$$

It follows from Proposition 1 that for $F_1, F_2 \in \mathcal{P}^{p\times m}$,

$$\begin{aligned} &\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] \\ &= \Bigg\| [\tilde{M}_2(s)\tilde{M}_2^*(s) + \tilde{N}_2(s)\tilde{N}_2^*(s)]^{-\frac{1}{2}} [-\tilde{N}_2(s)\ \tilde{M}_2(s)] \\ &\qquad \begin{bmatrix} M_1(s) \\ N_1(s) \end{bmatrix} [M_1^*(s)M_1(s) + N_1^*(s)N_1(s)]^{-\frac{1}{2}} \Bigg\| \end{aligned} \tag{3}$$

where $N_1M_1^{-1}$ is a right-coprime factorization of F_1 and $\tilde{M}_2^{-1}\tilde{N}_2$ is a left-coprime factorization of F_2. Then $\delta(F_1, F_2)$ can be obtained by taking the supremum over $\mathbf{C}^+$. One might wonder if an analogue of the maximum modulus principle exists in this case, i.e. if the supremum over $\mathbf{C}^+$ can be replaced by a supremum over $\{j\omega : \omega \in \mathbf{R}\}$. Unfortunately, examples can be constructed to rule out such an expectation. Therefore, the computation of pointwise gap metrics becomes a non-concave maximization problem over a two-dimensional domain which in principle can be solved by a two-dimensional brute force search technique. It is shown in [16] that this maximization problem can actually be solved by carrying out a one-dimensional brute force search together with a one-dimensional bisection search. A more efficient method to compute pointwise gap metrics is yet to be found.

4 The Pointwise Gap Metric in the Scalar Case

Suppose that F_i is a scalar real rational function and $N_iM_i^{-1}$ is a (right- or left-) coprime factorization for $i = 1, 2$. Then by (3),

$$\begin{aligned} &\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] \\ &= \Bigg| [|M_1(s)|^2 + |N_1(s)|^2]^{-\frac{1}{2}} [-N_2(s)\ M_2(s)] \\ &\qquad \begin{bmatrix} M_1(s) \\ N_1(s) \end{bmatrix} [|M_1(s)|^2 + |N_1(s)|^2]^{-\frac{1}{2}} \Bigg| \\ &= \frac{|M_2(s)N_1(s) - N_2(s)M_1(s)|}{\sqrt{|M_1(s)|^2 + |N_1(s)|^2}\sqrt{|M_2(s)|^2 + |N_2(s)|^2}}. \end{aligned}$$

If we use the arithmetic in $\mathbf{C} \cup \{\infty\}$ instead of $\mathbf{C}$, we have

$$\begin{aligned} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] &= \frac{|N_1(s)M_1^{-1}(s) - N_2(s)M_2^{-1}(s)|}{\sqrt{1 + |N_1(s)M_1^{-1}(s)|^2}\sqrt{1 + |N_2(s)M_2^{-1}(s)|^2}} \\ &= \frac{|F_1(s) - F_2(s)|}{\sqrt{1 + |F_1(s)|^2}\sqrt{1 + |F_2(s)|^2}}. \end{aligned}$$

This is simply the chordal metric between $F_1(s)$ and $F_2(s)$ which is the chordal distance of the stereographic projections of $F_1(s)$ and $F_2(s)$ on the Riemann sphere [13]. The pointwise gap metric for scalar real rational functions is then given by

$$\delta(F_1, F_2) = \sup_{s \in \mathbf{C}^+} \frac{|F_1(s) - F_2(s)|}{\sqrt{1 + |F_1(s)|^2}\sqrt{1 + |F_2(s)|^2}}. \tag{4}$$

This formula gives a very clear and intuitive geometric interpretation of the pointwise gap metric for scalar real rational functions: it is no more than the supremum of the difference between the values of the transfer functions over all $s \in \mathbf{C}^+$, where the difference is measured by the chordal metric.

The metric (4) was actually proposed by El-Sakkary [5] to study the robustness of SISO systems. We have just shown that it is simply a special case of pointwise gap metrics. Therefore, almost all results in [5] can be obtained by specializing the results given in this paper.

