Pointwise Gap Metrics on Transfer Matrices

Li Qiu, *Member, IEEE,* and Edward J. Davison, *Fellow, IEEE*

***Abstract*—A new family of metrics, called pointwise gap metrics, in the space of real rational matrices of fixed size is developed in this paper. These metrics are then used to study open-loop and closed-loop stability robustness of linear time-invariant finite-dimensional continuous-time systems. It is shown that pointwise gap metrics have the desired qualitative properties for the study of stability robustness. Necessary and sufficient conditions on the open- and closed-loop stability robustness are obtained in terms of the radii of the pointwise gap metric balls centered at the nominal plant and/or the nominal controller. Comparison of the new metric with the available metrics, e.g., the gap metric and the graph metric, is made. All of these metrics induce the same topology. Surprisingly, it is shown that many of the quantitative properties of pointwise gap metrics are the same as those of the gap metric, although they differ in value. A notable distinct property of pointwise gap metrics is that in the scalar case, they have a very simple expression which is potentially useful to access the relationship between the uncertainty of physical parameters and uncertainty measured by pointwise gap metrics.**

I. Introduction

The study of the stability robustness problem concerns the stability of uncertain systems. If such a study is to be carried out quantitatively, it is necessary to have a mechanism to measure the size of the uncertainty. A metric in the space of the systems under consideration can provide such a mechanism. In this paper, we consider the stability robustness of real rational matrices and assume that these matrices are transfer matrices of linear time-invariant finite-dimensional continuous-time systems. Hence, a real rational matrix is said to be stable if it is bounded in $\mathbb{C}^+$, where $\mathbb{C}^+ = \{s \in \mathbb{C}: \mathscr{R}(s) > 0\}$. The purpose of this paper is to develop a metric in the space of all real rational matrices of certain size which facilitates the study of stability robustness. Before proceeding, we have to know what are the requirements for such a metric.

Let $\mathscr{P}$ be the field of all real rational functions and let $\mathscr{S}$ be the ring of all stable real rational functions. Then $\mathscr{P}$ can be considered as the quotient field of $\mathscr{S}$. We denote naturally by $\mathscr{P}^{p \times m}$ and $\mathscr{S}^{p \times m}$ the sets of all $p \times m$ matrices over (field) $\mathscr{P}$ and (ring) $\mathscr{S}$, respectively.

Each element of $\mathscr{P}^{p \times m}$ corresponds to an open-loop system. If $F_0 \in \mathscr{S}^{p \times m}$, i.e., F_0 is stable, a desired property of the metric to be developed is that F_0 has certain stability robustness under the metric, which means that any uncertain system F which is close to F_0 should be stable. Formally, this requires that $\mathscr{S}^{p \times m}$ be an open subset of $\mathscr{P}^{p \times m}$.

To study closed-loop stability robustness, consider the standard feedback configuration shown in Fig. 1. In Fig. 1, $P \in \mathscr{P}^{p \times m}$ represents the plant and $C \in \mathscr{P}^{m \times p}$ the controller. By the closed-loop transfer matrix of the feedback configuration, we mean the transfer matrix from $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ to $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ which is given by

$$H(P, C) = \begin{bmatrix} I & -C \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - CP)^{-1} & C(I - PC)^{-1} \\ P(I - CP)^{-1} & (I - PC)^{-1} \end{bmatrix}. \quad (1)$$

In order for $H(P, C)$ to exist, $I - CP$ and $I - PC$ have to be invertible.[1] If this is the case, we say that the closed-loop system, or simply the pair (P, C), is well posed. If we denote the set of all well-posed pairs, which is a subset of $\mathscr{P}^{p \times m} \times \mathscr{P}^{m \times p}$, by $\mathscr{W}(p, m)$, then H defines a function from $\mathscr{W}(p, m)$ to $\mathscr{P}^{(p+m) \times (p+m)}$. This function is injective since

$$\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \left[I - (H(P, C))^{-1} \right].$$

Let us denote the range of $H(P, C)$, which is a subset of $\mathscr{P}^{(p+m) \times (p+m)}$, by $\mathscr{C}(p, m)$. Then H is a bijective function from $\mathscr{W}(p, m)$ to $\mathscr{C}(p, m)$. A desired property of the metric to be developed for closed-loop stability robustness analysis is that if $(P_0, C_0) \in \mathscr{W}(p, m)$ and (P, C) is close to (P_0, C_0), then $(P, C) \in \mathscr{W}(p, m)$ and $H(P, C)$ is close to $H(P_0, C_0)$. Formally, this is equivalent to the requirement that $\mathscr{W}(p, m)$ be an open subset of $\mathscr{P}^{p \times m} \times \mathscr{P}^{m \times p}$ and that the function H from $\mathscr{W}(p, m)$ to $\mathscr{C}(p, m)$ be continuous.

The closed-loop system given by the feedback configuration is said to be stable if the closed-loop transfer matrix $H(P, C)$ is stable. In this case, we say that C stabilizes P. The open-loop and closed-loop requirements together now imply that if (P_0, C_0) is well posed and $H(P_0, C_0)$ is stable, then (P, C) is well posed and $H(P, C)$ is stable for (P, C) close to (P_0, C_0).

The requirements given above are qualitative in nature since only the existence of stability robustness is asked; these requirements only depend on the topology induced by the metric, but not on the quantitative values of the metric. However, such existence arguments are usually not enough. We also want to use the metric to determine how much

[1] In fact, the invertibility of $I - CP$ and that of $I - PC$ are equivalent.

Manuscript received November 13, 1990; revised October 22, 1991. Paper recommended by Associate Editor, T. T. Georgiou. This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grant A4396.

L. Qiu was with the Department of Electrical Engineering, University of Toronto, Toronto, Ontario, Canada M5S 1A4. He is now with the Fields Institute for Research in Mathematical Sciences, Waterloo, Ontario, Canada N2L 525.

E. J. Davison is with the Department of Electrical Engineering, University of Toronto, Toronto, Ontario, Canada M5S 1A4.

IEEE Log Number 9108280.

0018-9286/92$03.00 © 1992 IEEE

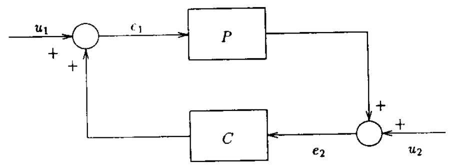


Fig. 1. The standard feedback system.

uncertainty can be tolerated on a stable transfer matrix F in order to maintain stability, and how much uncertainty can be tolerated on a pair $(P, C) \in \mathscr{W}(r, m)$ with stable $H(P, C)$ such that the closed-loop system is still stable. This requires that the metric developed should have certain desirable quantitative properties so that the questions above can be easily answered in terms of the size of uncertainty measured by the metric.

Two metrics serving our purpose have been developed in the last decade. One is the gap metric introduced by Zames and El-Sakkary [23] and the second one is the graph metric defined by Vidyasagar [21]. Both metrics induce the same topology in $\mathscr{P}^{p\times m}$ and this topology satisfies our requirements. However, they possess different quantitative properties. A computable formula of the gap metric is given by Georgiou [7]; a simple necessary and sufficient condition is derived by Georgiou and Smith [8] for the closed-loop stability robustness in terms of the radius of the gap metric ball centered at the nominal plant or the nominal controller; a necessary and sufficient condition is obtained by Qiu and Davison [16] for the closed-loop stability robustness when both the plant and the controller are subject to independent gap metric uncertainties; and a procedure is given in [8] to design a robust controller which maximizes the closed-loop stability robustness with respect to plant uncertainty. In contrast, the graph metric is difficult to compute, the quantitative conditions for the open-loop and closed-loop stability robustness in terms of the graph metric are conservative, and it is not clear how to incorporate it in the design of robust control systems.

For the stability robustness analysis of SISO systems, El-Sakkary [4] defines a metric in $\mathscr{P}$ which is based on the chord distance of the values of real rational functions on the Riemann sphere. This metric induces the same topology in $\mathscr{P}$ as the gap metric and the graph metric. The advantage of this metric is that it has a simple expression in terms of the values of real rational functions which may provide an access to the relationship between the uncertainty of physical parameters and the uncertainty measured by the metric.

In this paper, we develop a new family of metrics, called pointwise gap metrics. It is shown that pointwise gap metrics induce the same topology as the gap metric and the graph metric, but differ from them in value. Pointwise gap metrics are in principle computable directly from their definition. In the scalar case, this family of metrics degenerates to a single metric and this metric turns out to be exactly the same as the metric defined in [4]. Necessary and sufficient conditions on the open-loop and closed-loop stability robustness are obtained in terms of the radii of the pointwise gap metric balls centered at the nominal plant and/or the nominal controller. Surprisingly, the conditions are identical to the ones obtained for the gap metric [8], [16]. From this we can conclude that a closed-loop system is robust with respect to the gap metric if and only if it is robust with respect to any of the pointwise gap metrics. The same procedure in [8] can be used to design a controller with optimal closed-loop stability robustness measured by pointwise gap metrics.

Throughout this paper, we always assume that the singular values of a matrix $A \in \mathbb{C}^{p\times m}$ are ordered nonincreasingly and we denote the ith singular value of A by $\sigma_i(A)$, $i = 1, 2, \cdots, \min\{p, m\}$. In particular, $\sigma_1(A)$ is also denoted by $\bar{\sigma}(A)$ and $\sigma_{\min\{p,m\}}$ by $\underline{\sigma}(A)$. The matrix norms used in this paper, denoted by $\|\cdot\|$, belong to the family of unitarily invariant matrix norms, i.e., matrix norms satisfying:

a) $\|UAV\| = \|A\|$ for all $A \in \mathbb{C}^{p\times m}$ and all compatible unitary matrices U, V; and
b) $\|uv^*\| = \|u\|_2\|v\|_2$ for all $u \in \mathbb{C}^p$ and $v \in \mathbb{C}^m$.

For an introduction to unitarily invariant matrix norms, see [12]. Frequently used examples of unitarily invariant matrix norms are the spectral norm $\|A\|_s = \sigma_1(A)$, Frobenius norm $\|A\|_F = [\sum_{i=1}^{\min\{p,m\}} \sigma_i^2(A)]^{\frac{1}{2}}$, and the trace norm $\|A\|_t = \sum_{i=1}^{\min\{p,m\}} \sigma_i(A)$. A clear distinction has to be made in this paper between a real rational matrix and its value at a certain point in the complex plane. If F denotes a $p \times m$ real rational matrix, then $F(s)$, where $s \in \mathbb{C}$ is not a pole of F, is considered to be a matrix in $\mathbb{C}^{p\times m}$ and is just equal to the value of F at s. Thus, $F^*(s)$ means a matrix in $\mathbb{C}^{m\times p}$ which is the conjugate transpose of $F(s)$ and is equal to $F'(s^*)$. The norm on stable real rational matrices corresponding to an arbitrary unitarily invariant norm $\|\cdot\|$, also denoted by $\|\cdot\|$, is defined by

$$\|F\| = \sup_{s\in\mathbb{C}^+} \|F(s)\|.$$

In particular, the norm of F corresponding to the spectral norm is denoted by $\|F\|_s$. Since all unitarily invariant norms are equivalent, so are all norms on real rational matrices. A useful fact, as a consequence of the maximum modulus principle, is that $\sup_{s\in\mathbb{C}^+}\|F(s)\| = \sup_{\omega\in\mathbb{R}}\|F(j\omega)\|$ for a stable real rational matrix F.

II. Canonical Angles and Gaps

We introduce, in this section, the concepts of canonical angles and gaps between subspaces of the unitary space $\mathbb{C}^n$. The origins of these concepts are hard to trace, but a good set of references is given in [2]. See also [19] for a recent exposition. A detailed treatment of these concepts is also given in [15], with an emphasis in their application to the study of pointwise gap metrics.

Let us denote by $\Gamma_l(\mathbb{C}^n)$ the set of l-dimensional subspaces of $\mathbb{C}^n$. Let $\mathscr{X}$, $\mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$ and let X_1 and Y_1 be matrices with $X_1^*X_1 = I$, $Y_1^*Y_1 = I$, $\mathscr{R}(X_1) = \mathscr{X}$, and $\mathscr{R}(Y_1) = \mathscr{Y}$, where $\mathscr{R}(\cdot)$ means the range. Since $\|Y_1^*X_1\|_s \le 1$, all singular values of $Y_1^*X_1$ are bounded by 1. Hence, there exists an angle θ_i in $[0, (\pi/2)]$ such that $\cos\theta_i = \sigma_{l-i+1}(Y_1^* X_1)$ for $i = 1, 2, \cdots, l$. Furthermore, the singular values of $Y_1^* X_1$ are invariant to the choice of X_1 and Y_1 and are determined by $\mathscr{X}$ and $\mathscr{Y}$ completely. This leads to our definition: the angles $\{\theta_i, i = 1, 2, \cdots, l\}$ are called the *canonical angles* between $\mathscr{X}$ and $\mathscr{Y}$.

The canonical angles are always ordered nonincreasingly. The ith canonical angle between $\mathscr{X}$ and $\mathscr{Y}$ is denoted by $\theta_i(\mathscr{X}, \mathscr{Y})$.

Proposition 1 [1]: Let $\mathscr{X}, \mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$. Then there exist unitary matrices $[X = X_1 \ X_2]$, $Y = [Y_1 \ Y_2]$ such that $\mathscr{X} = \mathscr{R}(X_1)$, $\mathscr{Y} = \mathscr{R}(Y_1)$, and

$$Y^* X = \begin{bmatrix} C_1 & -S' \\ S & C_2 \end{bmatrix}$$

where

$$C_1 = \operatorname{diag}(c_1, c_2, \cdots, c_l)$$
$$C_2 = \operatorname{diag}(c_1, c_2, \ldots, c_{n-l})$$
$$S = \operatorname{diag}(s_1, s_2, \cdots, s_{\min\{l, n-l\}})$$

and $0 \le c_1 \le c_2 \le \cdots \le c_{\max\{l, n-l\}}$, $0 \le s_{\min\{l, n-l\}} \le \cdots \le s_2 \le s_1$.

Since $Y^* X$ is unitary,

$$C_1' C_1 + S' S = I \text{ and } C_2 C_2' + S S' = I.$$

The definition of s_i for $i = 1, 2, \cdots, \min\{l, n-l\}$ can be extended to $i = 1, 2, \cdots, \max\{l, n-l\}$ to achieve a symmetry in c_i and s_i. Let $S_1 = (S'S)^{\frac{1}{2}}$ and $S_2 = (SS')^{\frac{1}{2}}$. Then $S_1 = \operatorname{diag}(s_1, s_2, \cdots, s_l)$ and $S_2 = \operatorname{diag}(s_1, s_2, \cdots, s_{n-l})$, where $s_1, s_2, \cdots, s_{\min\{l, n-l\}}$ are defined in Proposition 1 and $s_{\min\{l, n-l\}+1}, \cdots, s_{\max\{l, n-l\}}$ are zeros. Consequently, we have

$$C_1^2 + S_1^2 = I \text{ and } C_2^2 + S_2^2 = I.$$

Let θ_i be the ith canonical angle between $\mathscr{X}$ and $\mathscr{Y}$ for $i = 1, 2, \cdots, l$. Since $c_i = \cos\theta_i$ by definition, it follows that $s_i = \sin\theta_i$. It is clear that there are at most $\min\{l, n - l\}$ nonzero canonical angles between $\mathscr{X}$ and $\mathscr{Y}$.

