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Pre-Classical Tools for Post-Modern Control

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Abstract— **In this paper, we present a systematic optimal and robust control theory (for SISO systems) in a language suitable for undergraduate teaching. The tools used are mostly simple polynomial arithmetics and elementary linear algebra. The theory covers almost all topics that post-modern control concerns: the computation of the RMS value (2-norm) of a signal or a system, the computation of the resonance peak (∞-norm) of a system, and the computation of the Hankel singular values and vectors, optimal transient stabilization (LQG control), optimal robust stabilization with respect to the Vinnicombe metric ($\mathcal{H}_\infty$ control), and system approximation. A systematic synthesis theory is presented based on the pole placement technique, which is equivalent to solving a polynomial Diophantine equation.**

I. Introduction

Recently we have been witnessing a great amount of attention paid to the innovation of undergraduate level control education. Several new textbooks have been published ([5], [14], [11], [7], [4]). The main effort seems to be in incorporating modern and post-modern control theory into the syllabus of a beginners' control course which has been dominated by classical materials for several decades. This effort is not easy and is potentially controversial because of the myth that the modern and post-modern control theory necessitates the use of advanced mathematical knowledge which a typical engineering undergraduate student does not have.

In this paper, we will examine a systematic control theory tailored for today's undergraduate level education. The theory is based on the post-modern philosophy, emphasizing analyticity, optimality, robustness, CAD suitability, and rigor, but uses pre-classical tools not much beyond the well-known Routh stability criterion.

Making available such a theory enables the advanced optimal and robust control of SISO systems, described by transfer functions, to be taught and applied using mostly polynomial arithmetics understandable by students and engineers with minimum mathematical sophistication. It changes the common perception that classical theory is associated with trial-and-error designs and approximate reasoning. It also demystifies the post-modern control theory and the advanced mathematics associated with it. It is our belief that materials for undergraduate teaching should be better connected with the most recent development in control theory and that one of the main reasons of the widening gap between control theory and control practice is the widening gap between theoretical development and the education.

Although by large the materials in this paper are not new, the treatment of almost all analysis and synthesis problems using certain orthonormal functions generated from the Routh table does have certain technical novelty. Our main purpose is to demonstrate the availability of simple systematic solutions to some of the standard analysis and synthesis problems in control. The motivation, interpretation, and the connections of these problems to practical control problems are not to be emphasized in this paper, though they serve as important parts of a comprehensive control course.

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II. Routh Stability Test and Orthonormal Functions

Consider polynomial

$$a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n, \qquad a_0 > 0.$$

Construct the Routh table

$$\begin{array}{c|llll}
s^n & r_{00} = a_0 & r_{01} = a_2 & r_{02} = a_4 & \cdots \\
s^{n-1} & r_{10} = a_1 & r_{11} = a_3 & r_{12} = a_5 & \cdots \\
s^{n-2} & r_{20} & r_{21} & r_{22} & \cdots \\
s^{n-3} & r_{30} & r_{31} & r_{32} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
s^2 & r_{(n-2)0} & r_{(n-2)1} & & \\
s^1 & r_{(n-1)0} & & & \\
s^0 & r_{n0} & & &
\end{array}$$

Each row starting from the third one is computed from its two preceding rows as

$$r_{ij} = -\frac{1}{r_{(i-1)0}} \begin{vmatrix} r_{(i-2)0} & r_{(i-2)(j+1)} \\ r_{(i-1)0} & r_{(i-1)(j+1)} \end{vmatrix}.$$

Here i goes from 2 to n and j goes from 0 to $\lfloor \frac{n-i}{2} \rfloor$. When computing the last element of certain row of the Routh table, one may find that the preceding row is one element short of what we need. For example, when we compute r_{n0}, we need $r_{(n-1)1}$ but $r_{(n-1)1}$ is not an element of the Routh table. In this case, we can simply augment the preceding row by a 0 in the end and keep the computation going. Keep in mind that this augmented 0 is not considered as part of the Routh table. Equivalently, whenever $r_{(i-1)(j+1)}$ is missing, simply let $r_{ij} = r_{(i-2)(j+1)}$. For example, r_{n0} is can be computed as

$$r_{n0} = -\frac{1}{r_{(n-1)0}} \begin{vmatrix} r_{(n-2)0} & r_{(n-2)1} \\ r_{(n-1)0} & 0 \end{vmatrix} = r_{(n-2)1}.$$

Theorem 1 (Routh Stability Criterion) *The following statements are equivalent:*

1) *$a(s)$ is stable.*
2) *All elements of the Routh table are positive, i.e., $r_{ij} > 0$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, \lfloor \frac{n-i}{2} \rfloor$.*
3) *All elements in the first column of the Routh table are positive, i.e., $r_{i0} > 0$, $i = 0, 1, \ldots, n$.*

The proof given by Routh is quite involved and is usually omitted in feedback control textbooks. There have been continuous efforts in finding simpler proofs. It appears that the proof given in [2] uses the most elementary arguments and is the most easily understandable.

