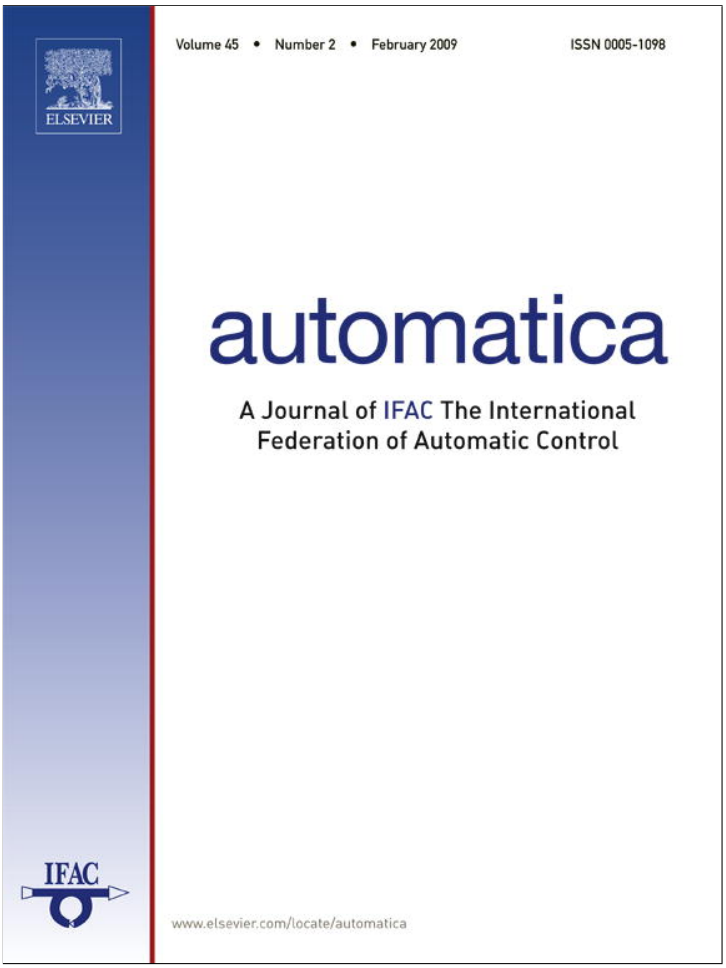

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Reconstruction of continuous-time systems from their non-uniformly sampled discrete-time systems☆

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ABSTRACT

A continuous-time system cannot be recovered *solely* from its uniformly sampled discrete-time model through the zero-order hold discretization or step-invariant transformation, but our studies indicate that it can be recovered *uniquely* from its non-uniformly sampled discrete-time model. In this paper, we discuss some related issues of non-uniformly sampled systems, including model derivation, controllability and observability, computation of single-rate models with different sampling periods, reconstruction of continuous-time systems, and parameter identification of non-uniformly sampled discrete-time systems. A numerical example is also given for illustration.

1. Introduction

Consider a continuous-time system with the following state space representation

$$P:\ \begin{cases}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{u}(t),\end{cases} \tag{1}$$

where $\boldsymbol{x}(t)\in\mathbb{R}^n$, $\boldsymbol{u}(t)\in\mathbb{R}^r$ and $\boldsymbol{y}(t)\in\mathbb{R}^m$ are the state, input and output vectors, respectively; $\boldsymbol{A}\in\mathbb{R}^{n\times n}$, $\boldsymbol{B}\in\mathbb{R}^{n\times r}$, $\boldsymbol{C}\in\mathbb{R}^{m\times n}$ and $\boldsymbol{D}\in\mathbb{R}^{m\times r}$ are constant matrices. Suppose that the sampling interval is τ. By using the step-invariant transformation or the zero-order hold (ZOH) discretization, i.e., taking $\boldsymbol{u}(t)=\boldsymbol{u}(k\tau)$, $k\tau\le t<(k+1)\tau$ and sampling the system in (1) give a discrete-time model (Chen & Francis, 1995):

$$P_\tau:\ \begin{cases}\boldsymbol{x}(k\tau+\tau)=\boldsymbol{G}_\tau\boldsymbol{x}(k\tau)+\boldsymbol{F}_\tau\boldsymbol{u}(k\tau),\\ \boldsymbol{y}(k\tau)=\boldsymbol{C}\boldsymbol{x}(k\tau)+\boldsymbol{D}\boldsymbol{u}(k\tau),\quad k=0,1,2,\ldots\end{cases} \tag{2}$$

where $\boldsymbol{x}(k\tau)=\boldsymbol{x}(t)\,|_{t=k\tau}$, $\boldsymbol{y}(k\tau)=\boldsymbol{y}(t)\,|_{t=k\tau}$, and

$$\boldsymbol{G}_\tau:=\exp(\boldsymbol{A}\tau),\qquad \boldsymbol{F}_\tau:=\int_0^\tau\exp(\boldsymbol{A}t)\mathrm{d}t\boldsymbol{B}. \tag{3}$$

The discrete-time system P_τ is shown in Fig. 1, where H and S denote the ZOH and sampler with the period τ, respectively. We call P_τ a single-rate sampled-data system since there exists only one sampling rate $1/\tau$ in the system. (Conventional sampled-data systems are single-rate ones.)

On the reconstruction/identification of continuous-time systems based on discrete-time data, conventional sampled-data methods use only one equidistant sampling interval, and hence the continuous-time models obtained are non-unique, e.g., one is the equivalent ramp invariant continuous-time model corresponding to a given discrete-time one (Bingulac & Cooper, 1990). In order to obtain a unique continuous-time model from a given discrete-time model, one generally requires some further information of the continuous-time system, e.g., pole locations (Sinha & Lastman, 1982).

It is well known that for a given continuous-time model P and sampling period τ, the corresponding discrete-time model P_τ is unique, but for given τ and P_τ, there exist infinite many continuous-time models giving rise to the same discrete-time model P_τ. Thus, one cannot recover or reconstruct the continuous-time system from its discretized model P_τ and τ without further

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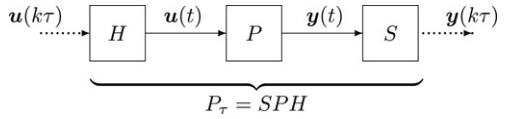


Fig. 1. The discrete-time sampled-data system.

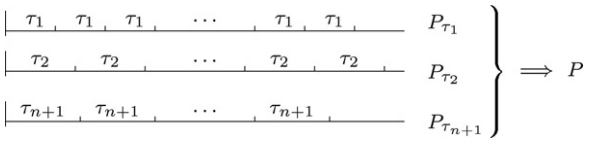


Fig. 2. $n+1$ equidistant sampling schemes yielding the continuous-time system P.

information. Recent studies indicated that for given $n+1$ sampling periods $\tau_1, \tau_2, \ldots, \tau_{n+1}$, the continuous-time model P, with possibly complex matrix parameters, may be reconstructed uniquely from the $n+1$ discrete-time models $P_{\tau_i}, i = 1, 2, \ldots, n+1$ (Chen & Miller, 2000); this case is illustrated in Fig. 2.

Using discrete-time system identification techniques (Ding & Chen, 2005a,b, 2007; Goodwin & Sin, 1984; Ljung, 1999), it is easy to identify the $n+1$ models P_{τ_i} from given (discrete-time) input–output data; however, the difficulty lies in that we have to conduct experiments for $n+1$ sampled-data configurations with *different* sampling periods. To simplify this process, we present a sampling pattern shown in Fig. 3 in which the sampling instants are spaced apart non-uniformly by $\tau_1, \tau_2, \ldots, \tau_{n+1}$, and the whole sampling process is repeated over the so-called frame period $T = \tau_1 + \tau_2 + \cdots + \tau_{n+1} = t_{n+1}$. It will be shown later that under this sampling scheme, we can reconstruct the continuous-time model from a *single* discretized model, thus combining $n+1$ different identification experiments into one.