The following two facts follow from Proposition 1 easily.

Proposition 2: Let $\mathscr{X}, \mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$. Suppose A is a unitary linear transformation on $\mathbb{C}^n$. Then

$$\theta_i(A\mathscr{X}, A\mathscr{Y}) = \theta_i(\mathscr{X}, \mathscr{Y})$$

for $i = 1, 2, \cdots, l$.

Proposition 3: Let $\mathscr{X}, \mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$. Then

$$\theta_i(\mathscr{X}^\perp, \mathscr{Y}^\perp) = \theta_i(\mathscr{X}, \mathscr{Y})$$

for $i = 1, 2, \cdots, \min\{l, n-l\}$.

Let $\|\cdot\|$ be a unitarily invariant matrix norm. Let θ_i, $i = 1, 2, \cdots, l$, be the ith canonical angles between subspaces $\mathscr{X}$ and $\mathscr{Y}$. The *gap* between $\mathscr{X}$ and $\mathscr{Y}$ corresponding to norm $\|\cdot\|$ is defined as

$$\gamma(\mathscr{X}, \mathscr{Y}) = \|\operatorname{diag}(\sin\theta_1, \sin\theta_2, \cdots, \sin\theta_l)\|.$$

Apparently, a gap γ depends on the underlying norm $\|\cdot\|$. If $\|\cdot\|$ is the spectral norm $\|\cdot\|_s$, then the corresponding gap is denoted by γ_s. The value of γ_s is always in $[0, 1]$. The following result concerns the computation of gaps.

Proposition 4: Let $\mathscr{X}, \mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$.

a) Suppose $X = [X_1 X_2]$, $Y = [Y_1 Y_2]$ are unitary matrices with $\mathscr{X} = \mathscr{R}(X_1)$, $\mathscr{Y} = \mathscr{R}(Y_1)$. Then

$$\gamma(\mathscr{X}, \mathscr{Y}) = \|Y_1^* X_2\| = \|Y_2^* X_1\|.$$

In particular,

$$\gamma_s(\mathscr{X}, \mathscr{Y}) = \bar{\sigma}(Y_1^* X_2) = \bar{\sigma}(Y_2^* X_1) = \sqrt{1 - \underline{\sigma}^2(Y_1^* X_1)} = \sqrt{1 - \underline{\sigma}^2(Y_2^* X_2)}.$$

b) $\gamma(\mathscr{X}^\perp, \mathscr{Y}^\perp) = \gamma(\mathscr{X}, \mathscr{Y})$.

c) Suppose A is a unitary linear transformation. Then

$$\gamma(A\mathscr{X}, A\mathscr{Y}) = \gamma(\mathscr{X}, \mathscr{Y}).$$

In our development, however, we need to relate $\gamma(A\mathscr{X}, A\mathscr{Y})$ to $\gamma(\mathscr{X}, \mathscr{Y})$ when A is merely a nonsingular linear transformation on $\mathbb{C}^n$. This will be done in Appendix A.

A gap is a function from $\Gamma_l(\mathbb{C}^n) \times \Gamma_l(\mathbb{C}^n)$ to $[0, \infty)$.

Proposition 5: γ is a metric on $\Gamma_l(\mathbb{C}^n)$ and the metric space $(\Gamma_l(\mathbb{C}^n), \gamma)$ is compact.

For a proof of Proposition 5, see [19] or [10]. A different proof is given in [15]. Since all unitarily invariant matrix norms are equivalent, all of the gaps defined above are uniformly equivalent metrics (see [18] for definitions). One of the implications that γ is a metric is that γ satisfies the triangular inequality, i.e., for $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \Gamma_l(\mathbb{C}^n)$

$$\gamma(\mathscr{X}, \mathscr{Z}) \le \gamma(\mathscr{X}, \mathscr{Y}) + \gamma(\mathscr{Y}, \mathscr{Z}). \quad (2)$$

However, this inequality is not tight enough for our application; it is a strict inequality unless $\mathscr{X} = \mathscr{Y}$ or $\mathscr{Y} = \mathscr{Z}$. Inequality (2) will be improved in Appendix B, leading to the proof of our main result.

III. Definition of Pointwise Gap Metrics

It is well known that every real rational matrix has both a right-coprime factorization and a left-coprime factorization over the ring of all stable real rational functions. This means that for each real rational matrix F, there exist stable real rational matrices M, N, U, V, and $\tilde{M}, \tilde{N}, \tilde{U}, \tilde{V}$ such that $F = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $\tilde{U}M + \tilde{V}N = I$, $\tilde{M}U + \tilde{N}V = I$. Right-coprime factorizations of a real rational matrix are not unique, but the set of all right-coprime factorizations can be parameterized by a free unimodular matrix. A stable square real rational matrix is said to be *unimodular* if its inverse exists and is stable. Let NM^{-1} be a right-coprime factorization of $F \in \mathscr{P}^{p\times m}$. Then the set of all right-coprime factorizations of F is given by $\{(ND)(MD)^{-1}: D \in \mathscr{S}^{m\times m}$ is unimodular$\}$. A similar parameterization exists for left-coprime factorizations. We refer to [21] and [6] for detailed exposition of coprime factorizations.

Let NM^{-1} be a right-coprime factorization of $F \in \mathscr{P}^{p\times m}$. Although such a factorization is not unique, the parameterization of all right-coprime factorizations given in the last paragraph shows that the subspace $\mathscr{R}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right) \in \mathbb{C}^{p+m}$ at each $s \in \mathbb{C}^+$ is uniquely determined by F. Denote this subspace by $\mathcal{G}(F, s)$. The right-coprimeness of N and M implies that the dimension of $\mathcal{G}(F, s)$ is m for all $s \in \mathbb{C}^+$.

Let γ be the gap between the m-dimensional subspaces of $\mathbb{C}^{p+m}$ corresponding to a unitarily invariant norm $\|\cdot\|$. Define a function δ: $\mathcal{P}^{p\times m} \times \mathcal{P}^{p\times m} \to [0, \infty)$ by

$$\delta(F_1, F_2) = \sup_{s\in\mathbb{C}^+} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)]. \quad (3)$$

Proposition 6: δ is a metric on $\mathcal{P}^{p\times m}$.

Proof: We only need to prove the triangular inequality. For $F_1, F_2, F_3 \in \mathcal{P}^{p\times m}$

$$\begin{aligned}\delta(F_1, F_2) + \delta(F_2, F_3) &= \sup_{s\in\mathbb{C}^+} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] \\ &\quad + \sup_{s\in\mathbb{C}^+} \gamma[\mathcal{G}(F_2, s), \mathcal{G}(F_3, s)] \\ &\geq \sup_{s\in\mathbb{C}^+} \{\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] \\ &\quad + \gamma[\mathcal{G}(F_2, s), \mathcal{G}(F_3, s)]\} \\ &\geq \sup_{s\in\mathbb{C}^+} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_3, s)] \\ &= \delta(F_1, F_3).\end{aligned}$$

This completes the proof. □

We call δ a *pointwise gap metric* on $\mathcal{P}^{p\times m}$ corresponding to the unitary invariant norm $\|\cdot\|$. If the underlying norm used to define the pointwise gap metric is $\|\cdot\|_s$, then the corresponding pointwise gap metric is written as δ_s. The space $\mathcal{P}^{p\times m}$ equipped with a pointwise gap metric forms a metric space. Since all of the gaps used are uniformly equivalent, all pointwise gap metrics are uniformly equivalent.

Let NM^{-1} be a right-coprime factorization of F and $\tilde{M}^{-1}\tilde{N}$ be a left-coprime factorization of F. Then an orthonormal basis of $\mathcal{G}(F, s)$ is given by the columns of

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} [M^*(s)M(s) + N^*(s)N(s)]^{-\frac{1}{2}}.$$

Since $-\tilde{N}M + \tilde{M}N = 0$, an orthonormal basis of the orthogonal complement of $\mathcal{G}(F, s)$ is given by the columns of

$$\begin{bmatrix} -\tilde{N}^*(s) \\ \tilde{M}^*(s) \end{bmatrix} [\tilde{M}(s)\tilde{M}^*(s) + \tilde{N}(s)\tilde{N}^*(s)]^{-\frac{1}{2}}.$$

It follows from Proposition 4a) that for $F_1, F_2 \in \mathcal{P}^{p\times m}$

$$\begin{aligned}\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] &= \Big\| [\tilde{M}_2(s)\tilde{M}_2^*(s) + \tilde{N}_2(s)\tilde{N}_2^*(s)]^{-\frac{1}{2}} \\ &\quad \cdot [-\tilde{N}_2(s)\tilde{M}_2(s)] \begin{bmatrix} M_1(s) \\ N_1(s) \end{bmatrix} \\ &\quad \cdot [M_1^*(s)M_1(s) + N_1^*(s)N_1(s)]^{-\frac{1}{2}} \Big\| \quad (4)\end{aligned}$$

where $N_1M_1^{-1}$ is a right-coprime factorization of F_1 and $\tilde{M}_2^{-1}\tilde{N}_2$ is a left-coprime factorization of F_2. Then $\delta(F_1, F_2)$ can be obtained by taking the supremum over $\mathbb{C}^+$. One might wonder if an analog of the maximum modulus principle exists in this case, i.e., if the supremum over $\mathbb{C}^+$ can be replaced by a supremum over $\mathbb{C}^0$, where $\mathbb{C}^0 = \{j\omega: \omega \in \mathbb{R}\}$. Unfortunately, an example given in the next section will rule out such an expectation. The computation of pointwise gap metrics, therefore, becomes a nonconcave maximization problem over a two-dimensional domain which in principle can be solved by a two-dimensional brute force search technique. It is shown in [15] that this maximization problem can actually be solved by carrying out a one-dimensional brute force search together with a one-dimensional bisection search. A more efficient method to compute pointwise gap metrics is yet to be found.

It is easy to see from (4) that $\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)]$ is a continuous function of s in $\mathbb{C}^+$. Moreover, the right-hand side of (4) is a continuous function of s in the compact set $\mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$, although $\mathcal{G}(F, s)$ is not initially defined for s in $\mathbb{C}^0 \cup \{\infty\}$. For the sake of convenience, we sometimes need to extend the definition of $\mathcal{G}(F, s)$ to all s in the set $\mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$ by letting $\mathcal{G}(F, s) = \mathcal{R}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right)$, where NM^{-1} is a right-coprime factorization of F. Under this extended definition, it is easy to see that $\sup_{s\in\mathbb{C}^+} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] = \max_{s\in\mathbb{C}^0\cup\mathbb{C}^+\cup\{\infty\}} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)]$.

IV. The Pointwise Gap Metric in the Scalar Case

Suppose that F_i is a scalar real rational function and $N_iM_i^{-1}$ is a (right- or left-) coprime factorization for $i = 1, 2$. Then by (4),

$$\begin{aligned}\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] &= \left| [|M_2(s)|^2 + |N_2(s)|^2]^{-\frac{1}{2}} [-N_2(s)\ M_2(s)] \begin{bmatrix} M_1(s) \\ N_1(s) \end{bmatrix} [|M_1(s)|^2 + |N_1(s)|^2]^{-\frac{1}{2}} \right| \\ &= \frac{|M_2(s)N_1(s) - N_2(s)M_1(s)|}{\sqrt{|M_1(s)|^2 + |N_1(s)|^2}\sqrt{|M_2(s)|^2 + |N_2(s)|^2}}.\end{aligned}$$

If we use the arithmetic in $\mathbb{C} \cup \{\infty\}$ instead of $\mathbb{C}$, we have

$$\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] = \frac{|N_1(s)M_1^{-1}(s) - N_2(s)M_2^{-1}(s)|}{\sqrt{1 + |N_1(s)M_1^{-1}(s)|^2}\sqrt{1 + |N_2(s)M_2^{-1}(s)|^2}} = \frac{|F_1(s) - F_2(s)|}{\sqrt{1 + |F_1(s)|^2}\sqrt{1 + |F_2(s)|^2}}.$$

This is simply the chordal metric between $F_1(s)$ and $F_2(s)$ which is the chordal distance of the stereographic projections of $F_1(s)$ and $F_2(s)$ on the Riemann sphere [11]. The pointwise gap metric for scalar real rational functions is then given by

$$\delta(F_1, F_2) = \sup_{s \in \mathbb{C}^+} \frac{|F_1(s) - F_2(s)|}{\sqrt{1 + |F_1(s)|^2}\sqrt{1 + |F_2(s)|^2}}. \tag{5}$$

This formula gives a very clear and intuitive geometric interpretation of the pointwise gap metric for scalar real rational functions: it is no more than the supremum of the difference between the values of the transfer over all $s \in \mathbb{C}^+$, where the difference is measured by the chordal metric.

The metric (5) was actually proposed by El-Sakkary [4] to study the robustness of SISO systems. We have just shown that it is simply a special case of pointwise gap metrics. Therefore, almost all results in [4] can be obtained by specializing the results given in this paper.

We complete this section by giving an example which shows that the supremum over $\mathbb{C}^+$ in the definition of the pointwise gap metrics does not in general occur in its boundary, i.e., $\mathbb{C}^0$.

Example 1: Let $F_1(s) = (s-1)/(s+1)$ and $F_2(s) = (2s-1)/(s+1)$. It follows from formula (5) that

$$\gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] = \frac{\left|\frac{s-1}{s+1} - \frac{2s-1}{s+1}\right|}{\sqrt{1 + \left|\frac{s-1}{s+1}\right|^2}\sqrt{1 + \left|\frac{2s-1}{s+1}\right|^2}}.$$

It is computed in [8] that

$$\sup_{s \in \mathbb{C}^0} \gamma[\mathcal{G}(F_1, s), \mathcal{G}(F_2, s)] = \sup_{\omega \in \mathbb{R}} \frac{\left|\frac{j\omega-1}{j\omega+1} - \frac{2j\omega-1}{j\omega+1}\right|}{\sqrt{1 + \left|\frac{j\omega-1}{j\omega+1}\right|^2}\sqrt{1 + \left|\frac{2j\omega-1}{j\omega+1}\right|^2}} = \frac{1}{\sqrt{10}}$$

but

$$\begin{aligned}\delta(F_1, F_2) &= \sup_{s \in \mathbb{C}^+} \frac{\left|\frac{s-1}{s+1} - \frac{2s-1}{s+1}\right|}{\sqrt{1 + \left|\frac{s-1}{s+1}\right|^2}\sqrt{1 + \left|\frac{2s-1}{s+1}\right|^2}} \\ &\geq \left.\frac{\left|\frac{s-1}{s+1} - \frac{2s-1}{s+1}\right|}{\sqrt{1 + \left|\frac{s-1}{s+1}\right|^2}\sqrt{1 + \left|\frac{2s-1}{s+1}\right|^2}}\right|_{s=1} \\ &= \frac{1}{\sqrt{5}} > \frac{1}{\sqrt{10}}.\end{aligned}$$

V. Qualitative Properties of Pointwise Gap Metrics

First, we will examine the open-loop qualitative properties of pointwise gap metrics. To start with, we give a lemma regarding the pointwise gap metric corresponding to the spectral norm. Note that a real rational matrix with all elements equal to zero is denoted by 0, with its size determined by the context.