5 Qualitative Properties of Pointwise Gap Metrics

First, we will examine the open loop qualitative properties of pointwise gap metrics. To start with, we give a lemma regarding the pointwise gap metric corresponding to the spectral norm.

Lemma 1 *For each stable real rational matrix F,*

$$\delta_s(F, 0) = \frac{\|F\|_s}{\sqrt{1 + \|F\|_s^2}}.$$

Now let δ be a pointwise gap metric corresponding to any unitarily invariant norm. Denote by $\mathcal{B}(F_0, r)$ the gap metric open ball centered at $F_0 \in \mathcal{P}^{p\times m}$ with radius r, i.e. $\mathcal{B}(F_0, r) = \{F \in \mathcal{P}^{p\times m} : \delta(F, F_0) < r\}$. In particular, $\mathcal{B}_s(F_0, r) = \{F \in \mathcal{P}^{p\times m} : \delta_s(F, F_0) < r\}$. Note that a real rational matrix with all elements equal to zero is denoted by $\mathbf{0}$, with its size determined by the context.

Theorem 1 *$\mathcal{S}^{p\times m}$ is an open subset of $(\mathcal{P}^{p\times m}, \delta)$. In particular, $\mathcal{S}^{p\times m} = \mathcal{B}_s(0, 1)$.*

Since the pointwise gap metrics corresponding to different unitarily invariant norms are equivalent, the first sentence of Theorem 1 is actually implied by the second sentence. Theorem 1 gives a major qualitative property of pointwise gap metrics; besides, it gives a precise quantitative characterization of the stable and the unstable real rational matrices in terms of the pointwise gap metric corresponding to the spectral norm: all stable matrices are in the open unit ball centered at the origin and the unstable ones are on the unit sphere.

More open loop properties of pointwise gap metrics are given in the following.

Proposition 4 *For any $F_1, F_2 \in \mathcal{S}^{p\times m}$,*

$$\frac{\|F_1 - F_2\|}{\sqrt{1 + \|F_1\|_s^2}\sqrt{1 + \|F_2\|_s^2}} \le \delta(F_1, F_2) \le \|F_1 - F_2\|. \tag{5}$$

An important consequence of Proposition 4 is that it identifies the topologies in $\mathcal{S}^{p\times m}$ induced by pointwise gap metrics and norms.

Corollary 1 *The topology in $\mathcal{S}^{p\times m}$ induced by a pointwise gap metric and that induced by a norm are the same.*

For each $F \in \mathcal{P}^{p\times m}$, we have $F' \in \mathcal{P}^{m\times p}$. Furthermore, if F is stable, so is F'. Hence $F \to F'$ is a bijection between $\mathcal{P}^{p\times m}$ and $\mathcal{P}^{m\times p}$ which preserves stability. The following result says that this bijection is actually an isometry under a pointwise gap metric.

Proposition 5 $\delta(F_1', F_2') = \delta(F_1, F_2)$.

The following two remarks, which are verified in [16] using examples, provide some topological insight to pointwise gap metrics.

(i) A linear space with a metric is called a *metric linear space* if the addition of vectors and the multiplication by scalars are continuous operations [17]. The space $\mathcal{P}^{p\times m}$ is a linear space over the field $\mathcal{P}$, but $(\mathcal{P}^{p\times m}, \delta)$ is not a metric linear space. In fact, both operations are not continuous.

(ii) If we consider the space $\mathcal{P}^{p\times m}$ as the $(p \times m)$-fold product space of $\mathcal{P}$, a natural question to ask is whether the topology in $\mathcal{P}^{p\times m}$ induced by a pointwise gap metric is the product topology of the topology in $\mathcal{P}$ induced by the pointwise gap metric in $\mathcal{P}$. The answer for this question is negative.