The non-uniformly sampling scheme proposed relates closely to multirate sampling. Multirate systems have had wide applications in chemical and petroleum processes (Gudi, Shah, & Gray, 1994, 1995; Tatiraju, Soroush, & Mutharasan, 1999) and a series of results have been achieved in theory, including controllability and observability (Francis & Georgiou, 1988; Kreisselmeier, 1999), robust control (Chen & Qiu, 1994), optimal control (Qiu & Chen, 1999), adaptive control (Ding & Chen, 2004a; Zhang, Middleton, & Evans, 1989), predictive control (Scattolinis & Schiavoni, 1995; Sheng, Chen, & Shah, 2002), modeling and identification (Ding & Chen, 2004b,c, 2005c,d; Ding, Chen, & Iwai, 2007; Ding & Ding, 2008), and so on.

In the area of multirate/non-uniformly sampled systems, Francis and Georgiou (1988) presented the conditions of preserving controllability/observability for sampled-data systems; Kreisselmeier (1999) explored a multirate sampling scheme to achieve observability/controllability in discrete-time systems, and Sheng et al. (2002) further discussed the results in non-uniformly sampled systems. Other studies of non-uniformly sampled systems include the real-time control by Albertos and Crespo (1999), generalized predictive control by Sheng et al. (2002), and subspace identification based fault detection and isolation by Li, Han, and Shah (2006).

To the best of our knowledge, few contributions have addressed modeling, estimation and reconstruction issues for non-uniformly sampled systems, which are the focus of this work. For the non-uniformly sampling pattern depicted in Fig. 3, our objective is two-fold:

- First, establish a mathematical model of the non-uniformly sampled system from input $\boldsymbol{u}(kT+t_i)$ to output $\boldsymbol{y}(kT+t_i)$, and study the related controllability and observability issues.
- Second, by means of a shift invariance property, derive each single-rate model P_{τ_i} from the non-uniformly sampled system, and from here reconstruct the continuous-time system. Finally, develop identification algorithms for estimating the parameters of the non-uniformly sampled system, based on the given input–output data $\{\boldsymbol{u}(kT+t_i), \boldsymbol{y}(kT+t_i) : i = 1, 2, \ldots, n+1, k = 0, 1, 2, \ldots\}$.

The rest of the paper is organized as follows. Section 2 derives mathematical models for non-uniformly sampled systems from continuous-time systems. Section 3 discusses the controllability and observability of the non-uniformly sampled systems; Section 4 computes single-rate models from the non-uniformly sampled models. Section 5 reconstructs the original continuous-time systems based on the single-rate discrete-time models obtained. Section 6 discusses the identification issues for non-uniformly sampled systems. Section 7 presents an illustrative example validating the methods proposed. Finally, Section 8 offers some concluding remarks.

2. Model derivations

This paper focuses on a class of non-uniformly (multirate) sampled systems depicted in Fig. 4, the input updating and output sampling pattern being shown in Fig. 3, where P is a continuous-time process with the representation in (1), $\mathcal{H}$ and $\mathcal{S}$ denote the non-uniform ZOH and sampler with the following characteristics:

$$\boldsymbol{u}(t) = \begin{cases} \boldsymbol{u}(kT), & kT \le t < kT + t_1, \\ \boldsymbol{u}(kT + t_1), & kT + t_1 \le t < kT + t_2, \\ \quad\vdots & \\ \boldsymbol{u}(kT + t_n), & kT + t_n \le t < (k+1)T, \end{cases}$$

with $\boldsymbol{y}(kT+t_i) = \boldsymbol{y}(t)\big|_{t=kT+t_i}, i = 0, 1, \ldots, n$, and $k = 0, 1, 2, \ldots$. $\{\tau_1, \tau_2, \ldots, \tau_{n+1}\}$ are the updating and sampling intervals, $t = kT + t_i$ are the updating and sampling instants and $T = \tau_1 + \tau_2 + \cdots + \tau_{n+1} = t_{n+1}$ is the frame period, where $t_0 = 0$, $t_i = t_{i-1} + \tau_i = \tau_1 + \tau_2 + \cdots + \tau_i$. The control input $\boldsymbol{u}$ is updated $n+1$ times at the instants $t = kT + t_i$ $(i = 0, 1, \ldots, n)$ over the kth period $[kT, (k+1)T)$ and the output $\boldsymbol{y}$ is sampled $n+1$ times at the instants $t = kT + t_i$ $(i = 0, 1, \ldots, n)$ over the kth period $[kT, (k+1)T)$. This is the non-uniform updating and sampling scheme.

Next, we derive a mathematical model of the non-uniformly sampled system in Fig. 4. Integrating (1) from $t = kT$ to $t = kT + t_i$ gives

$$\begin{aligned} \boldsymbol{x}(kT + t_i) &= \exp(\boldsymbol{A}t_i)\boldsymbol{x}(kT) \\ &\quad + \int_{kT}^{kT+t_i} \exp[\boldsymbol{A}(kT + t_i - s)]\boldsymbol{B}\boldsymbol{u}(s)\mathrm{d}s \\ &= \exp(\boldsymbol{A}t_i)\boldsymbol{x}(kT) \\ &\quad + \sum_{j=1}^{i} \int_{kT+t_{j-1}}^{kT+t_j} \exp[\boldsymbol{A}(kT + t_i - s)]\boldsymbol{B}\mathrm{d}s \\ &\quad \times \boldsymbol{u}(kT + t_{j-1}). \end{aligned}$$

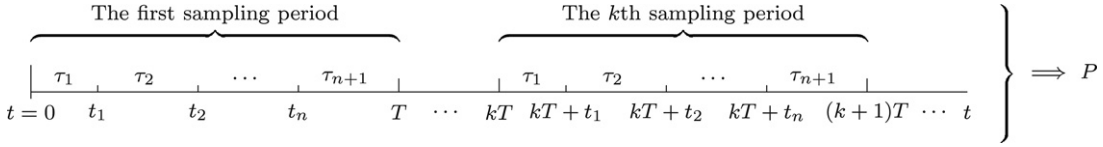


Fig. 3. A non-uniformly sampling scheme yielding the continuous-time system P.