Let $x(t), y(t)$, $t \geq 0$, be a two signal. Their inner product is defined as

$$\langle x(t), y(t) \rangle = \int_0^\infty x(t) y(t) dt.$$

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The RMS value or 2-norm of $x(t)$ is then defined as

$$\|x(t)\|_2 = \langle x(t), x(t)\rangle^{1/2} = \left[\int_0^\infty x^2(t)dt\right]^{1/2}.$$

Let $X(s)$ and $Y(s)$ be the Laplace transforms of $x(t)$ and $y(t)$. Their inner product is defined as

$$\langle X(s), Y(s)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(-j\omega)Y(j\omega)d\omega.$$

The RMS value or 2-norm of $X(s)$ is defined as

$$\|X(s)\|_2 = \langle X(s), X(s)\rangle^{1/2} = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega\right]^{1/2}.$$

Assume $x(t), y(t)$ have finite RMS values. The Parseval's identity tells us that

$$\langle x(t), y(t)\rangle = \langle X(s), Y(s)\rangle$$

and

$$\|x(t)\|_2 = \|X(s)\|_2.$$

The RMS value or 2-norm of a strictly proper stable system is defined to be that of its impulse response or transfer function.

Let us now fix a stable polynomial

$$a(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n, \quad a_0 > 0.$$

Consider the set of signals or systems

$$\mathcal{S}_{a(s)} = \left\{\frac{x_1 s^{n-1} + \cdots + x_{n-1}s + x_n}{a(s)} : x_i \in \mathbb{R}, i = 1, \ldots, n\right\}.$$

This set is clearly an n-dimensional vector space with an inner product inherited from that of arbitrary signals. An orthonormal basis of this space will be instrumental in the development later.

Let us construct the Routh table of $a(s)$. Since $a(s)$ is stable, the Routh table can always be constructed to the end and all $r_{i0}, i = 0, 1, \ldots, n$, are positive. For each row (except the first one) of the Routh table, define a polynomial

$$\begin{aligned} r_1(s) &= r_{10}s^{n-1} + r_{11}s^{n-3} + \cdots \\ r_2(s) &= r_{20}s^{n-2} + r_{21}s^{n-4} + \cdots \\ &\vdots \\ r_{n-1}(s) &= r_{(n-1)0}s \\ r_n(s) &= r_{n0}. \end{aligned}$$

Also define

$$\alpha_i = \frac{r_{(i-1)0}}{r_{i0}}, \quad i = 1, 2, \ldots, n.$$

Theorem 2 *The functions*

$$B_i(s) = \sqrt{2\alpha_i}\frac{r_i(s)}{a(s)}, \quad i = 1, 2, \ldots, n,$$

form an orthonormal basis of $\mathcal{S}_{a(s)}$.

This basis $\{B_i(s) : i = 1, 2, \ldots, n\}$ will be called the Routh basis of $\mathcal{S}_{a(s)}$. It appears that this basis was first discovered by [8] based on the technique in [2] for the computation of the RMS value, which is to be covered in the next section. It can also be shown that this orthonormal basis is exactly the one obtained by carrying out the Gram-Schmidt orthonormalization of the standard basis

$$\left\{\frac{1}{a(s)}, \frac{s}{a(s)}, \ldots, \frac{s^{n-1}}{a(s)}\right\}.$$

III. Computation of the RMS value

Given a strictly proper stable signal or system

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n-1} + \cdots + b_n}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}, \quad a_0 > 0.$$

Clearly $G(s) \in \mathcal{S}_{a(s)}$.