Lemma 1: For each stable real rational matrix F

$$\delta_s(F, 0) = \frac{\|F\|_s}{\sqrt{1 + \|F\|_s^2}}.$$

Proof: Since F is stable, a right-coprime factorization of F is given by FI^{-1}. A left-coprime factorization of 0 is given by $I^{-1}0$. By (4)

$$\begin{aligned}\gamma[\mathcal{G}(F, s), \mathcal{G}(0, s)] &= \left\|[0 \;\; I]\begin{bmatrix} I \\ F(s)\end{bmatrix}[I + F^*(s)F(s)]^{-\frac{1}{2}}\right\|_s \\ &= \|F(s)[I + F^*(s)F(s)]^{-\frac{1}{2}}\|_s \\ &= \frac{\|F(s)\|_s}{\sqrt{1 + \|F(s)\|_s^2}}.\end{aligned}$$

Therefore, by using the monotone increasing property of the function $x \to (x/\sqrt{1+x^2})$, we obtain

$$\delta(F_1, F_2) = \sup_{s \in \mathbb{C}^+} \frac{\|F(s)\|_s}{\sqrt{1 + \|F(s)\|_s^2}} = \frac{\|F\|_s}{\sqrt{1 + \|F\|_s^2}}.$$

□

Now let δ be a pointwise gap metric corresponding to any unitarily invariant norm. Denote by $\mathcal{B}(F_0, r)$ the gap metric open ball centered at $F_0 \in \mathcal{P}^{p \times m}$ with radius r, i.e., $\mathcal{B}(F_0, r) = \{F \in \mathcal{P}^{p \times m}: \delta(F, F_0) < r\}$. In particular, $\mathcal{B}_s(F_0, r) = \{F \in \mathcal{P}^{p \times m}: \delta_s(F, F_0) < r\}$.

Theorem 1: $\mathcal{S}^{p \times m}$ is an open subset of $(\mathcal{P}^{p \times m}, \delta)$. In particular, $\mathcal{S}^{p \times m} = \mathcal{B}_s(0, 1)$.

Proof: Since the pointwise gap metrics corresponding to different unitarily invariant norms are equivalent, it is enough to prove $\mathcal{S}^{p \times m} = \mathcal{B}_s(0, 1)$. It follows from Lemma 1 that $\delta_s(F, 0) < 1$ for all F in $\mathcal{S}^{p \times m}$. Now assume that F is unstable and let NM^{-1} be a right-coprime factorization of F. Then $M(\bar{s})$ is a singular matrix for some $\bar{s} \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$. Let x be a nonzero vector in the kernel of $M(\bar{s})$ and let $y = [M^*(\bar{s})M(\bar{s}) + N^*(\bar{s})N(\bar{s})]^{\frac{1}{2}}x$. Then by (4),

$$\begin{aligned}\gamma_s[\mathcal{G}(F, \bar{s}), \mathcal{G}(0, \bar{s})] &= \left\|[0 \;\; I]\begin{bmatrix} M(\bar{s}) \\ N(\bar{s})\end{bmatrix}[M^*(\bar{s})M(\bar{s}) + N^*(\bar{s})N(\bar{s})]^{-\frac{1}{2}}\right\|_s \\ &\geq \frac{\|N(\bar{s})[M^*(\bar{s})M(\bar{s}) + N^*(\bar{s})N(\bar{s})]^{-\frac{1}{2}}y\|_2}{\|y\|_2} \\ &= \frac{\|N(\bar{s})x\|_2}{\|[M^*(\bar{s})M(\bar{s}) + N^*(\bar{s})N(\bar{s})]^{\frac{1}{2}}x\|_2}\end{aligned}$$

$$= \left(\frac{x^* N^*(\bar{s}) N(\bar{s}) x}{x^* [M^*(\bar{s}) M(\bar{s}) + N^*(\bar{s}) N(\bar{s})] x} \right)^{\frac{1}{2}}$$
$$= 1.$$

Consequently, $\delta_s(F, 0) \geq 1$. Note that δ_s takes value in [0, 1]. This completes the proof. □

Theorem 1 gives a major qualitative property of pointwise gap metrics; in addition, it gives a precise quantitative characterization of the stable and the unstable real rational matrices in terms of the pointwise gap metric corresponding to the spectral norm: all stable matrices are in the open unit ball centered at the origin and the unstable ones are on the unit sphere.

More open-loop properties of pointwise gap metrics are given in the following.

Proposition 7: For any $F_1, F_2 \in \mathscr{S}^{p \times m}$

$$\frac{\|F_1 - F_2\|}{\sqrt{1 + \|F_1\|_s^2}\sqrt{1 + \|F_2\|_s^2}} \leq \delta(F_1, F_2) \leq \|F_1 - F_2\|. \tag{6}$$

Proof: Since F_i is stable for $i = 1, 2$, a right-coprime factorization of F_1 is given by $F_1 I^{-1}$ and a left-coprime factorization of F_2 is given by $I^{-1} F_2$. Then by (4), we obtain

$$\gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)]$$
$$= \left\| [I + F_2(s) F_2^*(s)]^{-\frac{1}{2}} [-F_2(s) \;\; I] \cdot \begin{bmatrix} I \\ F_1(s) \end{bmatrix} [I + F_1^*(s) F_1(s)]^{-\frac{1}{2}} \right\|$$
$$\geq \underline{\sigma}\left([I + F_2(s) F_2^*(s)]^{-\frac{1}{2}} \right) \|F_1(s) - F_2(s)\| \cdot \underline{\sigma}\left([I + F_1^*(s) F_1(s)]^{-\frac{1}{2}} \right)$$
$$= \frac{1}{\sqrt{1 + \|F_2(s)\|_s^2}} \|F_1(s) - F_2(s)\| \cdot \frac{1}{\sqrt{1 + \|F_1(s)\|_s^2}}.$$

Hence

$$\delta(F_1, F_2) = \sup_{s \in \mathbb{C}^+} \gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)]$$
$$\geq \inf_{s \in \mathbb{C}^+} \frac{1}{\sqrt{1 + \|F_2(s)\|_s^2}} \sup_{s \in \mathbb{C}^+} \|F_1(s) - F_2(s)\| \cdot \inf_{s \in \mathbb{C}^+} \frac{1}{\sqrt{1 + \|F_1(s)\|_s^2}}$$
$$= \frac{\|F_1 - F_2\|}{\sqrt{1 + \|F_1\|_s^2}\sqrt{1 + \|F_2\|_s^2}}.$$

On the other hand, since $[1 + F_1^*(s) F_1(s)]^{-\frac{1}{2}}$ and $[1 + F_2(s) F_2^*(s)]^{-\frac{1}{2}}$ are contractions,

$$\gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)] \leq \|F_1(s) - F_2(s)\|.$$

By taking the supremum over all $s \in \mathbb{C}^+$ on both sides, we obtain

$$\delta(F_1, F_2) \leq \|F_1 - F_2\|.$$

This completes the proof. □

An important consequence of Proposition 7 is that it identifies the topologies in $\mathscr{S}^{p \times m}$ induced by pointwise gap metrics and norms.

Corollary 1: The topology in $\mathscr{S}^{p \times m}$ induced by a pointwise gap metric and that induced by a norm are the same.

Proof: Let $\{F_i\}$ be a sequence in $\mathscr{S}^{p \times m}$ and $F \in \mathscr{S}^{p \times m}$. It is easy to see from the second inequality in (6) that $\|F_i - F\| \to 0$ implies $\delta(F_i, F) \to 0$. To show the converse, assume $\delta(F_i, F) \to 0$. Since F is stable, $\delta(F, 0) < 1$. Hence, there exist $\varepsilon > 0$ such that $\delta(F_i, 0) \leq 1 - \epsilon$ for all large enough i. It then follows from Lemma 1 that $\|F_i\|_s \leq (1 - \epsilon)^2 / (1 - (1 - \epsilon)^2)$. By the first inequality in (6), $\|F_i - F\| \to 0$. □

For each $F \in \mathscr{P}^{p \times m}$, we have $F' \in \mathscr{P}^{m \times p}$. Furthermore, if F is stable, so is F'. Hence, $F \to F'$ is a bijection between $\mathscr{P}^{p \times m}$ and $\mathscr{P}^{m \times p}$ which preserves stability. The following result says that this bijection is actually an isometry under a pointwise gap metric.

Proposition 8: $\delta(F_1', F_2') = \delta(F_1, F_2)$.

Proof: Let $N_i M_i^{-1}$ and $\tilde{M}_i^{-1} \tilde{N}_i$ be a right and a left-coprime factorization of F_i for $i = 1, 2$. Since

$$[-\tilde{N}_i \;\; \tilde{M}_i] \begin{bmatrix} M_i \\ N_i \end{bmatrix} = 0$$

the space spanned by $\begin{bmatrix} -\tilde{N}_i^*(s) \\ \tilde{M}_i^*(s) \end{bmatrix}$ is orthogonal to $\mathscr{G}(F_i, s)$. By Proposition 4 b) and c),

$$\gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)]$$
$$= \gamma\left[\mathscr{R}\left(\begin{bmatrix} -\tilde{N}_1^*(s) \\ \tilde{M}_1^*(s) \end{bmatrix} \right), \mathscr{R}\left(\begin{bmatrix} -\tilde{N}_2^*(s) \\ \tilde{M}_2^*(s) \end{bmatrix} \right) \right]$$
$$= \gamma\left[\mathscr{R}\left(\begin{bmatrix} \tilde{M}_1'(s^*) \\ \tilde{N}_1'(s^*) \end{bmatrix} \right), \mathscr{R}\left(\begin{bmatrix} \tilde{M}_2'(s^*) \\ \tilde{N}_2'(s^*) \end{bmatrix} \right) \right]$$
$$= \gamma[\mathscr{G}(F_1', s^*), \mathscr{G}(F_2', s^*)].$$

Therefore

$$\delta(F_1', F_2') = \sup_{s \in \mathbb{C}^+} \gamma[\mathscr{G}(F_1', s), \mathscr{G}(F_2', s)]$$
$$= \sup_{s \in \mathbb{C}^+} \gamma[\mathscr{G}(F_1', s^*), \mathscr{G}(F_2', s^*)]$$
$$= \sup_{s \in \mathbb{C}^+} \gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)]$$
$$= \delta(F_1, F_2).$$

□

It is known that the space $(\mathscr{S}^{p \times m}, \|\cdot\|)$ is a normed linear space. A linear space with a metric is called a *metric linear space* if the addition of vectors and the multiplication by scalars are continuous operations [17]. A nontrivial example of a metric linear space is $(\mathscr{S}^{p \times m}, \delta)$. The space $\mathscr{P}^{p \times m}$

is also a (real) linear space. The following example shows that the space $(\mathscr{P}^{p\times m}, \delta)$ is not a metric linear space.

Example 2: Let $F_\epsilon(s) = (s-1)/s$, $G_\epsilon(s) = (1-\epsilon)/s$, $F_0(s) = (s-1)/s$, and $G_0(s) = 1/s$. It is easy to verify by using formula (5) that under the pointwise gap metric $F_\epsilon \to F_0$, $G_\epsilon \to G_0$ as $\epsilon \to 0$, but $F_\epsilon + G_\epsilon = 1 - \epsilon/s \not\to F_0 + G_0 = 1$ as $\epsilon \to 0$.

If we consider the space $\mathscr{P}^{p\times m}$ as the $(p \times m)$-fold product space of $\mathscr{P}$, a natural question to ask is whether the topology in $\mathscr{P}^{p\times m}$ induced by a pointwise gap metric is the product topology of the topology in $\mathscr{P}$ induced by the pointwise gap metric in $\mathscr{P}$. The following example shows that the answer is negative.

Example 3: Consider

$$F_\epsilon(s) = \begin{bmatrix} \dfrac{1+\epsilon}{s-1} & \dfrac{1}{s-1} \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{bmatrix} \text{ and }$$

$$F_0(s) = \begin{bmatrix} \dfrac{1}{s-1} & \dfrac{1}{s-1} \\ \dfrac{1}{s-1} & \dfrac{1}{s-1} \end{bmatrix}.$$

It is easy to verify by using formula (5) that $F_{\epsilon ij} \to F_{0ij}$ as $\epsilon \to 0$. A right-coprime factorization of F_ϵ, $\epsilon \neq 0$, is given by

$$F_\epsilon(s) = \begin{bmatrix} \dfrac{1+\epsilon}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{bmatrix} \begin{bmatrix} \dfrac{s-1}{s+1} & 0 \\ 0 & \dfrac{s-1}{s+1} \end{bmatrix}^{-1}.$$

A left-coprime factorization of F_0 is given by

$$F_0(s) = \begin{bmatrix} \dfrac{s-1}{s+1} & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} \\ 0 & 0 \end{bmatrix}.$$

By using formula (4), we obtain

$$\delta(F_\epsilon, F_0) \geq \gamma[\mathscr{G}(F_\epsilon, 1), \mathscr{G}(F_0, 1)] = 1$$

which means $F_\epsilon \not\to F_0$ as $\epsilon \to 0$.

In the rest of this section, we work towards the qualitative closed-loop properties of pointwise gap metrics. Consider the feedback configuration shown in Fig. 1; in this case, different spaces of real rational matrices are involved. If the plant space is $\mathscr{P}^{p\times m}$, then the controller space is $\mathscr{P}^{m\times p}$ and the closed-loop transfer matrices are in the space $\mathscr{P}^{(p+m)\times(p+m)}$. We will allow the pointwise gap metrics in these spaces to be induced by different unitarily invariant matrix norms. In addition to the spaces mentioned above, another space we have to consider is the product space of plant–controller pairs $\mathscr{P}^{p\times m} \times \mathscr{P}^{m\times p}$. In the following, we do not need a metric in the product space $\mathscr{P}^{p\times m} \times \mathscr{P}^{m\times p}$, but we do need a topology in the product space. The topology is assumed to be the product topology generated from the gap metric topologies in $\mathscr{P}^{p\times m}$ and $\mathscr{P}^{m\times p}$.

The following result shows that the topology on block diagonal matrices induced by a pointwise gap metric is the product topology of topologies on individual diagonal blocks.

Proposition 9: Let $F_1, G_1 \in \mathscr{P}^{p_1\times m_1}$ and $F_2, G_2 \in \mathscr{P}^{p_2\times m_2}$. Then

$$\delta_s\left(\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}\right) = \max\{\delta_s(F_1, G_1), \delta_s(F_2, G_2)\}.$$

Proof: Let $N_i M_i^{-1}$ be a right-coprime factorization of F_i for $i = 1, 2$. Then a right-coprime factorization of $\begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$ is given by $\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}^{-1}$. Similarly, let $\tilde{M}_i^{-1}\tilde{N}_i$ be a left-coprime factorization of G_i for $i = 1, 2$. Then a left-coprime factorization of $\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$ is given by $\begin{bmatrix} \tilde{M}_1 & 0 \\ 0 & \tilde{M}_2 \end{bmatrix}^{-1}\begin{bmatrix} \tilde{N}_1 & 0 \\ 0 & \tilde{N}_2 \end{bmatrix}$. The rest of the proof is simply an algebraic manipulation based on (4).

The following two properties (Propositions 10 and 11) share a common idea in their proofs, which are given in Appendix A. The first property implies that the multiplication of real rational matrices by unimodular matrices is continuous.