A change of variable $t = kT + t_j - s$ yields

$$\begin{aligned}\boldsymbol{x}(kT + t_i) &= \exp(\boldsymbol{A}t_i)\boldsymbol{x}(kT) + \sum_{j=1}^{i}\exp[\boldsymbol{A}(t_i - t_j)] \\ &\quad \times \int_0^{\tau_j}\exp(\boldsymbol{A}t)\mathrm{d}t\boldsymbol{B}\boldsymbol{u}(kT + t_{j-1}) \\ &=: \boldsymbol{G}_i\boldsymbol{x}(kT) + \sum_{j=1}^{i}\exp[\boldsymbol{A}(t_i - t_j)]\boldsymbol{F}_{\tau_j}\boldsymbol{u}(kT + t_{j-1}).\end{aligned}$$

where

$$\begin{aligned}&\boldsymbol{G}_i := \exp(\boldsymbol{A}t_i) \in \mathbb{R}^{n\times n}, \quad i = 1, 2, \ldots, n+1, \\ &\boldsymbol{G} := \boldsymbol{G}_{n+1} = \exp(\boldsymbol{A}T) = \exp(\boldsymbol{A}t_{n+1}) \in \mathbb{R}^{n\times n}, \\ &\boldsymbol{F}_{\tau_i} := \int_0^{\tau_i}\exp(\boldsymbol{A}t)\mathrm{d}t\boldsymbol{B} \in \mathbb{R}^{n\times r}.\end{aligned}$$

Define

$$\boldsymbol{F}_i := \exp(\boldsymbol{A}(T - t_i))\boldsymbol{F}_{\tau_i} = \boldsymbol{G}\boldsymbol{G}_i^{-1}\boldsymbol{F}_{\tau_i} \in \mathbb{R}^{n\times r}, \tag{4}$$

$$\boldsymbol{F} := [\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_{n+1}] \in \mathbb{R}^{n\times(n+1)r}$$

$$\underline{\boldsymbol{u}}(kT) := \begin{bmatrix}\boldsymbol{u}(kT) \\ \boldsymbol{u}(kT + t_1) \\ \boldsymbol{u}(kT + t_2) \\ \vdots \\ \boldsymbol{u}(kT + t_n)\end{bmatrix} \in \mathbb{R}^{(n+1)r}$$

(The non-uniformly stacked input vector). (5)

When $i = n + 1$, we have

$$\begin{aligned}\boldsymbol{x}(kT + T) &= \boldsymbol{x}(kT + t_{n+1}) \\ &= \boldsymbol{G}\boldsymbol{x}(kT) + \sum_{j=1}^{n+1}\exp[\boldsymbol{A}(T - t_j)]\boldsymbol{F}_{\tau_j}\boldsymbol{u}(kT + t_{j-1}) \\ &= \boldsymbol{G}\boldsymbol{x}(kT) + \boldsymbol{F}\underline{\boldsymbol{u}}(kT).\end{aligned}$$

The outputs at the sampling instants can be expressed as

$$\begin{aligned}\boldsymbol{y}(kT + t_i) &= \boldsymbol{C}\boldsymbol{x}(kT + t_i) + \boldsymbol{D}\boldsymbol{u}(kT + t_i) \\ &=: \boldsymbol{C}\boldsymbol{G}_i\boldsymbol{x}(kT) + \boldsymbol{H}_i\underline{\boldsymbol{u}}(kT),\end{aligned}$$

where

$$\begin{aligned}&\boldsymbol{D}_{ij} := \boldsymbol{C}\boldsymbol{G}_i\boldsymbol{G}_j^{-1}\boldsymbol{F}_{\tau_1} \in \mathbb{R}^{m\times r}, \quad j = 1, 2, \ldots, i, \\ &\boldsymbol{H}_i := [\boldsymbol{D}_{i1}, \boldsymbol{D}_{i2}, \ldots, \boldsymbol{D}_{ii}, \boldsymbol{D}, \boldsymbol{0}, \ldots, \boldsymbol{0}] \in \mathbb{R}^{m\times(n+1)r}.\end{aligned}$$

Hence, we get a mathematical model of the non-uniformly sampled system as follows:

$$P_n : \left[\begin{array}{c}\boldsymbol{x}(kT + T) \\ \hline \underline{\boldsymbol{y}}(kT)\end{array}\right] = \left[\begin{array}{c|c}\boldsymbol{G} & \boldsymbol{F} \\ \hline \boldsymbol{\Gamma} & \boldsymbol{H}\end{array}\right]\left[\begin{array}{c}\boldsymbol{x}(kT) \\ \hline \underline{\boldsymbol{u}}(kT)\end{array}\right], \tag{6}$$

where

$$\boldsymbol{\Gamma} := \begin{bmatrix}\boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{G}_1 \\ \boldsymbol{C}\boldsymbol{G}_2 \\ \vdots \\ \boldsymbol{C}\boldsymbol{G}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{C} \\ \boldsymbol{C}\exp(\boldsymbol{A}t_1) \\ \boldsymbol{C}\exp(\boldsymbol{A}t_2) \\ \vdots \\ \boldsymbol{C}\exp(\boldsymbol{A}t_n)\end{bmatrix} \in \mathbb{R}^{(n+1)m\times n},$$

$$\boldsymbol{H} := \begin{bmatrix}\boldsymbol{H}_0 \\ \boldsymbol{H}_1 \\ \boldsymbol{H}_2 \\ \vdots \\ \boldsymbol{H}_n\end{bmatrix} \in \mathbb{R}^{(n+1)m\times(n+1)r},$$

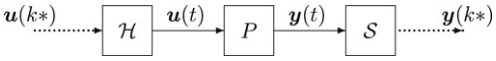


Fig. 4. The non-uniformly sampling systems.

$$\underline{\boldsymbol{y}}(kT) := \begin{bmatrix}\boldsymbol{y}(kT) \\ \boldsymbol{y}(kT + t_1) \\ \boldsymbol{y}(kT + t_2) \\ \vdots \\ \boldsymbol{y}(kT + t_n)\end{bmatrix} \in \mathbb{R}^{(n+1)m}$$

(The non-uniformly stacked output vector).

In order to identify the parameters of this model from the input–output data, the system in (6) needs to be controllable and observable. The next section is devoted to the study of controllability and observability of the model in (6).

3. Controllability and observability

For conventional single-rate sampled-data systems, it is well known that the process of discretization may result in loss of controllability and observability, and cannot gain controllability and observability (Chen & Francis, 1995). For non-uniformly sampled systems, assume controllability and observability of the continuous-time model in (1); under what conditions the controllability and observability are preserved for the model in (6)?

The sampling frequency $\omega_s := \frac{2\pi}{\tau}$ is pathological (relative to $\boldsymbol{A}$) if $\boldsymbol{A}$ has two eigenvalues with equal real parts and imaginary parts that differ by an integer multiple of ω_s. Otherwise, the sampling frequency is non-pathological, i.e., $\lambda_i - \lambda_j \neq \pm\frac{2k\pi\sqrt{-1}}{\tau}$ ($k = 1, 2, \ldots$) for any two eigenvalues, λ_i and λ_j, of $\boldsymbol{A}$ (Chen & Francis, 1995).

Lemma 1 (*Chen & Francis, 1995*). *Suppose that the continuous-time system P in (1) is controllable and observable. The discrete-time model P_τ in (2) is also controllable and observable if the sampling period τ is non-pathological.*

The proof can be found in Chen and Francis (1995). The conclusion of Lemma 1 can be extended to non-uniformly sampled systems.

Lemma 2. *For the non-uniformly sampled discrete-time model P_n in (6), if no two eigenvalues of $\boldsymbol{A}$, say, λ_i and λ_j, satisfy the equality,*

$$\lambda_i - \lambda_j = \pm\frac{2k\pi\sqrt{-1}}{T}, \quad k = 1, 2, 3, \ldots \tag{7}$$

then observability of $(\boldsymbol{C}, \boldsymbol{A})$ implies that of $(\boldsymbol{\Gamma}, \boldsymbol{G})$ and controllability of $(\boldsymbol{A}, \boldsymbol{B})$ implies that of $(\boldsymbol{G}, \boldsymbol{F})$.

Proof. This proof is similar to that of the single-rate case (Chen & Francis, 1995).