If we expand $b(s)$ as

$$b(s) = \beta_1 r_1(s) + \beta_2 r_2(s) + \cdots + \beta_n r_n(s), \tag{1}$$

then

$$G(s) = \frac{\beta_1}{\sqrt{2\alpha_1}}B_1(s) + \frac{\beta_2}{\sqrt{2\alpha_2}}B_2(s) + \cdots + \frac{\beta_n}{\sqrt{2\alpha_n}}B_n(s).$$

Consequently

$$\|G(s)\|_2^2 = \frac{\beta_1^2}{2\alpha_1} + \frac{\beta_2^2}{2\alpha_2} + \cdots + \frac{\beta_n^2}{2\alpha_n}.$$

It seems that finding all β_i requires solving a set of linear equations obtained by comparing the coefficients in (1). Actually, these equations have special structure which leads to a tabular solution. Construct the augmented Routh table:

r_{00}	r_{01}	$\cdots$	q_{00}	q_{01}	$\cdots$			
r_{10}	r_{11}	$\cdots$	r_{10}	r_{11}	$\cdots$	q_{10}	q_{11}	$\cdots$
r_{20}	r_{21}	$\cdots$	q_{20}	q_{21}	$\cdots$	r_{20}	r_{21}	$\cdots$
r_{30}	r_{31}	$\cdots$	r_{30}	r_{31}	$\cdots$	q_{30}	q_{31}	$\cdots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	
r_{n0}								

The augmented Routh table is formed by adding two blocks to the right of the usual Routh table. The first added block (the middle block of the augmented Routh table) is constructed in the following way: the first row is directly from the coefficients $b_1, b_3, b_5, \ldots$, of $b(s)$, i.e.,

$$q_{00} = b_1, q_{01} = b_3, q_{02} = b_5, \ldots;$$

the second, forth, sixth, ..., rows are copied from the corresponding rows of the Routh table; the third, fifth, seventh, ..., rows are obtained from their preceding two rows in exactly the same way as the rows of the Routh table:

$$q_{ij} = -\frac{1}{r_{(i-1)0}}\begin{vmatrix} q_{(i-2)0} & q_{(i-2)(j+1)} \\ r_{(i-1)0} & r_{(i-1)(j+1)} \end{vmatrix}. \tag{2}$$

The second added block (the right block of the augmented Routh table) is constructed in the following way: the first row is irrelevant; the second row is directly from the coefficients $b_2, b_4, b_6, \ldots$, of $b(s)$, i.e.,

$$q_{10} = b_2, q_{11} = b_4, q_{12} = b_6, \ldots;$$

the third, fifth, seventh, ..., rows are copied from the corresponding rows of the Routh table, the forth, sixth, eighth, ..., rows are obtained from their preceding two rows in exactly the same way as the rows of the Routh table using formula (2).

In summary, the following algorithm gives the 2-norm of a stable strictly proper transfer function.

Algorithm 1

Step 1 *Compute the augmented Routh table of $G(s)$.*

Step 2 *Set $\alpha_i = \frac{r_{(i-1)0}}{r_{i0}}$ and $\beta_i = \frac{q_{(i-1)0}}{r_{i0}}$.*

Step 3 $\|G(s)\|_2 = \left(\frac{\beta_1^2}{2\alpha_1} + \frac{\beta_2^2}{2\alpha_2} + \cdots + \frac{\beta_n^2}{2\alpha_n}\right)^{1/2}$.

The effort to find a simple method to compute the RMS value of a transfer function started in the late 40's by a group in MIT. The initial effort ended up with formulas for transfer functions up to 7th order, reported in [9]. Another team effort was carried out in the 50's by another group in MIT. This effort, documented in [12], led to an algorithm based on matrix equation for arbitrarily high order transfer functions and corrections to two formulas in [9]. Algorithm 1 in this section is not new and first appeared in [2]. What is new here is the observation that this algorithm directly follows from the availability of an orthonormal basis of $\mathcal{S}_{a(s)}$.

IV. Computation of the Resonance Peak

For a proper system

$$G(s)=\frac{b(s)}{a(s)}=\frac{b_0 s^n+b_1 s^{n-1}+\cdots+b_n}{a_0 s^n+a_1 s^{n-1}+\cdots+a_n}.$$

Its resonance peak or ∞-norm is defined as

$$\|G(s)\|_\infty=\sup_{\omega\in\mathbb{R}}\left|\frac{b(j\omega)}{a(j\omega)}\right|.$$

Compared to the computation of $\|G(s)\|_2$, that of $\|G(s)\|_\infty$ is really a trivial matter. One can read it from the Bode diagram of $G(s)$ or use the standard procedure for optimizing a univariate function that we learn from calculus.