Proposition 10: Let $F_1, F_2 \in \mathscr{P}^{p\times m}$ and let $G \in \mathscr{S}^{p\times p}$, $H \in \mathscr{S}^{m\times m}$ be unimodular matrices. Then

$$\delta(GF_1H, GF_2H) \leq \max\{\|H^{-1}\|_s, \|G\|_s\} \cdot \max\{\|H\|_s, \|G^{-1}\|_s\}\delta(F_1, F_2) \quad (7)$$

$$\delta(F_1, F_2) \leq \max\{\|H^{-1}\|_s, \|G\|_s\} \cdot \max\{\|H\|_s, \|G^{-1}\|_s\}\delta(GF_1H, GF_2H). \quad (8)$$

In our applications, the matrices G and H are orthogonal matrices. In this case, the inequalities in Proposition 10 degenerate to an equality.

Corollary 2: Let $F_1, F_2 \in \mathscr{P}^{p\times m}$ and let $G \in \mathbb{R}^{p\times p}$ and $H \in \mathbb{R}^{m\times m}$ be orthogonal matrices. Then

$$\delta(GF_1H, GF_2H) = \delta(F_1, F_2).$$

The second property shows that the addition of real rational matrices by a stable real rational matrix is continuous.

Proposition 11: Let $F_1, F_2 \in \mathscr{P}^{p\times m}$ and $G \in \mathscr{S}^{p\times m}$. Then

$$\delta(G + F_1, G + F_2) \leq \frac{1}{2}\left(2 + \|G\|_s^2 + \|G\|_s\sqrt{4 + \|G\|_s^2}\right)\delta(F_1, F_2) \quad (9)$$

$$\delta(F_1, F_2) \leq \frac{1}{2}\left(2 + \|G\|_s^2 + \|G\|_s\sqrt{4 + \|G\|_s^2}\right) \cdot \delta(G + F_1, G + F_2). \quad (10)$$

The following result shows that the inversion of invertible real rational matrices is an isometry.

Proposition 12: If $F_1, F_2 \in \mathscr{P}^{m\times m}$ are invertible, then

$$\delta(F_1^{-1}, F_2^{-1}) = \delta(F_1, F_2).$$

Proof: Let $N_i M_i^{-1}$ be a right coprime factorization of F_i for $i = 1,2$. Then $M_i N_i^{-1}$ is a right coprime factorization of F_i^{-1} for $i = 1,2$.

Now let us fix some s in $\mathbb{C}^+$. The following is obtained:

$$\mathscr{G}(F_i, s) = \mathscr{R}\left(\begin{bmatrix} M_i(s) \\ N_i(s) \end{bmatrix}\right)$$

$$\mathscr{G}(F_i^{-1}, s) = \mathscr{R}\left(\begin{bmatrix} N_i(s) \\ M_i(s) \end{bmatrix}\right) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathscr{G}(F_i, s)$$

where $i = 1,2$.

Since $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ is an orthogonal matrix, Proposition 4c) leads to

$$\gamma[\mathscr{G}(F_1^{-1}, s), \mathscr{G}(F_2^{-1}, s)] = \gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)].$$

The result then follows by taking the supremum over all $s \in \mathbb{C}^+$ on both sides of the above equality. □

The study of well-posedness of a plant–controller pair requires the following result on the set of invertible matrices. In its proof, we need the concept of normalized coprime factorizations, which will also be needed in Appendix B. A right-coprime factorization NM^{-1} of $F \in \mathscr{P}^{p\times m}$ is said to be *normalized* if $\begin{bmatrix} M(j\omega) \\ N(j\omega) \end{bmatrix}$ has orthonormal columns for each $\omega \in \mathbb{R}$. Similarly, a left-coprime factorization $\tilde{M}^{-1}\tilde{N}$ of F is said to be *normalized* if $[\tilde{M}(j\omega)\tilde{N}(j\omega)]$ has orthonormal rows for each $\omega \in \mathbb{R}$. It has been shown in [21] that every real rational matrix has a normalized right-coprime factorization and a normalized left-coprime factorization.

Proposition 13: The set of invertible matrices in $\mathscr{P}^{m\times m}$ is open.

Proof: Let $F_0 \in \mathscr{P}^{m\times m}$ be invertible and let $\tilde{M}_0^{-1}\tilde{N}_0$ be its normalized left-coprime factorization. The invertibility of F_0 implies that there exists $\bar{\omega} \in \mathbb{R}$ such that $\tilde{N}_0(j\bar{\omega})$ is nonsingular. Let $F \in \mathscr{P}^{m\times m}$ be noninvertible and let NM^{-1} be its normalized right-coprime factorization. Then $N(j\bar{\omega})$ is singular. Choose $x \in \mathbb{C}^m$ with $\|x\|_2 = 1$ and $N(j\bar{\omega})x = 0$. Since $\begin{bmatrix} M(j\bar{\omega}) \\ N(j\bar{\omega}) \end{bmatrix}$ has orthonormal columns

$$\|M(j\bar{\omega})x\|_2 = \left\|\begin{bmatrix} M(j\bar{\omega}) \\ N(j\bar{\omega}) \end{bmatrix} x\right\|_2 = 1.$$

Hence,

$$\begin{aligned}
\delta(F, F_0) &\geq \gamma[\mathscr{G}(F, j\bar{\omega}), \mathscr{G}(F_0, j\bar{\omega})] \\
&= \left\|[-\tilde{N}_0(j\bar{\omega}) \;\; \tilde{M}_0(j\bar{\omega})]\begin{bmatrix} M(j\bar{\omega}) \\ N(j\bar{\omega}) \end{bmatrix}\right\|_s \\
&\geq \left\|[-\tilde{N}_0(j\bar{\omega}) \;\; \tilde{M}_0(j\bar{\omega})]\begin{bmatrix} M(j\bar{\omega}) \\ N(j\bar{\omega}) \end{bmatrix} x\right\|_2 \\
&= \|-\tilde{N}_0(j\bar{\omega})M(j\bar{\omega})x\|_2 \\
&\geq \underline{\sigma}[\tilde{N}_0(j\bar{\omega})]\,\|M(j\bar{\omega})x\|_2 \\
&= \underline{\sigma}[\tilde{N}_0(j\bar{\omega})].
\end{aligned}$$

Therefore, if F_0 is invertible, then each element in $\mathscr{B}(F_0, \epsilon)$ is invertible as long as ϵ is small enough. Since F_0 is arbitrarily chosen in the set of $m \times m$ invertible real rational matrices, we conclude that this set is open. □

The following corollary of Proposition 12 and 13 is sometimes useful.

Corollary 3: The set of unimodular matrices in $\mathscr{P}^{m\times m}$ is open.

Proof: The set of unimodular matrices in $\mathscr{P}^{m\times m}$ is equal to

$$\mathscr{S}^{m\times m} \cap [\mathscr{S}^{m\times m} \cap (\text{the set of } m \times m \text{ invertible real rational matrices})]^{-1}.$$

Since $\mathscr{S}^{m\times m}$ and the set of invertible matrices are open and since $(\cdot)^{-1}$ is an isometry, the set of unimodular matrices must be open. □

Consider the following maps.

i) $(F_1, F_2) \to \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}$
ii) $F \to GF$ (G is real orthogonal)
iii) $F \to I + F$ (F is square)
iv) $F \to F^{-1}$ (F is invertible).

An interpretation of Propositions 9–12 and Corollary 2 is that maps i)–iv) are all continuous and have continuous inverses in their ranges. In other words, these maps are homeomorphisms between their domains and their ranges. (In fact, maps ii), iv) are isometries.) This interpretation allows us to establish our main result on the qualitative closed-loop properties of pointwise gap metrics.

Recall that $\mathscr{W}(p, m) \in \mathscr{P}^{p\times m} \times \mathscr{P}^{m\times p}$ is the set of all well-posed (P, C) pairs, $\boldsymbol{H}$ is a function from $\mathscr{W}(p, m)$ to $\mathscr{P}^{(p+m)\times(p+m)}$ defined by (1), and $\mathscr{C}(p, m)$ is the set of $\boldsymbol{H}(P, C)$ when $(P, C) \in \mathscr{W}(p, m)$.

Theorem 2: The set $\mathscr{W}(p, m)$ is an open subset of $\mathscr{P}^{p\times m} \times \mathscr{P}^{m\times p}$ and the map $\boldsymbol{H}$ is a homeomorphism between $\mathscr{W}(p, m)$ and $\mathscr{C}(p, m)$.

Proof: $\mathscr{W}(p, m)$ is the set of all (P, C) with $I - \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} P & 0 \\ 0 & C \end{bmatrix}$ invertible. Since $\mathscr{W}(p, m)$ is simply the inverse image of the set of all invertible matrices in $\mathscr{P}^{(p+m)\times(p+m)}$ which is an open set, under the composition of the continuous maps i)–iii), it must be open. The map $\boldsymbol{H}$ is a composition of maps i)–iv); since these maps are homeomorphisms, so is $\boldsymbol{H}$. □

Roughly speaking, Theorem 2 says that if (P_0, C_0) is well posed and P, C are sufficiently close to P_0, C_0, respectively, then (P, C) is well posed and $\boldsymbol{H}(P, C)$ is close to $\boldsymbol{H}(P_0, C_0)$. Conversely, if (P_0, C_0) and (P, C) are well posed and $\boldsymbol{H}(P, C)$ is close to $\boldsymbol{H}(P_0, C_0)$, then P, C must be close to P_0, C_0, respectively. By combining Theorem 2 and Theorem 1, we conclude that if (P_0, C_0) is well posed, $\boldsymbol{H}(P_0, C_0)$ is stable, and P, C are sufficiently close to P_0, C_0, respectively, then (P, C) is well posed and $\boldsymbol{H}(P, C)$ is also stable.

VI. Quantitative Properties of Pointwise Gap Metrics

Again in this section we first look at the open-loop problem and then switch to the closed-loop problem defined by the feedback configuration shown in Fig. 1. In the open-loop setup, the objective is to find a necessary and sufficient condition which guarantees that each element of the ball $\mathscr{B}(F_0, r)$ centered at a stable real rational matrix F_0 is stable.

Theorem 3: Let $F_0 \in \mathscr{S}^{p\times m}$. Then $\mathscr{B}(F_0, r) \subset \mathscr{S}^{p\times m}$ if and only if $r \le \sqrt{1 - \delta_s^2(F_0, 0)}$.

We will see later on, that this theorem is only a special case of a forthcoming theorem. The bound given in Theorem 3 takes another form by using Lemma 1

$$\sqrt{1 - \delta_s^2(F_0, 0)} = \frac{1}{\sqrt{1 + \|F_0\|_s^2}}.$$

Now let us consider the feedback configuration shown in Fig. 1. Let the pointwise gap metric in the plant space $\mathscr{P}^{p\times m}$ be δ_1 and that in the controller space $\mathscr{P}^{m\times p}$ be δ_2. The open balls in the two spaces take the form

$$\mathscr{B}_1(P_0, r) = \{P: \delta_1(P, P_0) < r\}$$

and

$$\mathscr{B}_2(C_0, r) = \{C: \delta_2(C, C_0) < r\}.$$

Define a partial function $\nu: \mathscr{P}^{p\times m} \times \mathscr{P}^{m\times p} \to \mathbb{R}$ by

$$\nu(P, C) = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \ C] \right\|_s^{-1}. \tag{11}$$

The domain of definition of ν includes all pairs (P, C) such that $H(P, C)$ is stable, since we have the identity

$$\begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \ C] = H(P, C) - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Theorem 4: Let (P_0, C_0) be given with $H(P_0, C_0)$ stable. Then the following statements are equivalent:

a) $r \le \nu(P_0, C_0)$;
b) $H(P, C_0)$ is stable for all $P \in \mathscr{B}_1(P_0, r)$;
c) $H(P_0, C)$ is stable for all $C \in \mathscr{B}_2(C_0, r)$;
d) $H(P, C)$ is stable for all $P \in \mathscr{B}_1(P_0, r_1)$, $C \in \mathscr{B}_2(C_0, r_2)$ with r_1, r_2 satisfying

$$r_1^2 + r_2^2 + 2r_1r_2\sqrt{1 - \nu^2(P_0, C_0)} = r^2.$$

Note that Theorem 3 is obtained from Theorem 4 on letting $C_0 = 0$. Appendix B is dedicated to the long proof of Theorem 4.

A graphic interpretation of Theorem 4 is shown in Fig. 2. A point (r_1, r_2) with $0 \le r_1, r_2 \le 1$ in the coordinate system represents a set of plant–controller pairs (P, C) which satisfy $\delta_1(P, P_0) = r_1$ and $\delta_2(C, C_0) = r_2$. The equivalence of a) and d) in Theorem 4 says that the largest area which contains only stable pairs is the shaded area in Fig. 2 whose upper-right boundary is given by the equation

$$r_1^2 + r_2^2 + 2r_1r_2\sqrt{1 - \nu^2(P_0, C_0)} = \nu^2(P_0, C_0).$$

An immediate consequence of Theorem 4 is given by the following corollary. Note that $(P, C) \in \mathscr{W}(p, m)$ if and only if $(C, P) \in \mathscr{W}(m, p)$ and that $H(P, C)$ is stable if and only if $H(C, P)$ is stable.

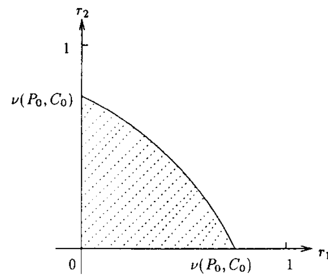


Fig. 2. The largest area containing only stable pairs.

Corollary 4: $\nu(P, C) = \nu(C, P)$.

Proof: Let δ_1 and δ_2 in Theorem 4 be the same. Then

$$\begin{aligned} r \le \nu(P_0, C_0) &\Leftrightarrow H(P, C_0) \text{ is stable for all } P \in \mathscr{B}_1(P_0, r) \\ &\Leftrightarrow H(C_0, P) \text{ is stable for all } P \in \mathscr{B}_2(P_0, r) \\ &\Leftrightarrow r \le \nu(C_0, P_0). \end{aligned}$$

This proves $\nu(P_0, C_0) = \nu(C_0, P_0)$. □

The same conclusion as Corollary 4 is obtained in [8] by using different arguments.

Theorem 4 shows that the function ν gives a good measure for the closed-loop stability robustness. A natural design problem is to find a controller C for a given plant P so that the closed-loop stability robustness is maximized, i.e., we want to find

$$\sup_C \nu(P, C) = \left[\inf_C \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \ C] \right\|_s \right]^{-1}$$

where C is taken from the set of all controllers which stabilize P. This problem falls into the $\mathscr{H}_\infty$ optimal control problem which has been extensively studied in the recent control literature; see, e.g., [6]. The general $\mathscr{H}_\infty$ optimal control problem can be solved by using iterative procedures. However, the particular problem we have here has been solved in [9] and [8] without iteration. See also [14].

VII. Multiplication Operators, Compactification of $\mathbb{C}^{p\times m}$, and Pointwise Gap Metrics

In this section, we give connections of pointwise gap metrics to the concepts of the multiplication operators from $\mathscr{H}_2^m$ to $\mathscr{H}_2^p$ and compactifications of $\mathbb{C}^{p\times m}$. Details on the concepts of unbounded operators and compactifications can be found in [13] and [18], respectively.