In the rest of this section, we work towards the qualitative closed loop properties of pointwise gap metrics. Consider the feedback configuration shown in Figure 1; in this case, different spaces of real rational matrices are involved. If the plant space is $\mathcal{P}^{p\times m}$, then the controller space is $\mathcal{P}^{m\times p}$ and the closed loop transfer matrices are in the space $\mathcal{P}^{(p+m)\times(p+m)}$. We will allow the pointwise gap metrics in these spaces to be induced by different unitarily invariant matrix norms. In addition to the spaces mentioned above, another space we have to consider is the product space of plant-controller pairs $\mathcal{P}^{p\times m}\times\mathcal{P}^{m\times p}$. In the following, we do not need a metric in the product space $\mathcal{P}^{p\times m}\times\mathcal{P}^{m\times p}$, but we do need a topology in the product space. The topology is assumed to be the product topology generated from the gap metric topologies in $\mathcal{P}^{p\times m}$ and $\mathcal{P}^{m\times p}$.

The following result shows that the topology on block diagonal matrices induced by a pointwise gap metric is the product topology of topologies on individual diagonal blocks.

Proposition 6 *Let $F_1, G_1 \in \mathcal{P}^{p_1\times m_1}$ and $F_2, G_2 \in \mathcal{P}^{p_2\times m_2}$. Then*

$$\delta_s\left(\begin{bmatrix} F_1 & 0 \\ 0 & F_2\end{bmatrix}, \begin{bmatrix} G_1 & 0 \\ 0 & G_2\end{bmatrix}\right) = \max\{\delta_s(F_1,G_1), \delta_s(F_2,G_2)\}.$$

The following result shows that the multiplication of real rational matrices by unimodular matrices is continuous.

Proposition 7 *Let $F_1, F_2 \in \mathcal{P}^{p\times m}$ and let $G \in \mathcal{S}^{p\times p}$, $H \in \mathcal{S}^{m\times m}$ be unimodular matrices. Then*

$$\delta(GF_1H, GF_2H) \le \max\{\|H^{-1}\|_s, \|G\|_s\}\max\{\|H\|_s, \|G^{-1}\|_s\}\delta(F_1,F_2)$$
$$\delta(F_1,F_2) \le \max\{\|H^{-1}\|_s, \|G\|_s\}\max\{\|H\|_s, \|G^{-1}\|_s\}\delta(GF_1H, GF_2H).$$

In our applications, the matrices G and H are orthogonal matrices. In this case, the inequalities in Proposition 7 degenerate to an equality.

Corollary 2 *Let $F_1, F_2 \in \mathcal{P}^{p\times m}$ and let $G \in \mathbf{R}^{p\times p}$ and $H \in \mathbf{R}^{m\times m}$ be orthogonal matrices. Then*

$$\delta(GF_1H, GF_2H) = \delta(F_1,F_2).$$

The following result shows that the addition of real rational matrices by a stable real rational matrix is continuous.

Proposition 8 *Let F_1, $F_2 \in \mathcal{P}^{p\times m}$ and $G \in \mathcal{S}^{p\times m}$. Then*

$$\delta(G+F_1, G+F_2) \le \frac{1}{2}(2+\|G\|_s^2+\|G\|_s\sqrt{4+\|G\|_s^2})\delta(F_1,F_2)$$
$$\delta(F_1,F_2) \le \frac{1}{2}(2+\|G\|_s^2+\|G\|_s\sqrt{4+\|G\|_s^2})\delta(G+F_1, G+F_2).$$

An important special case in our application happens when $G = I$. We have the following corollary for this case.

Corollary 3 *Let F_1, $F_2 \in \mathcal{P}^{p\times m}$. Then*

$$\begin{aligned}\delta(I+F_1, I+F_2) &\le \frac{3+\sqrt{5}}{2}\delta(F_1,F_2)\\ \delta(F_1,F_2) &\le \frac{3+\sqrt{5}}{2}\delta(I+F_1, I+F_2).\end{aligned}$$

Proposition 9 *If $F_1, F_2 \in \mathcal{P}^{m\times m}$ are invertible, then*

$$\delta(F_1^{-1}, F_2^{-1}) = \delta(F_1,F_2).$$

The study of well-posedness of a plant-controller pair requires the following result in the set of invertible matrices.