V. Hankel Singular Values and Vectors

Given a proper stable transfer function:

$$G(s)=\frac{b(s)}{a(s)},$$

take a function $\frac{x(s)}{a(s)}$ in $\mathcal{S}_{a(s)}$. Then

$$G(s)\frac{x(-s)}{a(-s)}=\frac{b(s)x(-s)}{a(s)a(-s)}$$

is a strictly proper rational function with poles at the roots of $a(s)$ and their mirror images respect to the imaginary axis. This rational function can be uniquely decomposed into

$$\frac{b(s)x(-s)}{a(s)a(-s)}=\frac{y(s)}{a(s)}+\frac{z(s)}{a(-s)}$$

where both terms on the right hand side are strictly proper. Throw away the unstable term $\frac{z(s)}{a(-s)}$. Then we are left with the stable term $\frac{y(s)}{a(s)}$ which belongs to $\mathcal{S}_{a(s)}$. This process defines a map from $\mathcal{S}_{a(s)}$ to $\mathcal{S}_{a(s)}$:

$$\frac{x(s)}{a(s)}\mapsto\frac{y(s)}{a(s)}.$$

This map is clearly a linear transformation on $\mathcal{S}_{a(s)}$. We call it the Hankel operator with symbol $G(s)$, denoted by $H_{G(s)}$.

A proper $G(s)=\frac{b(s)}{a(s)}$ can in general be decomposed as the sum of a constant term and a strictly proper term

$$G(s)=d+\frac{c(s)}{a(s)}$$

where $d=G(\infty)$ and $c(s)=b(s)-G(\infty)a(s)$. It can be easily seen that

$$H_{G(s)}\frac{x(s)}{a(s)}=H_{\frac{c(s)}{a(s)}}\frac{x(s)}{a(s)}.$$

This shows that the Hankel operator does not depend on d, the constant term in $G(s)$. Hence in the computation related to a Hankel operator, one can disregard the constant part of the symbol.

The Hankel operator can be represented by a matrix if a basis in $\mathcal{S}_{a(s)}$ is chosen. Naturally we can use the Routh basis given in Theorem 2. The matrix representation under this basis is called the Routh-Hankel matrix of $G(s)$ and is denoted by $R_{G(s)}$. The singular values of Routh-Hankel matrix $R_{G(s)}$ are called the Hankel singular values of $G(s)$ and are denoted by $\sigma_1(G(s)),\sigma_2(G(s)),\ldots,\sigma_n(G(s))$. Here we assume that the singular values are ordered in a nonincreasing way, i.e., we assume that $\sigma_1(G(s))\geq\sigma_2(G(s))\geq\cdots\geq\sigma_n(G(s))$. In particular, the largest Hankel singular value $\sigma_1(G(s))$ is called the Hankel norm of $G(s)$ and is denoted by $\|G(s)\|_H$. Let (u_i,v_i) be a pair of left and right singular vectors of $R_{G(s)}$ corresponding to singular value $\sigma_i(G(s))$ and let

$$\begin{aligned}U_i(s)&=\begin{bmatrix}B_1(s)&B_2(s)&\cdots&B_n(s)\end{bmatrix}u_i\\ V_i(s)&=\begin{bmatrix}B_1(s)&B_2(s)&\cdots&B_n(s)\end{bmatrix}v_i.\end{aligned}$$

Then $(U_i(s),V_i(s))$ is called a Schmidt pair of $H_{G(s)}$ corresponding to $\sigma_i(G(s))$.

If we are interested in computing the Hankel singular values and Schmidt pairs of $H_{G(s)}$, then the key is to find the Routh-Hankel matrix $R_{G(s)}$ from $G(s).=\frac{b(s)}{a(s)}$. There are several ways to do this. One of the ways is given in the following algorithm (with the assumption that $G(s)$ is strictly proper).

Algorithm 2

Step 1 *Construct the augmented Routh table of* $G(s)=\frac{b(s)}{a(s)}$.

Step 2 *Set* $\alpha_i=\frac{r_{(i-1)0}}{r_{i0}}$ *and* $\beta_i=\frac{q_{(i-1)0}}{r_{i0}}$ *from the augmented Routh table. Also set* $\alpha_0=1$.