We denote by $\mathscr{H}_2^p$ the Hardy space of all functions u which are analytic in $\mathbb{C}^+$, take values in $\mathbb{C}^p$, and satisfy the uniform square-integrability condition

$$\left[\sup_{\xi > 0} (2\pi)^{-1} \int_{-\infty}^{\infty} \|u(\xi + j\omega)\|_2^2 \, d\omega \right]^{\frac{1}{2}} < \infty.$$

It is well known that $\mathscr{H}_2^p$ is a Hilbert space with inner product

$$\langle u, v \rangle = (2\pi)^{-1} \int_{-\infty}^{\infty} v^*(j\omega) u(j\omega) \, d\omega$$

where $u(j\omega)$ and $v(j\omega)$ are extensions of $u, v \in \mathscr{H}_2^p$ to the imaginary axis.

For any $F \in \mathscr{P}^{r\times m}$, the *multiplication operator* from $\mathscr{H}_2^m$ to $\mathscr{H}_2^p$ due to F, denoted by M_F, is defined to be the possibly unbounded operator which maps $u \in \mathscr{H}_2^m$ to Fu if Fu is in $\mathscr{H}_2^p$. The operator M_F is unbounded if F is unstable since not every Fu is in $\mathscr{H}_2^p$ in this case. The domain, the range, and the graph of M_F are defined as[2]

$$\mathscr{D}(F) = \{u \in \mathscr{H}_2^m: Fu \in \mathscr{H}_2^p\}$$
$$\mathscr{R}(F) = \{Fu: u \in \mathscr{D}(F)\}$$
$$\mathscr{G}(F) = \left\{\begin{bmatrix} u \\ Fu \end{bmatrix}: u \in \mathscr{D}(F)\right\}.$$

It is clear that $\mathscr{D}(F)$, $\mathscr{R}(F)$, and $\mathscr{G}(F)$ are linear manifolds in $\mathscr{H}_2^m$, $\mathscr{H}_2^p$ and $\mathscr{H}_2^m \times \mathscr{H}_2^p$, respectively.

The graph of M_F can be characterized by right-coprime factorizations of F.

Lemma 2 [21]: Let $F \in \mathscr{P}^{p\times m}$ and NM^{-1} be any right coprime factorization of F. Then

$$\mathscr{G}(F) = \left\{\begin{bmatrix} M \\ N \end{bmatrix} u: u \in \mathscr{H}_2^m\right\}.$$

At any fixed $s \in \mathbb{C}^+$, the *evaluation operator* on $\mathscr{H}_2^p \times \mathscr{H}_2^m$ maps $v \in \mathscr{H}_2^p \times \mathscr{H}_2^m$ to $v(s) \in \mathbb{C}^{p+m}$. Since $\mathscr{G}(F)$ is a linear manifold in $\mathscr{H}_2^p \times \mathscr{H}_2^m$, the image of $\mathscr{G}(F)$ under the evaluation operator at any fixed $s \in \mathbb{C}^+$ must be a subspace of $\mathbb{C}^{p+m}$. By Lemma 2, this subspace is just $\mathscr{G}(F, s)$ defined in Section III. Therefore, a pointwise gap metric between F_1 and F_2 in $\mathscr{P}^{p\times m}$ can be interpreted using the graphs of the multiplication operators M_{F_1} and M_{F_2}. It is the supremum over all s in $\mathbb{C}^+$ of a gap between the images of the graphs of M_{F_1} and M_{F_2} under the evaluation operator at s.

It has been seen from Section IV that the value of a real rational function at each $s \in \mathbb{C}^+$ is an element of $\mathbb{C} \cup \{\infty\}$, and the pointwise gap metric between two real rational functions is simply the supremum over all $s \in \mathbb{C}^+$ of the chordal metric between the values of these two functions. It is known that $\mathbb{C} \cup \{\infty\}$ with the chordal metric is a one-point compactification of $\mathbb{C}$. The idea for the scalar case can be generalized to the matrix case. It has been shown in Section II that the set of all l-dimensional subspaces in $\mathbb{C}^n$, denoted by $\Gamma_l(\mathbb{C}^n)$, is a compact metric space if its topology is induced by a gap.

Proposition 14: $\Gamma_m(\mathbb{C}^{p+m})$ is a compactification of $\mathbb{C}^{p\times m}$.

We refer to [15] for the proof of Proposition 14.

Let $F \in \mathscr{P}^{p\times m}$ and NM^{-1} be a right-coprime factorization of F. Then for each $s \in \mathbb{C}^+$, F uniquely determines an m-dimensional subspace of $\mathbb{C}^{p+m}$ which is given by $\mathscr{R}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right)$. In other words, the value of $F(s)$ at each $s \in \mathbb{C}^+$ can be considered as an element in the compactification of $\mathbb{C}^{p\times m}$. A pointwise gap metric between F_1 and F_2 is simply the supremum over all $s \in \mathbb{C}^+$ of the distance between $F_1(s)$ and $F_2(s)$ measured by a metric in the compactification of $\mathbb{C}^{p\times m}$.

[2]The notation is slightly abused here; $\mathscr{D}(M_F)$, $\mathscr{R}(M_F)$, and $\mathscr{G}(M_F)$ are abbreviated as $\mathscr{D}(F)$, $\mathscr{R}(F)$, and $\mathscr{G}(F)$, respectively.

VIII. Comparison to the Gap Metric

A metric in the space of real rational matrices which has undergone extensive studies in recent years is the gap metric. The gap metric on real rational matrices is inherited from the gap metric on closed unbounded operators between Banach spaces which has been thoroughly studied in [13], and was introduced to control theory, particularly to the stability robustness study, in [23]. Significant contributions to the gap metric study (in control) include [3], [24], [7], [8].

Let $\mathscr{X}$ and $\mathscr{Y}$ be subspaces (closed linear manifolds) of a Hilbert space $\mathscr{H}$ and let $P_{\mathscr{X}}$ and $P_{\mathscr{Y}}$ be the orthogonal projections onto $\mathscr{X}$ and $\mathscr{Y}$, respectively. The gap between $\mathscr{X}$ and $\mathscr{Y}$ is defined as[3]

$$\gamma(\mathscr{X}, \mathscr{Y}) = \| P_{\mathscr{X}} - P_{\mathscr{Y}} \|.$$

If $\mathscr{H}$ is finite dimensional and $\mathscr{X}$, $\mathscr{Y}$ have the same dimension, then the above definition is equivalent to the definition in Section III for the case when the unitarily invariant norm used is the spectral norm. (See [1], [2], or [15] for justifications.)

Let $F \in \mathscr{P}^{p\times m}$ and let M_F be the multiplication operator from $\mathscr{H}_2^m$ to $\mathscr{H}_2^p$ due to F. It is shown in [23] that $\mathscr{G}(F)$, the graph of M_F, is a closed linear manifold (i.e., a subspace) of $\mathscr{H}_2^p \times \mathscr{H}_2^m$. In other words, the multiplication operator is a *closed operator*. The *gap metric* on $\mathscr{P}^{p\times m}$ is defined to be a function δ_g: $\mathscr{P}^{p\times m} \times \mathscr{P}^{p\times m} \to [0, 1]$ given by

$$\delta_g(F_1, F_2) = \gamma[\, \mathscr{G}(F_1), \mathscr{G}(F_2)]$$

for all $F_1, F_2 \in \mathscr{P}^{p\times m}$. A nice formula for the computation of the gap metric is given in [7].

The following example shows that the gap metric is not equal to any of the pointwise gap metrics.

Example 4: We use the same two real rational functions as in Example 1: $F_1 = (s-1)/(s+1)$ and $F_2 = (2s-1)/(s+1)$. It is obtained in [8] that $\delta_g(F_1, F_2) = 1/3$, whereas we have shown in Example 1 that $\delta(F_1, F_2) \geq 1/\sqrt{5}$.

The fact that the pointwise gap metric δ_s and the gap metric differ is perhaps surprising for the following reason. Consider a stable rational matrix $F \in \mathscr{P}^{p\times m}$ and the bounded multiplication operator from $\mathscr{H}_2^p$ to $\mathscr{H}_2^m$ due to F. It is well known that the norm of the multiplication operator is equal to $\sup_{s\in\mathbb{C}^+} \|F(s)\|_s$, which can be regarded as a "pointwise" norm. The difference in the definitions of the pointwise gap metric corresponding to the spectral norm and the gap metric bears an analogy to the difference in the definitions of the "pointwise" norm corresponding to the spectral norm and the operator norm. However, this analogy does not extend to the value of the metrics.

The following example shows that the gap metric is not uniformly equivalent to any of the pointwise gap metrics in the nonscalar case. It is not known yet whether or not in the scalar case the gap metric is uniformly equivalent to a pointwise gap metric.

[3]Here and only here in the entire paper, $\| \cdot \|$ means the induced norm of bounded operators.

Example 5: This example is also taken from [8]. Let

$$F_1(s) = \begin{bmatrix} \frac{\epsilon}{s-1} \\ 0 \end{bmatrix} \text{ and } F_2(s) = \begin{bmatrix} 0 \\ \frac{\epsilon}{s-1} \end{bmatrix}$$

where $\epsilon > 0$. Then right coprime factorizations of F_1 and F_2 are given by

$$\begin{bmatrix} M_1(s) \\ N_1(s) \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+1+\epsilon^2} \\ \frac{\epsilon}{s+1+\epsilon^2} \\ 0 \end{bmatrix} \text{ and }$$

$$\begin{bmatrix} M_2(s) \\ N_2(s) \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s+1+\epsilon^2} \\ 0 \\ \frac{\epsilon}{s+1+\epsilon^2} \end{bmatrix}.$$

We know from [8] that $\delta_g(F_1, F_2) = \epsilon\sqrt{2+\epsilon^2}/(1+\epsilon^2)$. On the other hand, the subspaces spanned by

$$\begin{bmatrix} M_1(1) \\ N_1(1) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\epsilon}{2+\epsilon^2} \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} M_2(1) \\ N_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\epsilon}{2+\epsilon^2} \end{bmatrix}$$

are orthogonal. Therefore, $\delta(F_1, F_2) = 1$ for all $\epsilon > 0$. As $\epsilon \to 0$, $\delta_g(F_1, F_2) \to 0$ while $\delta(F_1, F_2)$ remains at 1. This shows that there does not exist a positive constant c such that $\delta(F_1, F_2) \le c\delta_g(F_1, F_2)$.

Nevertheless, the gap metric is topologically equivalent to each of the pointwise gap metrics. This fact follows immediately from the following proposition. To make the idea precise, we have to consider the spaces $\bigcup_{p,m=1}^{\infty} \mathcal{P}^{p\times m}$ and $\bigcup_{p,m=1}^{\infty} \mathcal{S}^{p\times m}$ instead of the individual spaces $\mathcal{P}^{p\times m}$ and $\mathcal{S}^{p\times m}$. The topologies in $\bigcup_{p,m=1}^{\infty} \mathcal{P}^{p\times m}$ and $\bigcup_{p,m=1}^{\infty} \mathcal{S}^{p\times m}$ are uniquely determined by the topologies in the individual spaces and vice versa.

Proposition 15: There is a unique topology in $\bigcup_{p,m=1}^{\infty} \mathcal{P}^{p\times m}$ which has the following properties:

a) the relative topology in $\bigcup_{p,m=1}^{\infty} \mathcal{S}^{p\times m}$ is the norm topology;
b) $\bigcup_{p,m=1}^{\infty} \mathcal{W}(r, m)$ is open;
c) the map H is a homeomorphism between $\bigcup_{p,m=1}^{\infty} \mathcal{W}(p, m)$ and $\bigcup_{p,m=1}^{\infty} \mathcal{C}(p, m)$.

Proof: Let τ_1 and τ_2 be two topologies having properties a), b), c). Let $\mathcal{O} \in \tau_1$ and $P_0 \in \mathcal{O}$. Then there exists $C_0 \in \bigcup_{p,m=1}^{\infty} \mathcal{P}^{p\times m}$ such that $H(P_0, C_0)$ is stable. Furthermore, since H is a τ_1-homeomorphism, there exist $\mathcal{O}_1, \mathcal{O}_2 \in \tau_1$ with $P_0 \in \mathcal{O}_1 \subset \mathcal{O}$ and $C_0 \in \mathcal{O}_2$ such that $H(P, C)$ is stable for all $P \in \mathcal{O}_1$, $C \in \mathcal{O}_2$. The set $H(\mathcal{O}_1, \mathcal{O}_2)$ is in τ_1. Since the relative topologies of τ_1 and τ_2 in $\bigcup_{p,m=1}^{\infty} \mathcal{S}^{p\times m}$ are the same, the set $H(\mathcal{O}_1, \mathcal{O}_2)$ is in τ_2 too. By the fact that H is bijective and τ_2-continuous, the set $\mathcal{O}_1$ must be in τ_2. Since P_0 is arbitrarily chosen in $\mathcal{O}$, this implies that $\mathcal{O}$ is also in τ_2. This proves $\tau_1 \subset \tau_2$. In exactly the same way, we can show $\tau_2 \subset \tau_1$. This completes the proof. □

Note that in the above proof, we only used the fact that the H is τ_2-continuous to show that $\tau_1 \subset \tau_2$. Hence, the proof of Proposition 15 also shows the following result.

Proposition 16: The unique topology in $\bigcup_{p,m=1}^{\infty} \mathcal{P}^{p\times m}$ determined by Proposition 15 is the weakest topology with the following properties:

a) the relative topology in $\bigcup_{p,m=1}^{\infty} \mathcal{S}^{p\times m}$ is the norm topology;
b) $\bigcup_{p,m=1}^{\infty} \mathcal{W}(r, m)$ is open;
c) the map H is a continuous function from $\bigcup_{p,m=1}^{\infty} \mathcal{W}(p, m)$ to $\bigcup_{p,m=1}^{\infty} \mathcal{C}(p, m)$.

This result is important since, as discussed in Section I, the initial requirement for the desired metric in $\mathcal{P}^{p\times m}$ is that H is continuous, and if this is the case, we say that the closed-loop stability is a robust property. Therefore, Proposition 16 simply says that the topology induced by the gap metric or any of the pointwise gap metrics is the weakest topology such that the closed-loop stability is a robust property.

Similar statements to Proposition 15 are made in [23], [3], and [24]. Similar statements to Proposition 16 are made in [22] and [20].

Now let us come back to the comparison of the gap metric and the pointwise gap metrics. On summarizing, we have the following general remarks.

i) The gap metric and pointwise gap metrics are different and are not uniformly equivalent, but they are topologically equivalent. A natural consequence of this fact is that they have many similar qualitative properties; surprisingly, they also have many similar quantitative properties in spite of the fact that they differ in value. Theorem 3 and Theorem 4 can be restated for the gap metric [8], [16].
ii) The gap metrics can be computed by using a formula given in [7]. An efficient computational method for pointwise gap metrics is not yet available, although their definition allows them to be computed by a two-dimensional search. Easily computable lower and upper bounds for pointwise gap metrics are given in [15].
iii) The gap metrics relies on infinite-dimensional functional analysis, whereas the pointwise gap metrics are built from finite-dimensional linear algebra. This implies that pointwise gap metrics are conceptually simpler than the gap metric. The study of pointwise gap metrics seems more straightforward in some cases due to their conceptual simplicity.
iv) Pointwise gap metrics are transpose invariant, i.e., $\delta(F_1, F_2) = \delta(F_1', F_2')$, whereas the gap metric does not have this property.
v) A pointwise gap metric can be interpreted as being the supremum over $s \in \mathbb{C}^+$ of the distance between the values of real rational matrices, where the values are regarded to be in a compactification of the set of complex matrices, and the distance is given by a metric in this compactification. This interpretation is

potentially useful in applications since it may provide an access to the relationship between the metric and the physical parameters of the systems described by the real rational matrices. However, it appears that what the gap metric measures is so intrinsic that very little intuitive sense is provided.