Proposition 10 *The set of invertible matrices in $\mathcal{P}^{m\times m}$ is open.*

Consider the following maps:

$$\begin{array}{llll}\text{(i).} & (F_1,F_2) & \to & \begin{bmatrix} F_1 & 0 \\ 0 & F_2\end{bmatrix}\\ \text{(ii).} & F & \to & GF \quad (G \text{ is real orthogonal})\\ \text{(iii).} & F & \to & I+F \quad (F \text{ is square})\\ \text{(iv).} & F & \to & F^{-1} \quad (F \text{ is invertible}).\end{array}$$

An interpretation of Propositions 6-9 and Corollaries 2-3 is that maps (i)-(iv) are all continuous and have continuous inverses in their ranges. In other words, these maps are homeomorphisms between their domains and their ranges. (In fact, maps (ii), (iv) are isometries.) This interpretation allows us to establish our main result on the qualitative closed loop properties of pointwise gap metrics.

Recall that $\mathcal{W}(p,m) \in \mathcal{P}^{p\times m}\times\mathcal{P}^{m\times p}$ is the set of all well-posed (P,C) pairs, $\mathbf{H}$ is a function from $\mathcal{W}(p,m)$ to $\mathcal{P}^{(p+m)\times(p+m)}$ defined by (1), and $\mathcal{C}(p,m)$ is the set of $\mathbf{H}(P,C)$ when $(P,C) \in \mathcal{W}(p,m)$. The set $\mathcal{W}(p,m)$ must be open since it is the set of all (P,C) with $I - \begin{bmatrix} 0 & I \\ I & 0\end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & C\end{bmatrix}$ invertible and since such a set is simply the inverse image under the composition of the continuous maps (i)-(iii) of the set of all invertible matrices in $\mathcal{P}^{(p+m)\times(p+m)}$, which is an open set. The map $\mathbf{H}$ is a composition of maps (i)-(iv); since these maps are homeomorphisms, so is $\mathbf{H}$. Therefore we have proved the following theorem.

Theorem 2 *The set $\mathcal{W}(p,m)$ is an open subset of $\mathcal{P}^{p\times m}\times\mathcal{P}^{m\times p}$ and the map $\mathbf{H}$ is a homeomorphism between $\mathcal{W}(p,m)$ and $\mathcal{C}(p,m)$.*

Roughly speaking, Theorem 2 says that if (P_0, C_0) is well-posed and P, C are sufficiently close to P_0, C_0 respectively, then (P,C) is well-posed and $\mathbf{H}(P,C)$ is close to $\mathbf{H}(P_0,C_0)$. Conversely, if (P_0,C_0) and (P,C) are well-posed and $\mathbf{H}(P,C)$ is close to $\mathbf{H}(P_0,C_0)$, then P, C must be close to P_0, C_0 respectively.

6 Quantitative Properties of Pointwise Gap Metrics

Similar to the preceding section, this section starts with the open loop properties and then switchs to the closed loop properties.

Theorem 3 *Let $F_0 \in \mathcal{S}^{p\times m}$. Then $\mathcal{B}(F_0,r) \subset \mathcal{S}^{p\times m}$ if and only if $r \le \sqrt{1-\delta_s^2(F_0,0)}$.*

We will see later on, that this theorem is only a special case of a forthcoming theorem. The bound given in Theorem 3 takes another form by using Lemma 1:

$$\sqrt{1-\delta_s^2(F_0,0)} = \frac{1}{\sqrt{1+\|F_0\|_s^2}}.$$

Now let us consider the feedback configuration shown in Figure 1. Let the pointwise gap metric in the plant space $\mathcal{P}^{p\times m}$ be δ_1 and that in the controller space $\mathcal{P}^{m\times p}$ be δ_2. The open balls in the two spaces take the form

$$\mathcal{B}_1(P_0,r) = \{P : \delta_1(P,P_0) < r\}$$

and

$$\mathcal{B}_2(C_0,r) = \{C : \delta_2(C,C_0) < r\}.$$

Define a partial function $\nu : \mathcal{P}^{p\times m}\times\mathcal{P}^{m\times p} \to \mathbf{R}$ by

$$\nu(P,C) = \left\|\begin{bmatrix} I \\ P\end{bmatrix}(I-CP)^{-1}[I\ C]\right\|_s^{-1}. \tag{6}$$