The observability of $(\boldsymbol{C}, \boldsymbol{A})$ implies that for any eigenvalue λ of $\boldsymbol{A}$,

$$\operatorname{rank}\begin{bmatrix}\lambda\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C}\end{bmatrix} = n. \tag{8}$$

Define the function,

$$f(s) := \frac{\exp(Ts) - \exp(T\lambda)}{s - \lambda}$$

which is analytic since the pole at $s = \lambda$ is cancelled by a zero there. According to (7), we can draw that the zeros of $f(s)$ is not the eigenvalues of $\boldsymbol{A}$. Since the eigenvalues of the matrix $\boldsymbol{f}(\boldsymbol{A})$ are

precisely the values of $f(s)$ at the eigenvalues of $\boldsymbol{A}$, so $s = 0$ is not an eigenvalue of $\boldsymbol{f}(\boldsymbol{A})$, then $\boldsymbol{f}(\boldsymbol{A})$ is invertible. From the equality,

$$\exp(Ts) - \exp(T\lambda) = f(s)(s - \lambda),$$

we have

$$\boldsymbol{G} - \exp(T\lambda)\boldsymbol{I}_n = \boldsymbol{f}(\boldsymbol{A})(\boldsymbol{A} - \lambda\boldsymbol{I}_n),$$

note that the eigenvalue of $\boldsymbol{G}$ is $\exp(T\lambda)$. Thus,

$$\begin{bmatrix} \exp(T\lambda)\boldsymbol{I}_n - \boldsymbol{G} \\ \boldsymbol{\Gamma} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}(\boldsymbol{A}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{(n+1)m} \end{bmatrix} \begin{bmatrix} \lambda\boldsymbol{I}_n - \boldsymbol{A} \\ \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{G}_1 \\ \vdots \\ \boldsymbol{C}\boldsymbol{G}_n \end{bmatrix}.$$

Since

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{f}(\boldsymbol{A}) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{(n+1)m} \end{bmatrix} = (n+1)m + n,$$

using (8), we have

$$\operatorname{rank}\begin{bmatrix} \exp(T\lambda)\boldsymbol{I}_n - \boldsymbol{G} \\ \boldsymbol{\Gamma} \end{bmatrix} = n.$$

This implies observability of $(\boldsymbol{\Gamma}, \boldsymbol{G})$. The proof of controllability is similar and is omitted here. □

The proof of controllability and observability can be found in Francis and Georgiou (1988) and Ding and Chen (2005d) for uniformly sampled dual-rate systems and in Sheng et al. (2002) for non-uniformly sampled ones.

Lemma 2 shows that the choice of the frame sampling period T is *apparently* important for non-uniformly sampled systems to preserve controllability and observability of continuous-time systems. However, even if the condition in Lemma 2 is not satisfied (then T is called pathological), by proper choice of the sampling instants t_i, controllability and observability of P_n can still be preserved. Let us illustrate this by an example.

Consider a second-order system ($n = 2$) with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \qquad \boldsymbol{C} = [1, 0], \quad \omega > 0.$$

The pair $(\boldsymbol{A}, \boldsymbol{C})$ is in the observable canonical form and thus is observable. $\boldsymbol{A}$ has two eigenvalues $\pm\omega\sqrt{-1}$, so $\tau = \frac{2\pi}{\omega}$ is a pathological sampling period.

It is easily seen that $(\boldsymbol{C}, \boldsymbol{G}_\tau)$ is not observable for the uniformly sampling pattern with the sampling period $\tau = \frac{2\pi}{\omega}$. For the non-uniformly sampling case, let $\tau_1 < \frac{\pi}{\omega}$, $\tau_2 = \frac{\pi}{\omega} - \tau_1$, $\tau_3 = \frac{\pi}{\omega}$. Hence, $t_1 = \tau_1$, $t_2 = \tau_1 + \tau_2 = \frac{\pi}{\omega}$, $t_3 = \tau_1 + \tau_2 + \tau_3 = \frac{2\pi}{\omega} = T$. Note that T is pathological. One can check that

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}\boldsymbol{G} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\exp(\boldsymbol{A}t_1) \\ \boldsymbol{C}\exp(\boldsymbol{A}t_2) \\ \boldsymbol{C}\exp(\boldsymbol{A}T) \\ \boldsymbol{C}\exp(\boldsymbol{A}t_1)\exp(\boldsymbol{A}T) \\ \boldsymbol{C}\exp(\boldsymbol{A}t_2)\exp(\boldsymbol{A}T) \end{bmatrix} = 2 = n.$$

Thus, $(\boldsymbol{\Gamma}, \boldsymbol{G})$ is observable.

A general result is stated below.

Lemma 3. *Suppose that the continuous-time system P in (1) is controllable and observable. For the non-uniformly sampling pattern in Fig. 3 and any frame period T, let $\tau_1 = \tau_2 = \cdots = \tau_n =: \tau_0$ and assume that $\frac{\tau_{n+1}}{\tau_0}$ is irrational, then the non-uniformly sampled system P_n in (6) is always controllable and observable even if both the frame period T and sampling interval τ_0 are pathological.*

The proof can be done in a similar way as in Kreisselmeier (1999).

4. Computation of single-rate models

Assume that using some identification method, we have identified the parameter matrices $(\boldsymbol{G}, \boldsymbol{F}, \boldsymbol{\Gamma}, \boldsymbol{H})$ of the non-uniformly sampled system P_n in (6), a natural question is from here how to find the $n+1$ single-rate system models P_{τ_i} with sampling periods τ_i. To get P_{τ_i} from P_n in state-space data, we need to compute $(\boldsymbol{G}_{\tau_i}, \boldsymbol{F}_{\tau_i}, \boldsymbol{C}, \boldsymbol{D})$ from $(\boldsymbol{G}, \boldsymbol{F}, \boldsymbol{\Gamma}, \boldsymbol{H})$ according to (3). This is accomplished in a few steps.

4.1. Computation of $\boldsymbol{C}$ and $\boldsymbol{D}$

According to the structures of $\boldsymbol{\Gamma}$ and $\boldsymbol{H}$, $\boldsymbol{C}$ can be read directly from the first block row (first m rows) of $\boldsymbol{\Gamma}$, and $\boldsymbol{D}$ from the (1, 1) block (first m rows and first r columns) of $\boldsymbol{H}$. However, we note that $\boldsymbol{D}$ appears in $\boldsymbol{H}$ in $(n+1)$ sub-blocks. In order to reduce numerical computation errors, we may take their average as $\boldsymbol{D}$, i.e.,

$$\boldsymbol{D} = \frac{1}{n+1}\sum_{i=1}^{n+1} \boldsymbol{H}(im - m + 1 : im, ir - r + 1 : ir),$$

where the notation $\boldsymbol{H}(i : j, p : q)$ denotes the sub-matrix consisting of rows i to j and columns p to q of $\boldsymbol{H}$.