The graph metric is another metric which can be used in the stability robustness study of real rational matrice. It is shown in [7] that the graph metric and the gap metrics are uniformly equivalent. Therefore, the graph metric and pointwise gap metrics are not uniformly equivalent, but are topologically equivalent. However, it appears that the graph metric is not so convenient to use because it lacks the good quantitative properties which the gap metric and pointwise gap metrics have.

Appendix A
Proofs of Propositions 10 and 11

Before proving Propositions 10 and 11, we have to establish a relation between $\gamma(A\mathscr{X}, A\mathscr{Y})$ and $\gamma(\mathscr{X}, \mathscr{Y})$, where A is an arbitrary nonsingular linear transformaiton on $\mathbb{C}^n$. For this purpose, we have two results.

Proposition 17: Let $\mathscr{X}, \mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$ and let X_1 and Y_1 be matrices whose columns form orthonormal bases of $\mathscr{X}$ and $\mathscr{Y}$, respectively. Then

$$\gamma(\mathscr{X}, \mathscr{Y}) = \min_{Q \in \mathbb{C}^{l\times l}} \|X_1 - Y_1 Q\|.$$

Proof: Select Y_2 such that $Y = [Y_1 \ Y_2]$ is a unitary matrix. Then

$$\|X_1 - Y_1 Q\| = \|Y^*(X_1 - Y_1 Q)\| = \left\| \begin{bmatrix} Y_1^* X_1 \\ Y_2^* X_1 \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} Q \right\|.$$

Hence

$$\min_{Q \in \mathbb{C}^{l\times l}} \|X_1 - Y_1 Q\| = \left\| \begin{bmatrix} 0 \\ Y_2^* X_1 \end{bmatrix} \right\| = \gamma(\mathscr{X}, \mathscr{Y}).$$

□

Let A be a nonsingular linear transformation on $\mathbb{C}^n$. The *condition number* of A corresponding to the spectral norm $\|\cdot\|_s$, denoted by $\kappa_s(A)$, is defined as $\kappa_s(A) = \|A\|_s \|A^{-1}\|_s$.

Proposition 18: Let $\mathscr{X}, \mathscr{Y} \in \Gamma_l(\mathbb{C}^n)$ and let A be a nonsingular linear transformation. Then

$$\gamma(A\mathscr{X}, A\mathscr{Y}) \le \kappa_s(A)\gamma(\mathscr{X}, \mathscr{Y}) \quad (12)$$

$$\gamma(\mathscr{X}, \mathscr{Y}) \le \kappa_s(A)\gamma(A\mathscr{X}, A\mathscr{Y}). \quad (13)$$

Proof: Since (12) implies (13), we only need to prove (12). Let X_1, Y_1 be matrices with orthonormal columns and satisfy $\mathscr{R}(X_1) = \mathscr{X}$, $\mathscr{R}(Y_1) = \mathscr{Y}$. Then $AX_1 (X_1^*A^*AX_1)^{-\frac{1}{2}}$ and $AY_1(Y_1^*A^*AY_1)^{-\frac{1}{2}}$ are matrices whose columns are orthonormal and span $A\mathscr{X}$ and $A\mathscr{Y}$, respectively. By Proposition 17,

$$\begin{aligned}\gamma(A\mathscr{X}, A\mathscr{Y}) &= \min_{Q \in \mathbb{C}^{l\times l}} \|AX_1(X_1^*A^*AX_1)^{-\frac{1}{2}} - AY_1(Y_1^*A^*AY_1)^{-\frac{1}{2}}Q\| \\ &= \min_{Q \in \mathbb{C}^{l\times l}} \|A\big[X_1 - Y_1(Y_1^*A^*AY_1)^{-\frac{1}{2}} \cdot Q(X_1^*A^*AX_1)^{\frac{1}{2}}\big](X_1^*A^*AX_1)^{-\frac{1}{2}}\| \\ &\le \|A\|_s \min_{Q \in \mathbb{C}^{l\times l}} \|X_1 - Y_1(Y_1^*A^*AY_1)^{-\frac{1}{2}} \cdot Q(X_1^*A^*AX_1)^{\frac{1}{2}}\| \, \|(X_1^*A^*AX_1)^{-\frac{1}{2}}\|_s.\end{aligned}$$

Since

$$\min_{Q \in \mathbb{C}^{l\times l}} \|X_1 - Y_1(Y_1^*A^*AY_1)^{-\frac{1}{2}}Q(X_1^*A^*AX_1)^{\frac{1}{2}}\| = \min_{Q \in \mathbb{C}^{l\times l}} \|X_1 - Y_1 Q\|$$

and

$$\|(X_1^*A^*AX_1)^{-\frac{1}{2}}\|_s = \underline{\sigma}^{-\frac{1}{2}}(X_1^*A^*AX_1) = \underline{\sigma}^{-1}(AX_1) \le \underline{\sigma}^{-1}(A) = \|A^{-1}\|_s$$

it follows that

$$\gamma(A\mathscr{X}, A\mathscr{Y}) \le \|A\|_s \|A^{-1}\|_s \min_{Q \in \mathbb{C}^{l\times l}} \|X_1 - Y_1 Q\| = \kappa_s(A)\gamma(\mathscr{X}, \mathscr{Y}). \quad □$$

In the special case when A is unitary, Proposition 18 degenerates to Proposition 4 c).

We would like to remark here that inequalities (12) and (13) are not the tightest we can have. However, they are enough for our applications.

Proof of Proposition 10: Since (8) follows from (7), we only need to prove (7).

Let $N_i M_i^{-1}$ be a right-coprime factorization of F_i for $i = 1, 2$. Then $(GN_i)(H^{-1}M_i)^{-1}$ is a right-coprime factorization of GF_iH for $i = 1, 2$.

Now let us fix some s in $\mathbb{C}^+$. The following is obtained:

$$\mathscr{G}(F_i, s) = \mathscr{R}\left(\begin{bmatrix} M_i(s) \\ N_i(s) \end{bmatrix}\right)$$

$$\mathscr{G}(GF_iH, s) = \mathscr{R}\left(\begin{bmatrix} H^{-1}(s)M_i(s) \\ G(s)N_i(s) \end{bmatrix}\right) = \begin{bmatrix} H^{-1}(s) & 0 \\ 0 & G(s) \end{bmatrix} \mathscr{G}(F_i, s)$$

where $i = 1, 2$.

Since

$$\kappa_s\left(\begin{bmatrix} H^{-1}(s) & 0 \\ 0 & G(s) \end{bmatrix}\right) = \max\{\|H^{-1}(s)\|_s, \|G(s)\|_s\} \cdot \max\{\|H(s)\|_s, \|G^{-1}(s)\|_s\}$$

Proposition 18 leads to

$$\gamma[\mathscr{G}(GF_1H, s), \mathscr{G}(GF_2H, s)] \le \max\{\|H^{-1}(s)\|_s, \|G(s)\|_s\}$$

$$\cdot \max\{\|H(s)\|_s, \|G^{-1}(s)\|_s\}$$
$$\cdot \gamma[\mathscr{G}(F_1, s), \mathscr{G}_s(F_2, s)].$$

By taking the supremum over all $s \in \mathbb{C}^+$ on both sides of the above inequality, we obtain (7). □

Proof of Proposition 11: Since (10) follows from (9), we only need to prove (9).

Let $N_i M_i^{-1}$ be a right-coprime factorization of F_i for $i = 1, 2$. Then $(GM_i + N_i)M_i^{-1}$ is a right-coprime factorization of $G + F_i$ for $i = 1, 2$.

Now let us fix some s in $\mathbb{C}^+$. The following is obtained

$$\mathscr{G}(F_i, s) = \mathscr{R}\left(\begin{bmatrix} M_i(s) \\ N_i(s)\end{bmatrix}\right)$$

$$\mathscr{G}(G + F_i, s) = \mathscr{R}\left(\begin{bmatrix} M_i(s) \\ G(s)M_i(s) + N_i(s)\end{bmatrix}\right) = \begin{bmatrix} I & 0 \\ G(s) & I\end{bmatrix}\mathscr{G}(F_i, s)$$

where $i = 1, 2$.

If we can show that

$$\left\|\begin{bmatrix} I & 0 \\ G(s) & I\end{bmatrix}\right\|_s = \left[\frac{2 + \|G(s)\|_s^2 + \|G(s)\|_s\sqrt{4 + \|G(s)\|_s^2}}{2}\right]^{\frac{1}{2}}$$

then the condition number

$$\kappa_s\left(\begin{bmatrix} I & 0 \\ G(s) & I\end{bmatrix}\right) = \left\|\begin{bmatrix} I & 0 \\ G(s) & I\end{bmatrix}\right\|_s \left\|\begin{bmatrix} I & 0 \\ -G(s) & I\end{bmatrix}\right\|_s = \frac{1}{2}\Big(2 + \|G(s)\|_s^2 + \|G(s)\|_s \cdot \sqrt{4 + \|G(s)\|_s^2}\Big)$$

and so it follows from Proposition 18 that

$$\gamma[\mathscr{G}(G + F_1, s), \mathscr{G}(G + F_2, s)] \le \frac{1}{2}\Big(2 + \|G(s)\|_s^2 + \|G(s)\|_s\sqrt{4 + \|G(s)\|_s^2}\Big) \cdot \gamma[\mathscr{G}(F_1, s), \mathscr{G}(F_2, s)].$$

By taking the supremum over all $s \in \mathbb{C}^+$ on the both sides of above inequality, we obtain (9).

What remains to show is that for any $A \in \mathbb{C}^{p\times m}$

$$\left\|\begin{bmatrix} I & 0 \\ A & I\end{bmatrix}\right\|_s = \left(\frac{2 + \|A\|_s^2 + \|A\|_s\sqrt{4 + \|A\|_s^2}}{2}\right)^{\frac{1}{2}}.$$

Let a singular value decomposition of A be given by $U\Sigma V^*$ such that

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{\min\{p, m\}})$$

and $\sigma_1 = \|A\|_s$. Then

$$\begin{bmatrix} I & 0 \\ A & I\end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & U\end{bmatrix}\begin{bmatrix} I & 0 \\ \Sigma & I\end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & U^*\end{bmatrix}$$

which implies that

$$\left\|\begin{bmatrix} I & 0 \\ A & I\end{bmatrix}\right\|_s = \left\|\begin{bmatrix} I & 0 \\ \Sigma & I\end{bmatrix}\right\|_s.$$

By permutating the rows and the columns of the matrix $\begin{bmatrix} I & 0 \\ \Sigma & I\end{bmatrix}$, we can obtain the following block diagonal matrix:

$$\operatorname{diag}(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_{\min\{p, m\}}, 1, \cdots, 1)$$

where $\Upsilon_i = \begin{bmatrix} 1 & 0 \\ \sigma_i & 1\end{bmatrix}$, $i = 1, 2, \cdots, \min\{p, m\}$. Therefore,

$$\left\|\begin{bmatrix} I & 0 \\ A & I\end{bmatrix}\right\|_s = \|\Upsilon_1\|_s = \left(\frac{2 + \sigma_1^2 + \sigma_1\sqrt{4 + \sigma_1^2}}{2}\right)^{\frac{1}{2}}.$$

This completes the proof. □

Appendix B

Proof of Theorem 4

Some preparatory machinery is needed before we prove Theorem 4.

Let NM^{-1} and VU^{-1} be right coprime factorizations of P and C, respectively. It can be shown (see, e.g., [6]) that $H(P, C)$ is stable if and only if

$$\begin{bmatrix} M & V \\ N & U\end{bmatrix}^{-1}$$

is unimodular. Therefore, $H(P, C)$ is stable if and only if

$$\begin{bmatrix} M(s) & V(s) \\ N(s) & U(s)\end{bmatrix}$$

is nonsingular for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$. Let us denote

$$\mathscr{G}'(C, s) = \begin{bmatrix} 0 & I \\ I & 0\end{bmatrix}\mathscr{G}(C, s) = \mathscr{R}\left(\begin{bmatrix} V(s) \\ U(s)\end{bmatrix}\right).$$

Then $H(P, C)$ is stable if and only if

$$\mathscr{G}(P, s) \cap \mathscr{G}'(C, s) = \{0\}$$

for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$.

Lemma 3: Let $\mathscr{X} \in \Gamma_l(\mathbb{C}^n)$ and $\mathscr{Y} \in \Gamma_{n-l}(\mathbb{C}^n)$. Then $\mathscr{X} \cap \mathscr{Y} = \{0\}$ if and only if $\gamma_s(\mathscr{X}, \mathscr{Y}^\perp) < 1$.

Proof: Let X_1, Y_1, and Y_2 be matrices whose columns form orthonormal bases of $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Y}^\perp$, respectively. Then $\mathscr{X} \cap \mathscr{Y} = \{0\}$ if and only if $[X_1\ Y_1]$ is nonsingular. Note that

$$[Y_2\ Y_1]^*[X_1\ Y_1] = \begin{bmatrix} Y_2^* X_1 & 0 \\ Y_1^* X_1 & I\end{bmatrix}$$

and

$$\gamma_s(\mathscr{X}, \mathscr{Y}^\perp) = \sqrt{1 - \underline{\sigma}^2(Y_2^* X_1)}.$$

Therefore, $[X_1\ Y_1]$ is nonsingular if and only if $\gamma_s(\mathscr{X}, \mathscr{Y}^\perp) < 1$. □

Lemma 3 implies that $H(P, C)$ is stable if and only if

$$\gamma_s[\mathscr{G}(P, s), \mathscr{G}'(C, s)^\perp] < 1$$

for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$.

Recall from Section II that $\gamma_s(\mathscr{X}, \mathscr{Y}) = \arcsin \theta_1(\mathscr{X}, \mathscr{Y})$, where $\theta_1(\mathscr{X}, \mathscr{Y})$ is the largest canonical angle between $\mathscr{X}$ and $\mathscr{Y}$. Analogous to the singular value case, let $\bar{\theta}(\mathscr{X}, \mathscr{Y})$ be used to denote $\theta_1(\mathscr{X}, \mathscr{Y})$. It then follows that $H(P, C)$ is stable if and only if

$$\bar{\theta}[\mathscr{G}(P, s), \mathscr{G}'(C, s)^{\perp}] < \frac{\pi}{2}.$$

Our next result relates $\nu(P, C)$ with $\gamma_s[\mathscr{G}(P, s), \mathscr{G}'(C, s)^{\perp}]$.