Step 3 *Set*

$$A=\begin{bmatrix}-\frac{1}{\alpha_1}&\frac{1}{\sqrt{\alpha_1\alpha_2}}&&&\\ -\frac{1}{\sqrt{\alpha_1\alpha_2}}&0&\ddots&&\\ &\ddots&\ddots&\ddots&\\ &&\ddots&0&\frac{1}{\sqrt{\alpha_{n-1}\alpha_n}}\\ &&&-\frac{1}{\sqrt{\alpha_{n-1}\alpha_n}}&0\end{bmatrix}$$

Step 4 *Set*

$$R_0=\frac{1}{2}\begin{bmatrix}\sqrt{\frac{\alpha_0}{\alpha_1}}\beta_1\\ \sqrt{\frac{\alpha_0}{\alpha_2}}\beta_2\\ \vdots\\ \sqrt{\frac{\alpha_0}{\alpha_n}}\beta_n\end{bmatrix},\quad R_1=\frac{1}{2}\begin{bmatrix}\sqrt{\frac{\alpha_1}{\alpha_1}}\beta_1\\ \sqrt{\frac{\alpha_1}{\alpha_2}}\beta_2\\ \vdots\\ \sqrt{\frac{\alpha_1}{\alpha_n}}\beta_n\end{bmatrix}$$

$$R_i=\sqrt{\frac{\alpha_i}{\alpha_{i-2}}}R_{i-2}+\sqrt{\alpha_{i-1}\alpha_i}AR_{i-1}\quad i=2,3,\ldots,n.$$

Step 5 *Set*

$$R_{G(s)}=\begin{bmatrix}R_1&R_2&\cdots&R_n\end{bmatrix}.$$

An application of the Hankel operator is in the solution of the so-called Nehari problem, which will be useful in the optimal robust stabilization problem that we will study later. Actually the solution to the Nehari problem is the key step leading to the solution to the general $\mathcal{H}_\infty$ optimization. The Nehari problem is stated as follows: Given stable strictly proper system $G(s)=\frac{b(s)}{a(s)}$, find

$$\min_{Q(s)\in\mathcal{H}_\infty}\|G(-s)-Q(s)\|_\infty$$

and a minimizing $Q(s)\in\mathcal{H}_\infty$.

Theorem 3

$$\min_{Q(s)\in\mathcal{H}_\infty} \|G(-s)-Q(s)\|_\infty = \|G(s)\|_H$$

and if $(U_1(s), V_1(s))$ is a Schmidt pair of the $H_{G(s)}$ corresponding to the largest singular value $\sigma_1 = \|G(s)\|_H$, then the unique optimal $Q(s)$ is given by

$$Q(s) = G(-s) - \sigma_1 \frac{U_1(-s)}{V_1(s)}.$$

VI. Pole Placement

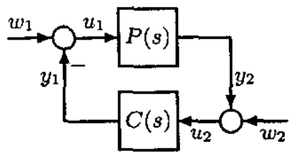

Fig. 1. Feedback system for stabilization

In the rest of the paper, we address synthesis issues. The first problem we will consider is the design of stabilizing controllers, i.e., given a plant $P(s)$, design a controller $C(s)$ such that the feedback system shown in Figure 1 is internally stable. We will start with a revisit to the pole placement problem. Then in the next few sections, we will look into a couple of optimal design problems. It will be seen that the solution to the optimal design problems can be obtained via pole placement. The material in this section is well-known, see [10, Section 4.5].

Let a plant be given by

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$$

where $a(s)$ and $b(s)$ are coprime and $a_0 \neq 0$. We first consider proper controllers of the form

$$C(s) = \frac{q(s)}{p(s)} = \frac{q_0 s^m + q_1 s^{m-1} + \cdots + q_m}{p_0 s^m + p_1 s^{m-1} + \cdots + p_m}$$

where $p(s)$ and $q(s)$ are coprime and $p_0 \neq 0$. Then the closed loop characteristic polynomial is

$$\begin{aligned} d(s) &:= d_0 s^{n+m} + d_1 s^{n+m-1} + \cdots + d_{n+m} \\ &= a(s)p(s) + b(s)q(s). \end{aligned}$$

The closed loop system is internally stable if and only if $d_0 \neq 0$ and $d(s)$ is a stable polynomial. Therefore if we can arbitrarily specify an $(n+m)$-th order stable polynomial $d(s)$ and then choose polynomials $p(s)$ and $q(s)$ so that (3) is satisfied, then we will be able to stabilize the closed loop system and a stabilizing controller is given by $C(s) = \frac{q(s)}{p(s)}$.