4.2. Computation of $\boldsymbol{G}_{\tau_i}$

By formulating the extended observability matrix using $\boldsymbol{\Gamma}$ and $\boldsymbol{G}$ or the extended controllability matrix using $\boldsymbol{G}$ and $\boldsymbol{F}$, the matrices $\boldsymbol{G}_{\tau_i}$ can be computed by the *shift invariance structure*. The following is to give an approach to compute $\boldsymbol{G}_{\tau_i}$. More specifically, define the extended observability matrix

$$\boldsymbol{Q}_o := \begin{bmatrix} \boldsymbol{\Gamma} \\ \hline \boldsymbol{\Gamma}\boldsymbol{G} \\ \hline \boldsymbol{\Gamma}\boldsymbol{G}^2 \\ \hline \vdots \\ \hline \boldsymbol{\Gamma}\boldsymbol{G}^N \\ \boldsymbol{C}\boldsymbol{G}^N\exp(\boldsymbol{A}t_{n+1}) \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\exp(\boldsymbol{A}t_1) \\ \boldsymbol{C}\exp(\boldsymbol{A}t_2) \\ \vdots \\ \boldsymbol{C}\exp(\boldsymbol{A}t_n) \\ \hline \boldsymbol{C}\boldsymbol{G} \\ \boldsymbol{C}\boldsymbol{G}\exp(\boldsymbol{A}t_1) \\ \boldsymbol{C}\boldsymbol{G}\exp(\boldsymbol{A}t_2) \\ \vdots \\ \boldsymbol{C}\boldsymbol{G}\exp(\boldsymbol{A}t_n) \\ \hline \vdots \\ \hline \boldsymbol{C}\boldsymbol{G}^N \\ \boldsymbol{C}\boldsymbol{G}^N\exp(\boldsymbol{A}t_1) \\ \boldsymbol{C}\boldsymbol{G}^N\exp(\boldsymbol{A}t_2) \\ \vdots \\ \boldsymbol{C}\boldsymbol{G}^N\exp(\boldsymbol{A}t_{n+1}) \end{bmatrix}$$

which is formed by $\boldsymbol{\Gamma}$ and $\boldsymbol{G}$ obtained by identification and assumed to be known. Likewise, use the entries of $\boldsymbol{Q}_o$ to form the matrices

$$\boldsymbol{\Gamma}_i := \begin{bmatrix} \boldsymbol{C}\exp(\boldsymbol{A}t_i) \\ \boldsymbol{C}\boldsymbol{G}\exp(\boldsymbol{A}t_i) \\ \boldsymbol{C}\boldsymbol{G}^2\exp(\boldsymbol{A}t_i) \\ \vdots \\ \boldsymbol{C}\boldsymbol{G}^N\exp(\boldsymbol{A}t_i) \end{bmatrix}, \quad 0 \le i \le n+1,$$

which are also available, noting that $\boldsymbol{G} = \exp(\boldsymbol{A}t_{n+1})$. It follows easily that

$$\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_0 \exp(\boldsymbol{A}t_i), \quad 1 \le i \le n+1,$$

and

$$\boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_{i-1} \exp(\boldsymbol{A}\tau_i), \quad 1 \le i \le n+1.$$

Using the assumptions of observability and that the sampling intervals τ_i is non-pathological, the matrices $\boldsymbol{\Gamma}_i$ have full column-rank for $N \ge n-1$. Thus, matrices $\exp(\boldsymbol{A}t_i)$ and $\exp(\boldsymbol{A}\tau_i)$ are given by

$$\exp(\boldsymbol{A}t_i) = (\boldsymbol{\Gamma}_0^{\mathrm{T}}\boldsymbol{\Gamma}_0)^{-1}\boldsymbol{\Gamma}_0^{\mathrm{T}}\boldsymbol{\Gamma}_i, \quad i = 1, 2, \ldots, n, \tag{9}$$

and

$$\boldsymbol{G}_{\tau_i} = \exp(\boldsymbol{A}\tau_i) = (\boldsymbol{\Gamma}_{i-1}^{\mathrm{T}}\boldsymbol{\Gamma}_{i-1})^{-1}\boldsymbol{\Gamma}_{i-1}^{\mathrm{T}}\boldsymbol{\Gamma}_i, \quad i = 1, 2, \ldots, n+1,$$

where the superscript T is the transpose. Of course, $\boldsymbol{G}_{\tau_i}$ may be also obtained by post-multiplying (9) by $\exp(-\boldsymbol{A}t_{i-1})$.

4.3. Computation of $\boldsymbol{F}_{\tau_i}$

From the definition of $\boldsymbol{F}_i$ in (4), we have

$$\boldsymbol{F}_{\tau_i} = \exp(-\boldsymbol{A}T)\boldsymbol{G}_i\boldsymbol{F}_i = \boldsymbol{G}^{-1}\exp(\boldsymbol{A}t_i)\boldsymbol{F}_i, \quad i = 1, 2, \ldots, n,$$

$$\boldsymbol{F}_{\tau_{n+1}} = \boldsymbol{G}^{-1}\exp(\boldsymbol{A}t_{n+1})\boldsymbol{F}_{n+1} = \boldsymbol{F}_{n+1},$$

since $\boldsymbol{G}$ and $\boldsymbol{F} = [\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_{n+1}]$ are obtained by identification and $\exp(\boldsymbol{A}t_i)$ are available by (9).

We comment that when the single-rate models P_{τ_i} are computed from P_n, the assumptions of observability ($\boldsymbol{\Gamma}_i$ having full column-rank) or non-pathological conditions of τ_i and T are required because one desires the single-rate models $P_{\tau_i} = (\boldsymbol{G}_{\tau_i}, \boldsymbol{F}_{\tau_i}, \boldsymbol{C}, \boldsymbol{D})$ to be observable, although this is not needed to preserve the observability of P_n.

5. Reconstruction of continuous-time systems

Recovery of the continuous-time P from P_τ is to determine the matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$. The matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ were obtained in the preceding section; in this section, we discuss how to find $\boldsymbol{A}$ and $\boldsymbol{B}$. From (2) and (3), the key of reconstructing the continuous-time system $P = (\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ is to compute the matrix $\boldsymbol{A}$ from $P_\tau = (\boldsymbol{G}_\tau, \boldsymbol{F}_\tau, \boldsymbol{C}, \boldsymbol{D})$. Once $\boldsymbol{A}$ is available, $\boldsymbol{B}$ is easily computed from (3) by

$$\boldsymbol{B} = \left[\int_0^\tau \exp(\boldsymbol{A}t)\mathrm{d}t\right]^{-1}\boldsymbol{F}_\tau$$

since this integral matrix is invertible under the non-pathological condition.

According to Chen and Miller (2000), the key of finding $\boldsymbol{A}$ is to obtain the eigenvalues of $\boldsymbol{A}$, or the poles of P [the poles of the transfer matrix $\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$, if it is controllable and observable]. The details are as follows. Since the function $f(s) = \mathrm{e}^s$ is an analytic function over the entire complex plane, by the Spectral Mapping Theorem, the eigenvalues of $f(\boldsymbol{A})$ equal the values of $f(s)$ at the eigenvalues of $\boldsymbol{A}$. Thus, from (3), the eigenvalues $\lambda_i[\boldsymbol{A}]$ of $\boldsymbol{A}$ and eigenvalues $\lambda_i[\boldsymbol{G}_\tau]$ of $\boldsymbol{G}_\tau$ have the mapping relation:

$$\boldsymbol{G}_\tau = \exp(\boldsymbol{A}\tau) \Longrightarrow \lambda_i[\boldsymbol{G}_\tau] = \exp(\lambda_i[\boldsymbol{A}\tau]), \quad i = 1, 2, \ldots, n.$$

Since e^s is periodic with period $\mathrm{j}2\pi$, i.e., $\exp(s + \mathrm{j}2\pi) = \mathrm{e}^s$ where $\mathrm{j} = \sqrt{-1}$, from the above equation, the possible eigenvalues of $\boldsymbol{A}$ are

$$\lambda_i[\boldsymbol{A}] = \frac{1}{\tau}\mathrm{Ln}\{\lambda_i[\boldsymbol{G}_\tau]\} = \frac{1}{\tau}\ln\{\lambda_i[\boldsymbol{G}_\tau]\} + \frac{\mathrm{j}2k\pi}{\tau}, \quad k \in \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of integers, $\ln\{\lambda_i[\boldsymbol{G}_\tau]\}$ is the principal logarithm of $\mathrm{Ln}\{\lambda_i[\boldsymbol{G}_\tau]\}$ and $-\pi < \arg\{\lambda_i[\boldsymbol{G}_\tau]\} \le \pi$.