The domain of definition of ν includes all pairs (P,C) such that $\mathbf{H}(P,C)$ is stable, since we have the identity

$$\begin{bmatrix} I \\ P\end{bmatrix}(I-CP)^{-1}[I\ C] = \mathbf{H}(P,C) - \begin{bmatrix} 0 & 0 \\ 0 & I\end{bmatrix}.$$

Theorem 4 *Let (P_0,C_0) be given with $\mathbf{H}(P_0,C_0)$ stable. Then the following statements are equivalent:*

(a) $r \le \nu(P_0,C_0)$;

(b) $\mathbf{H}(P,C_0)$ *is stable for all* $P \in \mathcal{B}_1(P_0,r)$;

(c) $\mathbf{H}(P_0,C)$ *is stable for all* $C \in \mathcal{B}_2(C_0,r)$;

(d) $\mathbf{H}(P,C)$ *is stable for all* P, C *with*

$$\delta_1^2(P,P_0)+\delta_2^2(C,C_0)+2\delta_1(P,P_0)\delta_2(C,C_0)\sqrt{1-\nu^2(P_0,C_0)} < r^2.$$

Note that Theorem 3 is obtained from Theorem 4 on letting $C_0 = 0$.

A graphic interpretation of Theorem 4 is shown in Figure 2. A point (r_1,r_2) with $0 \le r_1, r_2 \le 1$ in the coordinate system represents a set of plant-controller pairs (P,C) which satisfy $\delta_1(P,P_0) = r_1$ and $\delta_2(C,C_0) = r_2$. The equivalence of (a) and (d) in Theorem 4 says that the largest area which contains only stable pairs and whose boundary is given by an ellipse with equation

$$r_1^2+r_2^2+2r_1r_2\sqrt{1-\nu^2(P_0,C_0)} = r^2$$

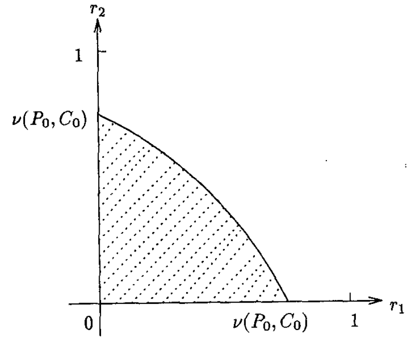

Figure 2: An Area Containing Only Stable Pairs

is the shaded area in Figure 2.

Theorem 4 shows that the function ν gives a good measure for the closed loop stability robustness. A natural design problem is to find a controller C for a given plant P so that the closed loop stability robustness is maximized, i.e. we want to find

$$\sup_C \nu(P, C) = \left[\inf_C \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \; C] \right\|_s \right]^{-1}$$

where C is taken from the set of all controllers which stabilize P. This problem falls into the $\mathcal{H}_\infty$ optimal control problem which has been extensively studied in the recent control literature. The solution for general $\mathcal{H}_\infty$ optimal control problem can be solved by using iterative procedures. However, the particular problem we have here has been solved in [11] and [8] without iteration.

7 Multiplication Operators, Compactification of $\mathbb{C}^{p\times m}$, and Pointwise Gap Metrics

In this section, we give connections of pointwise gap metrics to the concepts of the multiplication operators from $\mathcal{H}_2^m$ to $\mathcal{H}_2^p$ and compactifications of $\mathbb{C}^{p\times m}$. Details on the concepts of unbounded operators and compactifications can be found in [15] and [18] respectively.