It can be easily seen that for given coprime $a(s)$ and $b(s)$, as well as an arbitrarily chosen $d(s)$, the design of $p(s)$ and $q(s)$ amounts to solving a linear polynomial equation:

$$a(s)p(s) + b(s)q(s) = d(s). \tag{3}$$

This equation is called a Diophantine equation, which can be solved by comparing the coefficients of the both sides of (3). This linear equation has solution for arbitrary $d(s)$ if and only if $m \geq n-1$. The solution is unique if and only if $m = n-1$.

Next we consider pole placement using strictly proper controllers. Strictly proper controllers are more advantageous in some applications. In this case, a controller has a form

$$C(s) = \frac{q(s)}{p(s)} = \frac{q_1 s^{m-1} + \cdots + q_m}{p_0 s^m + p_1 s^{m-1} + \cdots + p_m}$$

where $p(s)$ and $q(s)$ are coprime and $p_0 \neq 0$. The closed loop characteristic polynomial is still (3). Again, it can be easily seen that for given coprime $a(s)$ and $b(s)$, as well as an arbitrarily chosen $d(s)$, the design of $p(s)$ and $q(s)$ amounts to solving a Diophantine equation of the form (3). This linear equation has solution for arbitrary $d(s)$ if and only if $m \geq n$. The solution is unique if and only if $m = n$.

A critical issue in the pole placement problem is what constitute good closed-loop poles. This issue cannot be addressed by the pole placement problem itself. In what follows, we will see that the "best" closed-loop poles are determined by the performance specifications.

VII. Optimal Transient Stabilization

Consider Fig. 1. Now we measure the performance of the closed loop system by

$$\begin{aligned} J_{\rho,\mu} &= \left(\|y_1(t)\|_2^2 + \rho^2\|y_2(t)\|_2^2\right)\Big|_{\substack{w_1(t)=\mu\delta(t)\\ w_2(t)=0}} \\ &\quad + \left(\|y_1(t)\|_2^2 + \rho^2\|y_2(t)\|_2^2\right)\Big|_{\substack{w_1(t)=0\\ w_2(t)=\delta(t)}}. \end{aligned}$$

Here ρ is a positive number used to give a relative weight to $y_1(t)$ and $y_2(t)$. Whereas μ is a positive number used to give a relative weight to $w_1(t)$ and $w_2(t)$.

Now the problem is to design a stabilizing controller so that $J_{\rho,\mu}$ is minimized for a given strictly proper plant $P(s) = \frac{b(s)}{a(s)}$. This is actually the SISO version of the LQG optimal control problem stated in a deterministic way and a special $\mathcal{H}_2$ optimal control problem. The design procedure is given by the following algorithm:

Algorithm 3

Step 1 (Spectral factorization) *Find a stable polynomial $d_\rho(s)$ such that*

$$a(-s)a(s) + \rho^2 b(-s)b(s) = d_\rho(-s)d_\rho(s).$$

Step 2 (Spectral factorization) *Find a stable polynomial $d_\mu(s)$ such that*

$$a(-s)a(s) + \mu^2 b(-s)b(s) = d_\mu(-s)d_\mu(s).$$

Step 3 (Pole placement) *The optimal controller $C(s) = \frac{q(s)}{p(s)}$ is the unique n-th order strictly proper pole placement controller such that*

$$a(s)p(s) + b(s)q(s) = d_\rho(s)d_\mu(s).$$

Step 4 ($\mathcal{H}_2$ norm computation) *The optimal performance index is given by*

$$\begin{aligned} J^*_{\rho,\mu} &= \mu^2 \left\|\frac{d_\rho(s)-a(s)}{d_\rho(s)}\right\|_2^2 + \rho^2\mu^2\left\|\frac{b(s)}{d_\rho(s)}\right\|_2^2 \\ &\quad + \mu^2\left\|\frac{d_\mu(s)-p(s)}{d_\mu(s)}\right\|_2^2 + \left\|\frac{q(s)}{d_\mu(s)}\right\|_2^2. \end{aligned}$$