Define the eigenvalue set of $\boldsymbol{A}$ as follows:

$$\mathrm{Eig}_\tau = \left\{\frac{1}{\tau}\ln\{\lambda_i[\boldsymbol{G}_\tau]\} + \frac{\mathrm{j}2k\pi}{\tau} : i = 1, 2, \ldots, n,\ k \in \mathbb{Z}\right\}.$$

From (3), if one knows that $\boldsymbol{A}$ has all real eigenvalues, then so has $\boldsymbol{G}_\tau$. This can help eliminate every element of Eig_τ except the real element $\frac{1}{\tau}\ln\{\lambda_i[\boldsymbol{G}_\tau]\}$, $i = 1, 2, \ldots, n$. In such a case, the eigenvalues of $\boldsymbol{A}$ can be determined uniquely from its corresponding *single* discrete-time P_τ or $\boldsymbol{G}_\tau$. To allow complex eigenvalues in $\boldsymbol{A}$, suppose we know an upper bound, $\omega_{\max}$, for the imaginary parts of eigenvalues $\lambda_i[\boldsymbol{A}]$, i.e.,

$$\mathrm{Im}(\lambda_i[\boldsymbol{A}]) \le \omega_{\max}, \quad i = 1, 2, \ldots, n.$$

Then if we sample fast enough, reconstruction of eigenvalues of $\boldsymbol{A}$ is possible from its corresponding *single* discrete-time P_τ. In fact, P is reconstructible if the sampling period $\tau < \pi/\omega_{\max}$ because the upper bound can be used to show that Eig_τ has exactly n elements. This is reminiscent of the well-known Shannon's Sampling theorem. However, from here we also know that the poles of the continuous-time P cannot be uniquely determined from the poles of its *single* discrete-time P_τ without other information such as the pole locations as above.

The purpose here is to recover P by introducing several discretized models $P_{\tau_j} = (\boldsymbol{G}_{\tau_j}, \boldsymbol{F}_{\tau_j}, \boldsymbol{C}, \boldsymbol{D})$ for different choices of τ_j; but, how to choose these sampling periods τ_j so that from these models P_{τ_j} we can reconstruct P? That is, can one determine the poles of the continuous-time system P or the eigenvalues of $\boldsymbol{A}$ uniquely from several $\boldsymbol{G}_{\tau_j}$ by appropriately choosing sampling periods $\tau_j, j = 1, 2, \ldots, l$? The answer is yes. The system $\boldsymbol{G}_{\tau_j}$'s give rise to the (possible) eigenvalue set of $\boldsymbol{A}$ as follows:

$$\mathrm{Eig}_{\tau_j} = \left\{\frac{1}{\tau_j}\ln\{\lambda_i[\boldsymbol{G}_{\tau_j}]\} + \frac{\mathrm{j}2k\pi}{\tau_j}, i = 1, 2, \ldots, n, k \in \mathbb{Z}\right\},$$

$j = 1, 2, \ldots, l$.

If their intersection set

$$\mathrm{Eig}_{\tau_1} \bigcap \mathrm{Eig}_{\tau_2} \bigcap \cdots \bigcap \mathrm{Eig}_{\tau_l} \tag{10}$$

has exactly n elements by choosing the sampling periods τ_j properly, then the eigenvalues of $\boldsymbol{A}$ are reconstructible. Clearly, the choice of sampling periods plays an important role in the reconstruction of P. This can be summarized as the following lemma.

Lemma 4. *For at most $n+1$ sampling periods τ_j ($j = 1, 2, \ldots, n+1$), the eigenvalues $\lambda_i[\boldsymbol{A}]$ of the continuous-time system can be uniquely determined from the set in (10) if the ratios of any two of τ_j are irrational.*

Proof. If we can prove that the eigenvalue intersection set of $\boldsymbol{A}$,

$$\mathrm{Eig}_{\tau_1} \bigcap \mathrm{Eig}_{\tau_2} \bigcap \cdots \bigcap \mathrm{Eig}_{\tau_{n+1}},$$

has exactly n elements $\{\mu_1, \mu_2, \ldots, \mu_n\}$, then the conclusion of Lemma 4 is true. So suppose that it does not, i.e., that there exists a μ in this set which is not in $\{\mu_1, \mu_2, \ldots, \mu_n\}$. Then from the definition of Eig_{τ_j}, there must exist $i_1, i_2, \ldots, i_{n+1} \in \{1, 2, \ldots, n\}$ and $k_1, k_2, \ldots, k_{n+1} \in \mathbb{Z}$ satisfying

$$\mu = \mu_{i_1} + \mathrm{j}\frac{2\pi k_1}{\tau_1} = \mu_{i_2} + \mathrm{j}\frac{2\pi k_2}{\tau_2} = \cdots = \mu_{i_{n+1}} + \mathrm{j}\frac{2\pi k_{n+1}}{\tau_{n+1}}.$$

Thus,

$$\mathrm{Re}(\mu_{i_1}) = \mathrm{Re}(\mu_{i_2}) = \cdots = \mathrm{Re}(\mu_{i_{n+1}}).$$

It follows that two of the μ_{i_j}'s must be equal; without loss of generality we assume that $\mu_{i_1} = \mu_{i_2}$. Hence

$$\frac{2\pi k_1}{\tau_1} = \frac{2\pi k_2}{\tau_2}.$$

Due to the assumption that μ is not an eigenvalue of $\boldsymbol{A}$, neither k_1 nor k_2 can be zero, which means that

$$\frac{\tau_1}{\tau_2} = \frac{k_1}{k_2},$$

a rational number. This is a contradiction. Hence Lemma 4 is proven. □

Once the eigenvalues $\lambda_i[\boldsymbol{A}]$ of $\boldsymbol{A}$ are found, it is relatively routine to compute $\boldsymbol{A}$ from $\boldsymbol{G}_\tau$. The following is to give a way to find $\boldsymbol{A}$ from $\boldsymbol{G}_\tau$. When $\boldsymbol{A}$ has n distinct eigenvalues, so does $\boldsymbol{G}_\tau$ because τ is non-pathological with respect to $\boldsymbol{A}$; then $\boldsymbol{G}_\tau$ and $\boldsymbol{A}$ share same eigenvectors. Let $\boldsymbol{T}$ be the matrix consisting of the eigenvectors of $\boldsymbol{G}_\tau$ and

$$\boldsymbol{\Lambda} := \begin{bmatrix} \lambda_1[\boldsymbol{G}_\tau] & & \\ & \ddots & \\ & & \lambda_n[\boldsymbol{G}_\tau] \end{bmatrix}.$$

We have

$$\boldsymbol{A} = \frac{1}{\tau}\mathrm{Ln}[\boldsymbol{G}_\tau] = \frac{1}{\tau}\mathrm{Ln}[\boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1}] = \frac{1}{\tau}\boldsymbol{T}\mathrm{Ln}[\boldsymbol{\Lambda}]\boldsymbol{T}^{-1}$$
$$= \boldsymbol{T}\begin{bmatrix} \mathrm{Ln}\{\lambda_1[\boldsymbol{G}_\tau]\}^{1/\tau} & & \\ & \ddots & \\ & & \mathrm{Ln}\{\lambda_n[\boldsymbol{G}_\tau]\}^{1/\tau} \end{bmatrix}\boldsymbol{T}^{-1}.$$

Replacing $\mathrm{Ln}\{\lambda_i[\boldsymbol{G}_\tau]\}^{1/\tau}$ by $\lambda_i[\boldsymbol{A}]$ yields

$$\boldsymbol{A} = \boldsymbol{T}\begin{bmatrix} \lambda_1[\boldsymbol{A}] & & \\ & \ddots & \\ & & \lambda_n[\boldsymbol{A}] \end{bmatrix}\boldsymbol{T}^{-1}.$$

Since $\boldsymbol{G}_\tau$ has been obtained in the preceding section, $\boldsymbol{T}$ can be computed according to $\boldsymbol{G}_\tau$.