Proposition 19: Suppose $H(P, C)$ is stable. Then

$$\max_{s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}} \gamma_s[\mathscr{G}(P, s), \mathscr{G}'(C, s)^{\perp}] = \sqrt{1 - \nu^2(P, C)}. \quad (14)$$

Proof: Let NM^{-1} and VU^{-1} be normalized right-coprime factorizations of P and C, respectively. Let $\tilde{U}^{-1}\tilde{V}$ be a normalized left coprime factorization of C. Since

$$[\tilde{U} \ -\tilde{V}]\begin{bmatrix} V \\ U \end{bmatrix} = 0$$

it follows that

$$\mathscr{G}'(C, s)^{\perp} = \mathscr{R}\left(\begin{bmatrix} \tilde{U}^*(s) \\ -\tilde{V}^*(s) \end{bmatrix}\right).$$

Hence, an orthonormal basis of $\mathscr{G}'(C, s)^{\perp}$ is given by the columns of

$$\begin{bmatrix} \tilde{U}^*(s) \\ -\tilde{V}^*(s) \end{bmatrix}[\tilde{U}(s)\tilde{U}^*(s) + \tilde{V}(s)\tilde{V}^*(s)]^{-\frac{1}{2}}$$

and an orthonormal basis of $\mathscr{G}(P, s)$ is given by the columns of

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}[M^*(s)M(s) + N^*(s)N(s)]^{-\frac{1}{2}}.$$

Proposition 4a) leads to

$$\sqrt{1 - \gamma_s^2[\mathscr{G}(P, s), \mathscr{G}'(C, s)^{\perp}]}$$
$$= \underline{\sigma}\left\{[\tilde{U}(s)\tilde{U}^*(s) + \tilde{V}(s)\tilde{V}^*(s)]^{-\frac{1}{2}}[\tilde{U}(s) \ -\tilde{V}(s)] \cdot \begin{bmatrix} M(s) \\ N(s) \end{bmatrix}[M^*(s)M(s) + N^*(s)N(s)]^{-\frac{1}{2}}\right\}.$$

By using the fact that $\|[\tilde{U}(s) \ -\tilde{V}(s)]\|_s \le 1$ and $\left\|\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right\|_s \le 1$ and the fact that $H(P, C)$ is stable if and only if $\tilde{U}M - \tilde{V}N$ is unimodular [21], we obtain

$$\sqrt{1 - \gamma_s^2[\mathscr{G}(P, s), \mathscr{G}'(C, s)^{\perp}]} \ge \underline{\sigma}[\tilde{U}(s)M(s) - \tilde{V}(s)N(s)] \ge \|(\tilde{U}M - \tilde{V}N)^{-1}\|_s^{-1}.$$

Let $\bar{\omega} \in \mathbb{R} \cup \{\infty\}$ such that

$$\underline{\sigma}[\tilde{U}(j\bar{\omega})M(j\bar{\omega}) - \tilde{V}(j\bar{\omega})N(j\bar{\omega})] = \|(\tilde{U}M - \tilde{V}N)^{-1}\|_s^{-1}.$$

Since $\tilde{U}(j\bar{\omega})\tilde{U}^*(j\bar{\omega}) + \tilde{V}(j\bar{\omega})\tilde{V}^*(j\bar{\omega}) = I$ and $U^*(j\bar{\omega})U(j\bar{\omega}) + V^*(j\bar{\omega})V(j\bar{\omega}) = I$, it follows that

$$\sqrt{1 - \gamma_s^2[\mathscr{G}(P, j\bar{\omega}), \mathscr{G}'(C, j\bar{\omega})^{\perp}]} = \underline{\sigma}[\tilde{U}(j\bar{\omega})M(j\bar{\omega}) - \tilde{V}(j\bar{\omega})N(j\bar{\omega})] = \|(\tilde{U}M - \tilde{V}N)^{-1}\|_s^{-1}.$$

Therefore,

$$\sqrt{1 - \max_{s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}} \gamma_s^2[\mathscr{G}_P(s), \mathscr{G}'_C(s)^{\perp}]}$$
$$= \|(\tilde{U}M - \tilde{V}N)^{-1}\|_s^{-1}$$
$$= \left\|\begin{bmatrix} M \\ N \end{bmatrix}(\tilde{U}M - \tilde{V}N)^{-1}[\tilde{U} \ \tilde{V}]\right\|_s^{-1}$$
$$= \left\|\begin{bmatrix} I \\ P \end{bmatrix}(I - CP)^{-1}[I \ C]\right\|_s^{-1}$$
$$= \nu(P, C).$$

This completes the proof. □

Again by using canonical angles, we can restate (14) as

$$\max_{s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}} \bar{\theta}[\mathscr{G}(P, s), \mathscr{G}'(C, s)^{\perp}] = \arccos \nu(P, C).$$

Our next result is the key to the proof of Theorem 4.

Proposition 20: Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \Gamma_l(\mathbb{C}^n)$. Then

$$\bar{\theta}(\mathscr{X}, \mathscr{Z}) \le \bar{\theta}(\mathscr{X}, \mathscr{Y}) + \bar{\theta}(\mathscr{Y}, \mathscr{Z}). \quad (15)$$

Proof: If we assume the following inequality:

$$\gamma_s(\mathscr{Y}, \mathscr{Z}) \ge \left|\gamma_s(\mathscr{X}, \mathscr{Z})\sqrt{1 - \gamma_s^2(\mathscr{X}, \mathscr{Y})} - \gamma_s(\mathscr{X}, \mathscr{Y})\sqrt{1 - \gamma_s^2(\mathscr{X}, \mathscr{Z})}\right| \quad (16)$$

then by applying the "arcsin" function, we obtain

$$\bar{\theta}(\mathscr{Y}, \mathscr{Z}) \ge |\bar{\theta}(\mathscr{X}, \mathscr{Z}) - \bar{\theta}(\mathscr{X}, \mathscr{Y})|$$

which leads to (15).

In the following, we strive for the proof of inequality (16). Let X, Y, Z be matrices with orthonormal columns whose ranges are $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$, respectively. Without loss of generality, we can assume that $X = \begin{bmatrix} I \\ 0 \end{bmatrix}$. Partition Y and Z consistently as $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ and $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$. If $\gamma_s(\mathscr{X}, \mathscr{Y}) = 1$, then $\underline{\sigma}(Y_1) = 0$. It follows that

$$\underline{\sigma}(Z^*Y) = \underline{\sigma}(Z_1^*Y_1 + Z_2^*Y_2) \le \bar{\sigma}(Z_2^*Y_2) \le \bar{\sigma}(Z_2) = \gamma_s(\mathscr{X}, \mathscr{Z}).$$

This shows that

$$\gamma_s(\mathscr{Y}, \mathscr{Z}) = \sqrt{1 - \underline{\sigma}^2(Z^*Y)} \ge \sqrt{1 - \gamma_s^2(\mathscr{X}, \mathscr{Z})},$$

i.e., (16) holds if $\gamma_s(\mathscr{X}, \mathscr{Y}) = 1$. Similarly, we can show that (16) holds if $\gamma_s(\mathscr{X}, \mathscr{Z}) = 1$. Now we assume that $\gamma_s(\mathscr{X}, \mathscr{Y}) < 1$ and $\gamma_s(\mathscr{X}, \mathscr{Z}) < 1$. This implies $\underline{\sigma}(Y_1) > 0$ and $\underline{\sigma}(Z_1) > 0$. We can also assume, without loss of generality, that Y_1 and Z_1 are positive definite Hermitian matrices since if they are not, we can always make them so by multiplying Y and Z from the right by unitary matrices, and such multiplications do not change the ranges of Y and Z. Therefore, we have $Y_1 = (I - Y_2^*Y_2)^{\frac{1}{2}}$ and $Z_1 = (I - Z_2^*Z_2)^{\frac{1}{2}}$. An orthonormal basis of $\mathscr{X}^\perp$ is then given by the columns of $\begin{bmatrix} -Z_2^* \\ (I - Z_2Z_2^*)^{\frac{1}{2}} \end{bmatrix}$. By Proposition 4a)

$$\begin{aligned}
\gamma_s(\mathscr{Y}, \mathscr{Z})
&= \left\| \begin{bmatrix} -Z_2 & (I - Z_2Z_2^*)^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} (I - Y_2^*Y_2)^{\frac{1}{2}} \\ Y_2 \end{bmatrix} \right\|_s \\
&= \|(I - Z_2Z_2^*)^{\frac{1}{2}} Y_2 - Z_2(I - Y_2^*Y_2)^{\frac{1}{2}}\|_s \\
&= \|(I - Z_2Z_2^*)^{\frac{1}{2}} \left[Y_2(I - Y_2^*Y_2)^{-\frac{1}{2}} \right. \\
&\quad \left. - (I - Z_2Z_2^*)^{-\frac{1}{2}} Z_2 \right] (I - Y_2^*Y_2)^{\frac{1}{2}}\|_s \\
&\geq \underline{\sigma}\left[(I - Z_2Z_2^*)^{\frac{1}{2}}\right] \Big| \|Y_2(I - Y_2^*Y_2)^{-\frac{1}{2}}\|_s \\
&\quad - \|(I - Z_2Z_2^*)^{-\frac{1}{2}} Z_2\|_s \Big| \underline{\sigma}\left[(I - Y_2^*Y_2)^{\frac{1}{2}}\right] \\
&= \underline{\sigma}(Z_1) \left| \frac{\bar{\sigma}(Y_2)}{\sqrt{1 - \bar{\sigma}^2(Y_2)}} - \frac{\bar{\sigma}(Z_2)}{\sqrt{1 - \bar{\sigma}^2(Z_2)}} \right| \underline{\sigma}(Y_1) \\
&= \underline{\sigma}(Z_1) \left| \frac{\sqrt{1 - \underline{\sigma}^2(Y_1)}}{\underline{\sigma}(Y_1)} - \frac{\sqrt{1 - \underline{\sigma}^2(Z_1)}}{\underline{\sigma}(Z_1)} \right| \underline{\sigma}(Y_1) \\
&= \left| \underline{\sigma}(Z_1)\sqrt{1 - \underline{\sigma}^2(Y_1)} - \underline{\sigma}(Y_1)\sqrt{1 - \underline{\sigma}^2(Z_1)} \right| \\
&= \Big| \gamma_s(\mathscr{X}, \mathscr{Z})\sqrt{1 - \gamma_s^2(\mathscr{X}, \mathscr{Y})} \\
&\quad - \gamma_s(\mathscr{X}, \mathscr{Y})\sqrt{1 - \gamma_s^2(\mathscr{X}, \mathscr{Z})} \Big|.
\end{aligned}$$

This completes the proof. □

Finally, two more lemmas.

Lemma 4: Let $X, \Delta X \in \mathbb{C}^{n \times m}$ satisfy $\text{rank}(X) = m$ and $\|\Delta X\| < \underline{\sigma}(X)$. Then

$$\gamma[\mathscr{R}(X + \Delta X), \mathscr{R}(X)] \leq \frac{\|\Delta X\|}{\underline{\sigma}(X)}.$$

Proof: Let Y be a matrix with orthonormal columns and $\mathscr{R}(Y) = \mathscr{R}(X + \Delta X)^\perp$. By Proposition 4a), we obtain

$$\begin{aligned}
\gamma[\mathscr{R}(X + \Delta X), \mathscr{R}(X)]
&= \|Y^*X(X^*X)^{-\frac{1}{2}}\| \\
&= \|Y^*(X + \Delta X - \Delta X)(X^*X)^{-\frac{1}{2}}\| \\
&= \|Y^*\Delta X(X^*X)^{-\frac{1}{2}}\| \\
&\leq \bar{\sigma}(Y^*)\|\Delta X\|\bar{\sigma}\left[(X^*X)^{-\frac{1}{2}}\right] \\
&= \frac{\|\Delta X\|}{\underline{\sigma}(X)}.
\end{aligned}$$

□

Lemma 5 [8]: Assume that (P, C) is a stable pair, and let NM^{-1} be a normalized right-coprime factorization of P and let $\tilde{U}^{-1}\tilde{V}$ be a normalized left-coprime factorization of C. Then

$$\nu(P, C) \leq \inf_{s \in \mathbb{C}^+} \underline{\sigma}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right)$$

and

$$\nu(P, C) \leq \inf_{s \in \mathbb{C}^+} \underline{\sigma}\left(\begin{bmatrix} \tilde{U}(s) & \tilde{V}(s) \end{bmatrix}\right).$$

Proof of Theorem 4: We will first prove the equivalence of a) and d), and then observe how the equivalence of a), b), c) follows. Note that for each pair $r_1, r_2 \in [0, 1]$

$$r_1^2 + r_2^2 + 2r_1r_2\sqrt{1 - \nu^2(P_0, C_0)} \leq \nu^2(P_0, C_0)$$

if and only if

$$\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0) \leq \frac{\pi}{2}.$$

I a) ⇒ d): Assume that γ_1 and γ_2 are the gaps corresponding to the same unitarily invariant matrix norms as δ_1 and δ_2, respectively. For each (P, C) with $P \in \mathscr{B}_1(P_0, r_1)$ and $C \in \mathscr{B}_2(C_0, r_2)$

$$\gamma_s[\mathscr{G}(P, s), \mathscr{G}(P_0, s)] \leq \gamma_1[\mathscr{G}(P, s), \mathscr{G}(P_0, s)] < r_1$$

and

$$\begin{aligned}
\gamma_s[\mathscr{G}'(C, s)^\perp, \mathscr{G}'(C_0, s)^\perp]
&= \gamma_s[\mathscr{G}(C, s), \mathscr{G}(C_0, s)] \\
&\leq \gamma_2[\mathscr{G}(C, s), \mathscr{G}(C_0, s)] < r_2
\end{aligned}$$

for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$. It follows from Propositions 19 and 20 that

$$\begin{aligned}
\bar{\theta}[\mathscr{G}(P, s), \mathscr{G}'(C, s)^\perp]
&\leq \bar{\theta}[\mathscr{G}(P, s), \mathscr{G}(P_0, s)] \\
&\quad + \bar{\theta}[\mathscr{G}(P_0, s), \mathscr{G}'(C_0, s)^\perp] \\
&\quad + \bar{\theta}[\mathscr{G}'(C, s)^\perp, \mathscr{G}'(C_0, s)^\perp] \\
&< \arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0)
\end{aligned}$$

for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$. Therefore, $H(P, C)$ is stable if

$$\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0) \leq \frac{\pi}{2}.$$

II d) ⇒ a): Assume that a pair (r_1, r_2) is given which satisfies

$$\arcsin r_1 + \arcsin r_2 + \arccos \nu(P_0, C_0) > \frac{\pi}{2}.$$

We have to show that there exist $P \in \mathscr{B}_1(P_0, r_1)$ and $C \in \mathscr{B}_2(C_0, r_2)$ such that (P, C) is unstable. The assumption made implies that there exist t_1 and t_2 with $0 < t_1 < r_1$ and

$0 < t_2 < r_2$ such that

$$\arcsin t_1 + \arcsin t_2 + \arccos \nu(P_0, C_0) = \frac{\pi}{2}.$$

Let

$$\begin{aligned} \phi_0 &= \arccos \nu(P_0, C_0) \\ \phi_1 &= \arcsin t_1 \\ \phi_2 &= \arcsin t_2. \end{aligned}$$

Then we have

$$\phi_0 + \phi_1 + \phi_2 = \frac{\pi}{2}.$$

Let $N_0 M_0^{-1}$ be a normalized right coprime factorization of P_0 and let $\tilde{U}_0^{-1} \tilde{V}_0$ be a normalized left coprime factorization of C_0. Since

$$\begin{aligned} \nu(P_0, C_0) &= \|(\tilde{U}_0 M_0 - \tilde{V}_0 N_0)^{-1}\|_\infty^{-1} \\ &= \inf_{\omega \in \mathbb{R}} \underline{\sigma}[\tilde{U}_0(j\omega) M_0(j\omega) - \tilde{V}_0(j\omega) N_0(j\omega)] \end{aligned}$$

there must exist $\bar{\omega} \in [0, \infty]$ such that

$$\underline{\sigma}[\tilde{U}_0(j\bar{\omega}) M_0(j\bar{\omega}) - \tilde{V}_0(j\bar{\omega}) N_0(j\bar{\omega})] = \nu(P_0, C_0).$$