We denote by $\mathcal{H}_2^p$ the Hardy space of all functions u which are analytic in $\mathbb{C}^+$, take values in $\mathbb{C}^p$, and satisfy the uniform square-integrability condition:

$$\left[\sup_{\xi > 0} (2\pi)^{-1} \int_{-\infty}^{\infty} \|u(\xi + j\omega)\|_2^2 d\omega \right]^{\frac{1}{2}} < \infty.$$

For any $F \in \mathcal{P}^{r\times m}$, the *multiplication operator* from $\mathcal{H}_2^m$ to $\mathcal{H}_2^p$ due to F, denoted by M_F, is defined to be the possibly unbounded operator which maps $u \in \mathcal{H}_2^m$ to Fu if Fu is in $\mathcal{H}_2^p$. The operator M_F is unbounded if F is unstable since not every Fu is in $\mathcal{H}_2^p$ in this case. The domain, the range and the graph of M_F are defined as[1]

$$\begin{aligned} \mathcal{D}(F) &= \{u \in \mathcal{H}_2^m : Fu \in \mathcal{H}_2^p\} \\ \mathcal{R}(F) &= \{Fu : u \in \mathcal{D}(F)\} \\ \mathcal{G}(F) &= \left\{ \begin{bmatrix} u \\ Fu \end{bmatrix} : u \in \mathcal{D}(F) \right\}. \end{aligned}$$

It is clear that $\mathcal{D}(F)$, $\mathcal{R}(F)$ and $\mathcal{G}(F)$ are linear manifolds in $\mathcal{H}_2^m$, $\mathcal{H}_2^p$ and $\mathcal{H}_2^m \times \mathcal{H}_2^p$ respectively.

The graph of M_F can be characterized by right-coprime factorizations of F.

Lemma 2 [19] *Let $F \in \mathcal{P}^{p\times m}$ and NM^{-1} be any right coprime factorization of F. Then*

$$\mathcal{G}(F) = \left\{ \begin{bmatrix} M \\ N \end{bmatrix} u : u \in \mathcal{H}_2^m \right\}.$$

[1]The notation is slightly abused here; $\mathcal{D}(M_F)$, $\mathcal{R}(M_F)$ and $\mathcal{G}(M_F)$ are abbreviated as $\mathcal{D}(F)$, $\mathcal{R}(F)$ and $\mathcal{G}(F)$ respectively.

At any fixed $s \in \mathbb{C}^+$, the *evaluation operator* on $\mathcal{H}_2^p \times \mathcal{H}_2^m$ maps $v \in \mathcal{H}_2^p \times \mathcal{H}_2^m$ to $v(s) \in \mathbb{C}^{p+m}$. Since $\mathcal{G}(F)$ is a linear manifold in $\mathcal{H}_2^p \times \mathcal{H}_2^m$, the image of $\mathcal{G}(F)$ under the evaluation operator at any fixed $s \in \mathbb{C}^+$ must be a subspace of $\mathbb{C}^{p+m}$. By Lemma 2, this subspace is just $\mathcal{G}(F, s)$ defined in Section 3. Therefore, a pointwise gap metric between F_1 and F_2 in $\mathcal{P}^{p\times m}$ can be interpreted using the graphs of the multiplication operators M_{F_1} and M_{F_2}. It is the supremum over all s in $\mathbb{C}^+$ of a gap between the images of the graphs of M_{F_1} and M_{F_2} under the evaluation operator at s.

It has been seen from Section 4 that the value of a real rational function at each $s \in \mathbb{C}^+$ is an element of $\mathbb{C} \cup \{\infty\}$, and the pointwise gap metric between two real rational functions is simply the supremum over all $s \in \mathbb{C}^+$ of the chordal metric between the values of these two functions. It is known that $\mathbb{C} \cup \{\infty\}$ with the chordal metric is a one-point compactification of $\mathbb{C}$. The idea for the scalar case can be generalized to the matrix case. It has been shown in Section 2 that the set of all l-dimensional subspaces in $\mathbb{C}^n$, denoted by $\Gamma_l(\mathbb{C}^n)$, is a compact metric space if its topology is induced by a gap.