If $\boldsymbol{A}$ is not diagonalizable, but $\boldsymbol{A}$ and $\boldsymbol{G}_\tau$ have the same Jordan structure, by which we can find $\boldsymbol{A}$ from $\boldsymbol{G}_\tau$.

6. Parameter identification algorithms

In the preceding two sections, assuming that the parameter matrices $(\boldsymbol{G}, \boldsymbol{F}, \boldsymbol{\Gamma}, \boldsymbol{H})$ of the non-uniformly sampled system in (6) are known, we have discussed computation of the single-rate models with different sampling intervals and recovery of the continuous-time system. Next, we develop the identification algorithms for non-uniformly sampled systems, including the two cases with either known or unknown states.

6.1. The case with known state

In practice, the system output and/or states are often contaminated by disturbances and after introducing uncorrelated noise vectors $\{\boldsymbol{w}(kT), \boldsymbol{v}(kT)\}$, the system in (6) becomes

$$P_n: \left[\begin{array}{c} \boldsymbol{x}(kT+T) \\ \hline \underline{\boldsymbol{y}}(kT) \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{G} & \boldsymbol{F} \\ \hline \boldsymbol{\Gamma} & \boldsymbol{H} \end{array}\right]\left[\begin{array}{c} \boldsymbol{x}(kT) \\ \hline \underline{\boldsymbol{u}}(kT) \end{array}\right] + \left[\begin{array}{c} \boldsymbol{w}(kT) \\ \hline \boldsymbol{v}(kT) \end{array}\right]. \quad (11)$$

Define the parameter matrix $\boldsymbol{\theta}$, information vector $\boldsymbol{\varphi}_0(kT)$, generalized output vector $\boldsymbol{Z}_0(kT)$ and noise vector $\boldsymbol{E}(kT)$ as

$$\boldsymbol{\theta}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{G} & \boldsymbol{F} \\ \boldsymbol{\Gamma} & \boldsymbol{H} \end{bmatrix}, \quad \boldsymbol{\varphi}_0(kT) = \begin{bmatrix} \boldsymbol{x}(kT) \\ \underline{\boldsymbol{u}}(kT) \end{bmatrix},$$
$$\boldsymbol{Z}_0(kT) = \begin{bmatrix} \boldsymbol{x}(kT+T) \\ \underline{\boldsymbol{y}}(kT) \end{bmatrix}, \quad \boldsymbol{E}(kT) = \begin{bmatrix} \boldsymbol{w}(kT) \\ \boldsymbol{v}(kT) \end{bmatrix}.$$

The system in (11) may be written as a linear regression model,

$$\boldsymbol{Z}_0(kT) = \boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\varphi}_0(kT) + \boldsymbol{E}(kT). \quad (12)$$

Suppose the input and output data are available and persistently exciting. If the system states $\boldsymbol{x}(kT)$ are measured [i.e., $\boldsymbol{Z}(kT)$ and $\boldsymbol{\varphi}_0(kT)$ are known], then the parameter matrix $\boldsymbol{\theta}$ can be estimated by the least squares algorithm:

$$\hat{\boldsymbol{\theta}}(kT+T) = \hat{\boldsymbol{\theta}}(kT) + \boldsymbol{P}_0(kT+T)\boldsymbol{\varphi}_0(kT) \times [\boldsymbol{Z}_0^{\mathrm{T}}(kT) - \boldsymbol{\varphi}_0^{\mathrm{T}}(kT)\hat{\boldsymbol{\theta}}(kT)], \quad (13)$$

$$\boldsymbol{P}_0(kT+T) = \boldsymbol{P}_0(kT) - \frac{\boldsymbol{P}_0(kT)\boldsymbol{\varphi}_0(kT)\boldsymbol{\varphi}_0^{\mathrm{T}}(kT)\boldsymbol{P}_0(kT)}{1+\boldsymbol{\varphi}_0^{\mathrm{T}}(kT)\boldsymbol{P}_0(kT)\boldsymbol{\varphi}_0(kT)}, \quad (14)$$

where $\boldsymbol{P}_0(kT)$ denotes the covariance matrix and $\hat{\boldsymbol{\theta}}(kT)$ represents the estimate of $\boldsymbol{\theta}$ with

$$\begin{bmatrix} \hat{\boldsymbol{G}}(kT) & \hat{\boldsymbol{F}}(kT) \\ \hat{\boldsymbol{\Gamma}}(kT) & \hat{\boldsymbol{H}}(kT) \end{bmatrix} = \hat{\boldsymbol{\theta}}^{\mathrm{T}}(kT).$$

6.2. The case with unknown states

If the states are not measured, it is clear that the identification expression in (12) contains both the unknown state vector $\boldsymbol{x}(kT)$ in $\boldsymbol{\varphi}_0(kT)$ and unknown parameter matrix $\boldsymbol{\theta}$ and thus, the least squares algorithm in (13) and (14) cannot be applied to identify the models in (12). In order to identify/estimate the parameter matrix $\boldsymbol{\theta}$ and state vector $\boldsymbol{x}(kT)$, we derive combined state and parameter estimation algorithms according to the hierarchical identification principle (Ding & Chen, 2005a,b,d). The basic idea is as follows: when recursive estimating the parameter matrix $\boldsymbol{\theta}$, the unknown state vector $\boldsymbol{x}(kT)$ in $\boldsymbol{\varphi}_0(kT)$ is replaced by its corresponding estimate $\hat{\boldsymbol{x}}(kT)$, and $\boldsymbol{\varphi}_0(kT)$ by $\boldsymbol{\varphi}(kT)$; in the same way, when estimating the state vector $\boldsymbol{x}(kT+T)$, the unknown parameter matrix $\boldsymbol{\theta}$ is also replaced by its estimates $\hat{\boldsymbol{\theta}}(kT)$. Based on this idea, we easily derive the following hierarchical identification algorithm consisting of both parameter and state estimation steps as follows.