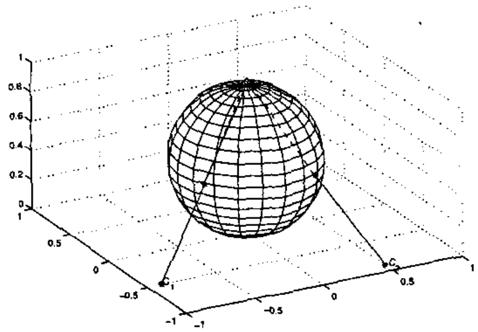

Fig. 2. Riemann sphere and stereographic projections

VIII. Optimal Robust Stabilization

Let us put the complex plane horizontally and place a sphere with unit diameter at the origin of the complex plane, as in Figure 2. This sphere is called the Riemann sphere and is denoted by $\mathbb{S}$. The origin of the complex plane is its south pole S and the pole in the top is its north pole N. For a point c in the complex plane, connect it and the north pole by a straight line. Then this straight line will intersect the Riemann sphere at one and only one point. This point is called the stereographic projection of c on the Riemann sphere and is denoted by $\phi(c)$. Notice that the stereographic projection defines a one-one correspondence between $\mathbb{C}\cup\{\infty\}$ and the Riemann sphere $\mathbb{S}$. The chordal distance between c_1 and c_2, denoted by $\delta(c_1, c_2)$, is defined as the length of the chord connecting $\phi(c_1)$ and $\phi(c_2)$. Simple derivation gives

$$\delta(c_1, c_2) = \frac{|c_1 - c_2|}{\sqrt{1+|c_1|^2}\sqrt{1+|c_2|^2}}. \qquad (4)$$

The frequency response of a system can be plotted as a Riemann plot. The Riemann plot is simply the steroegraphic projection of its Nyquist plot on the Riemann sphere. One advantage of the Riemann plot over the Nyquist plot is that the difficulty associated with the frequency response taking value at the infinity is completely gone. This come of course with a price: a three dimensional plot is needed instead of a 2-dimensional plot. Figure 3 gives a couple of Riemann plots.

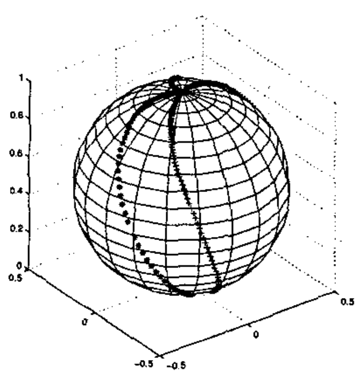

Fig. 3. Riemann plots

For a polynomial $p(s)$, define its inertia to be three numbers

$$\nu[p(s)] = \{\nu_-[p(s)], \nu_0[p(s)], \nu_+[p(s)]\}$$

which are equal to the numbers of its roots with negative, zero, and positive real parts respectively. Clearly, $p(s)$ is stable if and only if $\nu[p(s)] = \{\deg p(s), 0, 0\}$.

Let two transfer functions $G_i(s), i = 1, 2$, be given as

$$G_i(s) = \frac{b_i(s)}{a_i(s)}$$

where $a_i(s), b_i(s), i = 1, 2$, are polynomials. Assume that the orders of $G_i(s)$ are n_i respectively, i.e., $\deg a_i(s) = n_i$.

Definition 1 *Two systems $G_1(s)$ and $G_2(s)$ are said to be comparable, denoted by $G_1(s) \sim G_2(s)$, if*

$$\nu[a_2(-s)a_1(s) + b_2(-s)b_1(s)] = \{n_1, 0, n_2\}.$$

Definition 2 *The Vinnicombe metric between two systems $G_1(s)$ and $G_2(s)$ is defines as*

$$\delta(G_1(s), G_2(s)) = \begin{cases} \max_{\omega\in\mathbb{R}} \delta[G_1(j\omega), G_2(j\omega)] & \text{if } G_1(s) \sim G_2(s) \\ 1 & \text{otherwise.} \end{cases}$$

Roughly speaking, $G_1(s)$ and $G_2(s)$ are considered to be close if their Riemann plots are close on the Riemann sphere, as long as the comparability condition is satisfied.

With the distance function δ between systems, we can describe an uncertain system as a ball of systems:

$$\mathcal{B}_\delta[G(s), r] = \{\tilde{G}(s) : \delta[\tilde{G}(s), G(s)] \le r\}.$$

The center $G(s)$ is called the nominal system and the radius r is called the radius of uncertainty.