Proposition 11 *$\Gamma_m(\mathbb{C}^{p+m})$ is a compactification of $\mathbb{C}^{p\times m}$.*

Let $F \in \mathcal{P}^{p\times m}$ and NM^{-1} be a right-coprime factorization of F. Then for each $s \in \mathbb{C}^+$, F uniquely determines an m-dimensional subspace of $\mathbb{C}^{p+m}$ which is given by $\mathcal{R}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right)$. In other words, the value of $F(s)$ at each $s \in \mathbb{C}^+$ can be considered as an element in the compactification of $\mathbb{C}^{p\times m}$. A pointwise gap metric between F_1 and F_2 is simply the supremum over all $s \in \mathbb{C}^+$ of the distance between $F_1(s)$ and $F_2(s)$ measured by a metric in the compactification of $\mathbb{C}^{p\times m}$.

8 Comparison to the Gap Metric

A metric in the space of real rational matrices which has undergone extensive studies in recent years is the gap metric. The gap metric on real rational matrices is inherited from the gap metric on closed unbounded operators between Banach spaces which has been thoroughly studied in [15], and was introduced to control theory, particularly to the stability robustness study, in [20]. Significant contributions to the gap metric study (in control) include [4], [22], [7], [8]. The following remarks concern the comparison between the gap metric and pointwise gap metrics; for details, see [16].

(i) The gap metric and pointwise gap metrics are different and are not uniformly equivalent, but they are topologically equivalent. A natural consequence of this fact is that they have many similar qualitative properties; surprisingly, they also have many similar quantitative properties in spite of the fact that they differ in value. Theorem 3 can be restated for the gap metric; a result which is slightly weaker than Theorem 4 for the gap metric is proved in [8].

(ii) The gap metric can be computed by using a formula given in [7]. A good computational method for pointwise gap metrics is not yet available.

(iii) The gap metric relies on infinite dimensional functional analysis, whereas the pointwise gap metrics are built from finite dimensional linear algebra. This implies that pointwise gap metrics are conceptually simpler than the gap metric. The study of pointwise gap metrics seems more straightforward in some cases due to their conceptual simplicity.

(iv) Pointwise gap metrics are transpose invariant, i.e. $\delta(F_1, F_2) = \delta(F_1', F_2')$, whereas the gap metric does not have this property.

(v) A pointwise gap metric can be interpreted as being the supremum over $s \in \mathbb{C}^+$ of the distance between the values of real rational matrices, where the values are regarded to be in a compactification of the set of complex matrices, and the distance is given by a metric in this compactification. This interpretation is potentially useful in applications since it may provide an access to the relationship between the metric and the physical parameters of the systems described by the real rational matrices. However, it appears that what the gap metric measures is so intrinsic that very little intuitive sense is provided.

The graph metric is another metric which can be used in the stability robustness study of real rational matrices. It can be shown that the graph metric and pointwise gap metrics are different, are not uniformly equivalent, but are topologically equivalent. However, it appears that the graph metric is not so convenient to use because it lacks the good quantitative properties which the gap metric and pointwise gap metrics have.

9 Conclusion

A new family of metrics, called pointwise gap metrics, has been developed in this paper and the open and closed loop stability robustness of lumped linear time invariant finite dimensional continuous time systems has been successfully analyzed using these metrics. If we define the notion of stability of real rational matrices to be based on the stability of discrete time systems, then virtually all the results of this paper apply with only trivial changes.

A problem which is not solved satisfactorily in this paper is in the computation of pointwise gap metrics. A partial list of other interesting related problems is given in the following.

(i) Model reduction in terms of the pointwise gap metrics. Formally, this problem is equivalent to minimization problem:

$$\inf_{P_n} \delta(P, P_n)$$

where P_n runs over the set of all real rational matrices with order bounded by n. Cf. model reduction using Hankel norm [10].

(ii) Weighted pointwise gap metrics. This problem arises if we define a metric by

$$\delta_{W_1,W_2}(F_1, F_2) = \delta(W_1F_1W_2, W_1F_2W_2)$$

where W_1 and W_2 are unimodular real rational matrices. The introduction of the weighting matrices enables us to exploit some structural information on the system uncertainty. Cf. weighted norm [21] and weighted gap metric [9].