Let a singular value decomposition of $\begin{bmatrix} M_0(j\bar{\omega}) \\ N_0(j\bar{\omega}) \end{bmatrix}$ be given by

$$\begin{bmatrix} M_0(j\bar{\omega}) \\ N_0(j\bar{\omega}) \end{bmatrix} = X \begin{bmatrix} I \\ 0 \end{bmatrix} Y^*$$

where X, Y are unitary matrices. In this case, we can always choose X, Y and an additional unitary matrix Z such that

$$Z^*[\tilde{U}_0(j\bar{\omega}) \quad -\tilde{V}_0(j\bar{\omega})] X = [A \quad B]$$

where $A = \mathrm{diag}\,(a_1, \ a_2, \cdots, a_m) \in \mathbb{R}^{m \times m}$, $B = \mathrm{diag}\left(\sqrt{1 - a_1^2}, \sqrt{1 - a_2^2}, \cdots, \sqrt{1 - a_{\min\{p, m\}}^2}\right) \in \mathbb{R}^{m \times p}$ and $a_1 = \underline{\sigma}(A)$. Then

$$\begin{aligned} a_1 &= \underline{\sigma}(A) = \underline{\sigma}(ZAY^*) \\ &= \underline{\sigma}[\tilde{U}_0(j\bar{\omega}) M_0(j\bar{\omega}) - \tilde{V}_0(j\bar{\omega}) N_0(j\bar{\omega})] \\ &= \nu(P_0, C_0) = \cos \phi_0. \end{aligned}$$

Let

$$W = \left[\begin{array}{cccc|cccc} \sin \phi_1 & 0 & \cdots & 0 & \cos \phi_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline \cos \phi_1 & 0 & \cdots & 0 & -\sin \phi_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{array}\right]$$

which is clearly a real orthogonal matrix. Then

$$W \begin{bmatrix} I \\ 0 \end{bmatrix} = \left[\begin{array}{cccc} \sin \phi_1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline \cos \phi_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{array}\right]$$

and

$$[A \quad B] W' = \left[\begin{array}{cccc|cccc} \sqrt{1 - x^2} & 0 & \cdots & 0 & x & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 & 0 & \sqrt{1 - a_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & \cdot \end{array}\right]$$

where

$$x = \cos\phi_0 \cos\phi_1 - \sin\phi_0 \sin\phi_1.$$

Hence, $x = \cos(\phi_0 + \phi_1) = \sin\phi_2$.

For a fixed number $\bar{\omega} \in (0, \infty)$, define a map $\psi_{\bar{\omega}}$ from $\mathbb{C}$ to real rational functions by

$$\psi_{\bar{\omega}}(\lambda) = \begin{cases} \alpha \dfrac{s-\beta}{s+\beta} & \text{if } \lambda \text{ is not real} \\ \lambda & \text{if } \lambda \text{ is real} \end{cases}$$

where $\alpha \in \mathbb{R}$ and $\beta \in (0, \infty)$ are determined from

$$\lambda = \alpha \frac{j\bar{\omega} - \beta}{j\bar{\omega} + \beta}.$$

This map maps a complex number λ to a stable all-pass real rational function whose value at $\bar{\omega}$ is λ. In the following, when the map $\psi_{\bar{\omega}}$ is applied to a matrix, we assume that it is an elementwise map.

Let u_1 be the first column of XW', u_2 the first row of Y^*, u_3 the first column of Z, and u_4 the $(m+1)$th row of WX^*. If $\bar{\omega}$ is 0 or ∞, they can be made all real.

Let

$$\Delta(s) = \begin{cases} u_1 \sin\phi_1 \left(\dfrac{s-1}{s+1}\right)^{4n} u_2 & \text{if } \bar{\omega} = 0 \text{ or } \infty \\ \psi_{\bar{\omega}}(u_1) \sin\phi_1 \left(\dfrac{s-\bar{\omega}}{s+\bar{\omega}}\right)^{4n} \psi_{\bar{\omega}}(u_2) & \text{if } \bar{\omega} \in (0, \infty) \end{cases}$$

$$\tilde{\Delta}(s) = \begin{cases} u_3 \sin\phi_2 \left(\dfrac{s-1}{s+1}\right)^{4n} u_4 & \text{if } \bar{\omega} = 0 \text{ or } \infty \\ \psi_{\bar{\omega}}(u_3) \sin\phi_2 \left(\dfrac{s-\bar{\omega}}{s+\bar{\omega}}\right)^{4n} \psi_{\bar{\omega}}(u_4) & \text{if } \bar{\omega} \in (0, \infty) \end{cases}$$

and let

$$\begin{bmatrix} M(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix} - \Delta(s)$$

$$[\tilde{U}(s) \ \ -\tilde{V}(s)] = [\tilde{U}_0(s) \ \ -\tilde{V}_0(s)] - \tilde{\Delta}(s).$$

Then there exists $K_1 > 0$ such that M and $\tilde{U}$ are invertible for all $n > K_1$. Choose $n > K_1$. We know from Lemma 5 that for each $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$

$$\|\Delta(s)\|_s \le \sin\phi_1 = t_1 < \nu(P_0, C_0) \le \inf_{s\in\mathbb{C}^+} \underline{\sigma}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right)$$

and

$$\|\tilde{\Delta}(s)\|_s \le \sin\phi_2 = t_2 < \nu(P_0, C_0) \le \inf_{s\in\mathbb{C}^+} \underline{\sigma}([\tilde{U}_0(s) \ \ -\tilde{V}_0(s)])$$

which imply that $\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}$ has full column rank and $[\tilde{U}(s) - \tilde{V}(s)]$ has full row rank for all $s \in \mathbb{C}^0 \cup \mathbb{C}^+ \cup \{\infty\}$. Hence, M, N are right-coprime and $\tilde{U}$, $\tilde{V}$ are left-coprime.

Let $P = NM^{-1}$ and $C = \tilde{U}^{-1}\tilde{V}$. Since

$$[\tilde{U}(j\bar{\omega}) \ \ -\tilde{V}(j\bar{\omega})] \begin{bmatrix} M(j\bar{\omega}) \\ N(j\bar{\omega}) \end{bmatrix}$$

is singular, it follows that $H(P, C)$ is unstable.

In this case, by Lemma 4, we have

$$\begin{aligned} &\gamma_1[\mathscr{G}(P, s), \mathscr{G}(P_0, s)] \\ &= \gamma_1\left[\mathscr{R}\left(\begin{bmatrix} M(s) \\ N(s) \end{bmatrix}\right), \mathscr{R}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right)\right] \\ &\le \begin{cases} \left|\sin\phi_1 \left(\dfrac{s-1}{s+1}\right)^{4n}\right| \underline{\sigma}^{-1}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right) & \text{if } \bar{\omega} = 0 \text{ or } \infty \\ \left|\sin\phi_1 \left(\dfrac{s-\bar{\omega}}{s+\bar{\omega}}\right)^{4n}\right| \underline{\sigma}^{-1}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right) & \text{if } \bar{\omega} \in (0, \infty) \end{cases} \end{aligned}$$

where γ_1 is the gap corresponding to the same unitarily invariant norm as δ_1. Since $\underline{\sigma}\left(\begin{bmatrix} M_0(j\omega) \\ N_0(j\omega) \end{bmatrix}\right) = 1$ for all $\omega \in [0, \infty]$, there exists an open set $\mathscr{O}$ in the Riemann sphere such that $\{j\omega: \ \omega \in [-\infty, \infty]\} \subset \mathscr{O}$ and $\underline{\sigma}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right) > 2\sin\phi_1/(r_1 + \sin\phi_1)$ for all $s \in \mathbb{C}^+ \cap \mathscr{O}$. This implies that $\gamma_1[\mathscr{G}(P, s), \mathscr{G}(P_0, s)] < (r_1 + \sin\phi_1)/2$ for all $s \in \mathbb{C}^+ \cap \mathscr{O}$. On the other hand, since $\inf_{s\in\mathbb{C}^+} \underline{\sigma}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right) > 0$, there exists $K_2 > 0$ such that if $n > K_2$, then either

$$\left|\left(\frac{s-1}{s+1}\right)^{4n}\right| \underline{\sigma}^{-1}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right) < 1$$

(for the case $\bar{\omega} = 0$ or ∞) or

$$\left|\left(\frac{s-\bar{\omega}}{s+\bar{\omega}}\right)^{4n}\right| \underline{\sigma}^{-1}\left(\begin{bmatrix} M_0(s) \\ N_0(s) \end{bmatrix}\right) < 1$$

(for the case $\bar{\omega} \in (0, \infty)$) for all $s \in \mathbb{C}^+ \setminus \mathscr{O}$. This implies $\gamma_1[\mathscr{G}(P, s), \mathscr{G}(P_0, s)] < \sin\phi_1$ for all $s \in \mathbb{C}^+ \setminus \mathscr{O}$. Therefore, there exists $K_2 > 0$ such that $\delta_1(P, P_0) \le (r_1 + \sin\phi_1)/2 < r_1$ for $n > K_2$.

Similarly, we can show that there exist $K_3 > 0$ such that $\delta_2(C, C_0) = \delta_2(-C', -C_0') < r_2$ for $n > K_3$.

Consequently, if we choose $n > \max\{K_1, K_2, K_3\}$, then we have shown that $H(P, C)$ is unstable, $\delta_1(P, P_0) < r_1$, and $\delta_2(C, C_0) < r_2$.

We have proved a) $\Leftrightarrow$ d). The two balls used in d) are open balls. If we replace one of them by a closed ball, the equivalence is still valid. We can verify this by simple modification of the proof of a) $\Rightarrow$ d); d) $\Rightarrow$ a) remains true automatically. If we let the closed ball have radius zero, then we obtain the equivalence of a), b), and c), which completes the proof. □

Acknowledgment

The authors would like to thank Prof. B. Francis for providing some key references and Prof. C. Davis for helpful discussions.

References

[1] Å. Björck and G. H. Golub, "Numerical method for computing angles between linear subspaces," *Math. Comput.*, vol. 27, pp. 579-593, 1973.

[2] C. Davis and W. M. Kahan, "The rotation of eigenvectors by a perturbation. III," *SIAM J. Numer. Anal.*, vol. 7, pp. 1-46, 1970.

[3] A. K. El-Sakkary, "The gap metric: Robustness of stabilization of feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 240-247, 1985.

[4] ____, "Estimating robustness on the Riemann sphere," *Int. J. Contr.*, vol. 49, pp. 561-567, 1989.

[5] B. A. Francis, "On robustness of the stability of feedback systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 817-818, 1980.

[6] ____, *A Course in $\mathscr{H}_\infty$ Control Theory*. New York: Springer-Verlag, 1986.

[7] T. T. Georgiou, "On the computation of the gap metric," *Syst. Contr. Lett.*, vol. 11, pp. 253-257, 1988.

[8] T. T. Georgiou and M. C. Smith, "Optimal robustness in the gap metric," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 673-685, 1990.

[9] K. Glover and D. C. McFarlane, "Robust stabilization of normalized coprime factor plant description with $\mathscr{H}_\infty$-bounded uncertainty," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 821-830, 1989.

[10] I. Gohberg, P. Lancaster, and L. Rodman, *Matrix Polynomials*. New York: Academic, 1982.

[11] E. Hille, *Analytic Function Theory, Vol. 1*. New York: Blaisdell, 1959.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, England: Cambridge University Press, 1985.

[13] T. Kato, *Perturbation Theory for Linear Operator*. New York: Springer-Verlag, 1966.

[14] D. C. McFarlane and K. Glover, *Robust Controller Design Using Normalized Coprime Factor Plant Descriptions*. Berlin: Springer-Verlag, 1990.

[15] L. Qiu, "Tools for quantitative stability robustness analysis," Ph.D. dissertation, Univ. Toronto, 1990.

[16] L. Qiu and E. J. Davison, "Feedback stability under simultaneous Gap metric uncertainties in plant and controller," *Syst. Contr. Lett.*, vol. 11, pp. 9-12, 1992.

[17] S. Rolewicz, *Metric Linear Spaces*. Dordrecht, Holland: Reidel, and Warszawa, Poland: PWN-Polish, 1985.

[18] H. L. Royden, *Real Analysis*. New York: MacMillan, 1968.

[19] G. W. Stewart and J. G. Sun, *Matrix Perturbation Theory*. San Diego: Academic, 1990.

[20] M. Vidyasagar, "The graph metric for unstable plants and robustness estimates for feedback stability," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 403-418, 1984.

[21] ____, *Control System Synthesis: A Factorization Approach*. Cambridge, MA: M.I.T. Press, 1985.

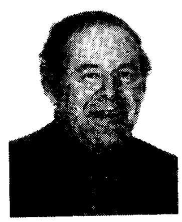
[22] M. Vidyasagar, H. Schneider, and B. A. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 880-894, 1982.

[23] G. Zames and A. K. El-Sakkary, "Unstable systems and feedback: the gap metric," in *Proc. 16th Allerton Conf.*, 1980, pp. 380-385.

[24] S. Q. Zhu, "Graph topology and gap topology for unstable systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 848-855, 1989.

Li Qiu (S'85-M'91) received the B.Eng. degree in electrical engineering from Hunan University, Changsha, Hunan, China, in 1981, and the M.A.Sc. and Ph.D. degrees in electrical engineering from the University of Toronto, Toronto, Ont., Canada, in 1987 and 1990, respectively.

From 1982 to 1984, he worked as a full time Teaching Assistant in Hunan University. Since 1990, he has held research positions in the University of Toronto and the Canadian Space Agency. At present, he is holding a joint postdoctoral position in the Department of Electrical and Computer Engineering and the Fields Institute for Research in Mathematical Sciences, the University of Waterloo, Waterloo, Ont., Canada. His current research interests include linear and nonlinear control theory, robust control, control of flexible structures, and numerical issues in control.



Edward J. Davison (S'60-M'66-SM'76-F'78) received the Ph.D. degree and the Sc.D. degree from Cambridge University, Cambridge, England, in 1964 and 1977, respectively.

At present he is Professor of Electrical Engineering at the University of Toronto.

Dr. Davison has received several awards including the National Research Council of Canada's E.W.R. Steacie Memorial Fellowship 1974-1977, the Canada Council Killam Research Fellowship 1979-1980, 1981-1983, the Athlone Fellowship (Cambridge University) 1961-1963, two IEEE Transactions on Automatic Control Outstanding Paper Awards, and a Current Contents Classic Paper Citation. He was elected a Fellow of the Royal Society of Canada in 1977, an Honorary Professor of Beijing Institute of Aeronautics and Astronautics in 1986, and is a designated Consulting Engineer of the Province of Ontario since 1979. He has served on numerous positions in the IEEE Control Systems Society including President in 1983 and Consulting Editor of the IEEE Transactions on Automatic Control, in 1985. He has served as Associate Editor of the IEEE Transactions on Automatic Control, *Automatica*, *Large Scale Systems: Theory and Application*, and as Chairman of the IFAC Theory Committee. He presently is Vice-Chairman of the IFAC Technical Board and a member of the IFAC Council, and serves on the editorial board of a number of journals.