For the feedback system in Figure 1, let

$$P(s) = \frac{b(s)}{a(s)}, \quad C(s) = \frac{q(s)}{p(s)}.$$

Define

$$b_{P,C} = \min_{\omega\in\mathbb{R}} \frac{|a(j\omega)p(j\omega) + b(j\omega)q(j\omega)|}{\sqrt{|a(j\omega)|^2 + |b(j\omega)|^2}\sqrt{|p(j\omega)|^2 + |q(j\omega)|^2}}.$$

Consider the feedback system shown in Figure 4. Here $\tilde{P}(s)$ is a perturbed version of $P(s)$ and $\tilde{C}(s)$ is a perturbed version of $C(s)$. If we know that the system shown in Figure 1 is stable, what can we say about the stability of the system shown in Figure 4?

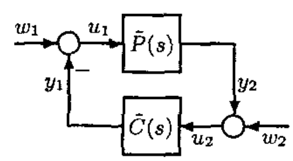

Fig. 4. Feedback system for stabilization

Theorem 4

1) *The feedback system in Figure 4 is stable for all $\tilde{P}(s) \in \mathcal{B}_\delta[P(s), r_P]$ and $\tilde{C}(s) \in \mathcal{B}_\delta[C(s), r_C]$ if and only if*

$$\arcsin r_P + \arcsin r_C < \arcsin b_{P,C}.$$

2) *If the inequality in 1) is satisfied, then*

$$\min_{\tilde{P}\in B_\delta(P,r_P),\tilde{C}\in B_\delta(C,r_C)} b_{\tilde{P},\tilde{C}} = \sin(\arcsin b_{P,C} - \arcsin r_P - \arcsin r_C).$$

Theorem 4 show that $b_{P,C}$ indeed gives a measure of the robustness of the feedback system. The bigger $b_{P,C}$ is, the more robust the feedback system. One natural design problem is then as follows: Given $P(s)$, design $C(s)$ so that $b_{P,C}$ is maximized. This problem is called optimal robust stabilization problem and it is a special $\mathcal{H}_\infty$ optimal control problem.

A procedure to solve the optimal robust stabilization problem is given as follows:

Algorithm 4

Step 1 *(Spectral factorization) Find stable $d(s)$ such that*

$$a(-s)a(s) + b(-s)b(s) = d(-s)d(s).$$

Step 2 *(Diophantine equation) Find $x(s)$ and $y(s)$ such that*

$$a(s)x(s) + b(s)y(s) = d^2(s).$$

It can be shown that

$$x(s)b(-s) - y(s)a(-s) = z(s)d(s).$$

for some polynomial $z(s)$. Let

$$T(s) = \frac{z(-s)}{d(s)}.$$

Step 3 *(Hankel SVD) Compute the largest Hankel singular values σ_1 of $T(s)$ and its corresponding Schmidt pair $(U_1(s), V_1(s))$. Then*

$$\sup_{C(s)} b_{P,C} = \frac{1}{\sqrt{1+\sigma_1^2}}.$$

Let $e(s)$ be any nonzero multiple of the numerator of $U(s)$, or any nonzero multiple of the numerator of $V(s)$, which are the same.

Step 4 *(Pole placement) The optimal controller is the unique $(n-1)$-st order pole placement controller $C(s) = \frac{q(s)}{p(s)}$ such that*

$$a(s)p(s) + b(s)q(s) = d(s)e(s).$$

IX. System Approximation

The problem of Hankel approximation is to find a lower order system to approximate a high order system so that the Hankel norm of the error is minimized. Precisely, if we are given a stable transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}, \quad a_0 > 0,$$

we wish to find

$$\min_{\mathrm{order}\tilde{G}(s)\le r} \|G(s) - \tilde{G}(s)\|_H$$

and a minimizing $\tilde{G}(s)$. Here we assume that $r < n$.

The solution to the Hankel approximation problem is well-known [1], see also the excellent exposition [15]. The solution relies on the computation of the Hankel singular values and the Schmidt pairs of $G(s)$. The method in Section V for the computation of the Hankel singular values and Schmidt pairs immediately applies. Due to the page limitation in this paper, we skip the details.

X. Conclusions

This paper gives a tutorial on our recent effort in developing a complete and systematic optimal and robust control theory for SISO systems using only pre-classical tools growing out of the Routh stability test. The analysis problems addressed are the computations of the RMS value (2-norm), the resonance peak (∞-norm), and the Hankel singular values and vectors. The synthesis problems addressed are optimal transient stabilization (LQG control), optimal robust stabilization ($\mathcal{H}_\infty$ control), and system approximation. Our experience shows that these materials are well suited for undergraduate teaching.

A parallel development for discrete time systems is given in [16].

XI. References

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