(iii) Structured pointwise gap metrics. When additional structure of the uncertainty is available, one might consider a complete structured pointwise gap metric analysis. Let $\mathcal{M}$ be the space of all P with the following form

$$P = \mathrm{diag}(P_1, P_2, \ldots, P_l)$$

where P_i, $i = 1, 2, \ldots, l$, are matrices over $\mathcal{P}$ with various sizes. Define a metric in $\mathcal{M}$ as

$$d(P, Q) = \max\{\delta(P_1, Q_1), \delta(P_2, Q_2), \ldots, \delta(P_l, Q_l)\}.$$

Let C_0 be a controller which stabilizes P_0 in such a space. Find

$$\inf\{d(P, P_0) : P \in \mathcal{M} \text{ and } \mathbf{H}(P, C_0) \text{ is stable}\}.$$

This is an analogue to the μ-problem. See [3], in which the diagonal structure of P is justified.

(iv) Pointwise gap metrics for distributed or time-varying systems, etc. In light of Section 7, pointwise gap metrics can be defined for unbounded linear operators from a function space to another function space with the same variable. Thus in principle, many of the results of this paper are extendible to distributed systems, time-varying systems, etc; however such an extension is not trivial because two major properties of real rational functions, i.e. the existence of coprime factorizations and the existence of stabilizing controllers, need not necessarily hold. Cf. the gap metric for distributed systems [22], [8].

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References

[1] Å. Björck and G.H. Golub, "Numerical method for computing angles between linear subspaces", *Math. of Computation*, vol. 27, pp. 579-593, 1973.

[2] C. Davis and W. M. Kahan, "The rotation of eigenvectors by a perturbation. III", *SIAM J. Numer. Anal.*, vol. 7, pp. 1-46, 1970.

[3] J.C. Doyle, "Analysis of feedback systems with structured uncertainties", *Proc. IEE, Pt. D.*, vol. 129, pp. 242-250, 1982.

[4] A.K. El-Sakkary, "The gap metric: robustness of stabilization of feedback systems", *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 240-247, 1985.

[5] A.K. El-Sakkary, "Estimating robustness on the Riemann sphere", *Int. J. Control*, vol. 49, pp. 561-567, 1989.

[6] B.A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory*, New York: Springer-Verlag, 1986.

[7] T.T. Georgiou, "On the computation of the gap metric", *Systems & Control Letters*, vol. 11, pp. 253-257, 1988.

[8] T.T. Georgiou and M.C. Smith, "Optimal robustness in the gap metric", *IEEE Trans. Automat. Contr.*, vol. AC-35, pp. 673-685, 1990.

[9] T.T. Georgiou and M.C. Smith, "Robust control of feedback systems with combined plant and controller uncertainty", *Proc. American Contr. Conf.*, pp. 2009-2013, 1990.

[10] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their L^∞-error bounds", pp. 1115-1193, *Int. J. Control*, vol. 39, pp. 1115-1193, 1984.

[11] K. Glover and D.C. McFarlane, "Robust stabilization of normalized coprime factor plant description with $\mathcal{H}_\infty$-bounded uncertainty", *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 821-830, 1989.

[12] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, New York: Academic Press, 1982.

[13] E. Hille, *Analytic Function Theory, Vol. 1*, New York: Blaisdell Publishing Company, 1959.

[14] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge, England: Cambridge University Press, 1985.

[15] T. Kato, *Perturbation Theory for Linear Operator*, New York: Springer-Verlag, 1966.

[16] L. Qiu, "Tools for quantitative stability robustness analysis", Ph.D thesis, University of Toronto, 1990.

[17] S. Rolewicz, *Metric Linear Spaces*, Dordrecht, Holland: D. Reidel Publishing Company and Warszawa, Poland: PWN-Polish Scientific Publishers, 1985.

[18] H.L. Royden, *Real Analysis*, New York: MacMillan Publishing Co., Inc., 1968.

[19] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, Cambridge, Massachusetts: The MIT Press, 1985.

[20] G. Zames and A.K. El-Sakkary, "Unstable systems and feedback: the gap metric", *Proc. 16th Allerton Conf.*, pp. 380-385, 1980.

[21] G. Zames, "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms, and approximate inverses", *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 301-320, 1981.

[22] S.Q. Zhu, "Graph topology and gap topology for unstable systems", *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 848-855, 1989.