- **The first step:** The state estimation algorithm

  Let $\hat{\boldsymbol{x}}(kT)$ be the estimates of $\boldsymbol{x}(kT)$. Assume that at time kT, we have gotten the estimates $\hat{\boldsymbol{x}}(kT)$ and input–output $\underline{\boldsymbol{u}}(kT)$ and $\underline{\boldsymbol{y}}(kT)$, and parameter estimation $\hat{\boldsymbol{\theta}}(kT)$ obtained in the second step:

$$\hat{\boldsymbol{\theta}}^{\mathrm{T}}(kT) = \begin{bmatrix} \hat{\boldsymbol{G}}(kT) & \hat{\boldsymbol{F}}(kT) \\ \hat{\boldsymbol{\Gamma}}(kT) & \hat{\boldsymbol{H}}(kT) \end{bmatrix}. \quad (15)$$

  Replacing unknown $(\boldsymbol{G}, \boldsymbol{F}, \boldsymbol{\Gamma}, \boldsymbol{H})$ in (11) by the estimates $(\hat{\boldsymbol{G}}(kT), \hat{\boldsymbol{F}}(kT), \hat{\boldsymbol{\Gamma}}(kT), \hat{\boldsymbol{H}}(kT))$. Applying the Kalman filtering principle to (11), it is easy to derive the estimation algorithm of the state $\boldsymbol{x}(kT+T)$:

$$\hat{\boldsymbol{x}}(kT+T) = \hat{\boldsymbol{G}}(kT)\hat{\boldsymbol{x}}(kT) + \hat{\boldsymbol{F}}(kT)\underline{\boldsymbol{u}}(kT) + \boldsymbol{L}_1(kT)[\underline{\boldsymbol{y}}(kT) - \hat{\boldsymbol{\Gamma}}(kT)\hat{\boldsymbol{x}}(kT) - \hat{\boldsymbol{H}}(kT)\underline{\boldsymbol{u}}(kT)], \quad (16)$$

$$\boldsymbol{L}_1(kT) = \hat{\boldsymbol{G}}(kT)\boldsymbol{P}_1(kT)\hat{\boldsymbol{\Gamma}}^{\mathrm{T}}(kT) \times [\boldsymbol{R}_v + \hat{\boldsymbol{\Gamma}}(kT)\boldsymbol{P}_1(kT)\hat{\boldsymbol{\Gamma}}^{\mathrm{T}}(kT)]^{-1}, \quad (17)$$

$$\boldsymbol{P}_1(kT+T) = [\hat{\boldsymbol{G}}(kT) - \boldsymbol{L}_1(kT)\hat{\boldsymbol{\Gamma}}(kT)] \times \boldsymbol{P}_1(kT)\hat{\boldsymbol{G}}^{\mathrm{T}}(kT) + \boldsymbol{R}_w, \quad (18)$$

  where $\boldsymbol{L}_1(kT)$ and $\boldsymbol{P}_1(kT)$ are the algorithm gain and covariance matrix, respectively, the estimates $(\hat{\boldsymbol{G}}(kT), \hat{\boldsymbol{F}}(kT), \hat{\boldsymbol{\Gamma}}(kT), \hat{\boldsymbol{H}}(kT))$ of $(\boldsymbol{G}, \boldsymbol{F}, \boldsymbol{\Gamma}, \boldsymbol{H})$ are formed by using the entries of the obtained $\hat{\boldsymbol{\theta}}(kT)$ by (15) in the second step, and the covariance matrices $\boldsymbol{R}_v$ and $\boldsymbol{R}_w$ of $\boldsymbol{w}(kT)$ and $\boldsymbol{v}(kT)$ are replaced by their estimates,

$$\hat{\boldsymbol{R}}_w(kT) = \frac{1}{k}\sum_{i=1}^{k}[\hat{\boldsymbol{x}}(iT+T) - \hat{\boldsymbol{G}}(kT)\hat{\boldsymbol{x}}(iT) - \hat{\boldsymbol{F}}(kT)\underline{\boldsymbol{u}}(iT)] \times [\hat{\boldsymbol{x}}(iT+T) - \hat{\boldsymbol{G}}(kT)\hat{\boldsymbol{x}}(iT) - \hat{\boldsymbol{F}}(kT)\underline{\boldsymbol{u}}(iT)]^{\mathrm{T}}, \quad (19)$$

$$\hat{\boldsymbol{R}}_v(kT) = \frac{1}{k}\sum_{i=1}^{k}[\underline{\boldsymbol{y}}(iT) - \hat{\boldsymbol{\Gamma}}(kT)\hat{\boldsymbol{x}}(iT) - \hat{\boldsymbol{H}}(kT)\underline{\boldsymbol{u}}(iT)] \times [\underline{\boldsymbol{y}}(iT) - \hat{\boldsymbol{\Gamma}}(kT)\hat{\boldsymbol{x}}(iT) - \hat{\boldsymbol{H}}(kT)\underline{\boldsymbol{u}}(iT)]^{\mathrm{T}}. \quad (20)$$

The state estimator in (16)–(20) is derived by using the obtained parameter estimates based on the Kalman filtering principle. But this estimator involves heavy computational efforts; for computational efficiency, a stochastic approximation algorithm may be used for state estimation without computing the covariance matrices:

$$\hat{\boldsymbol{x}}(kT+T) = \hat{\boldsymbol{G}}(kT)\hat{\boldsymbol{x}}(kT) + \hat{\boldsymbol{F}}(kT)\underline{\boldsymbol{u}}(kT) + \rho(kT)\hat{\boldsymbol{\Gamma}}^{\mathrm{T}}(kT)[\underline{\boldsymbol{y}}(kT) - \hat{\boldsymbol{\Gamma}}(kT)\hat{\boldsymbol{x}}(kT) - \hat{\boldsymbol{H}}(kT)\underline{\boldsymbol{u}}(kT)], \quad (21)$$

where $\rho(kT)$ is the convergence factor satisfying

$$\rho(kT) \geq 0, \quad \sum_{k=1}^{\infty}\rho(kT) = \infty, \qquad \sum_{k=1}^{\infty}\rho^2(kT) < \infty.$$

- **The second step:** The parameter estimation algorithm

  Using $\hat{\boldsymbol{x}}(kT+T)$, $\hat{\boldsymbol{x}}(kT)$, $\underline{\boldsymbol{u}}(kT)$ and $\underline{\boldsymbol{y}}(kT)$ to form $\boldsymbol{\varphi}(kT)$ and $\boldsymbol{Z}(kT)$ as

$$\boldsymbol{\varphi}(kT) = \begin{bmatrix}\hat{\boldsymbol{x}}(kT)\\ \underline{\boldsymbol{u}}(kT)\end{bmatrix}, \qquad \boldsymbol{Z}(kT) = \begin{bmatrix}\hat{\boldsymbol{x}}(kT+T)\\ \underline{\boldsymbol{y}}(kT)\end{bmatrix} \quad (22)$$

which are available, replacing $\boldsymbol{\varphi}_0(kT)$ and $\boldsymbol{Z}_0(kT)$ in (12) by $\boldsymbol{\varphi}(kT)$ and $\boldsymbol{Z}(kT)$, and applying the least squares principle lead to the estimation algorithm of the parameter matrix $\boldsymbol{\theta}$:

$$\hat{\boldsymbol{\theta}}(kT+T) = \hat{\boldsymbol{\theta}}(kT) + \boldsymbol{P}_2(kT+T)\boldsymbol{\varphi}(kT) \times [\boldsymbol{Z}^{\mathrm{T}}(kT) - \boldsymbol{\varphi}^{\mathrm{T}}(kT)\hat{\boldsymbol{\theta}}(kT)], \quad (23)$$

$$\boldsymbol{P}_2(kT+T) = \boldsymbol{P}_2(kT) - \frac{\boldsymbol{P}_2(kT)\boldsymbol{\varphi}(kT)\boldsymbol{\varphi}^{\mathrm{T}}(kT)\boldsymbol{P}_2(kT)}{1+\boldsymbol{\varphi}^{\mathrm{T}}(kT)\boldsymbol{P}_2(kT)\boldsymbol{\varphi}(kT)}. \quad (24)$$

$$\begin{bmatrix}\hat{\boldsymbol{G}}(kT) & \hat{\boldsymbol{F}}(kT)\\ \hat{\boldsymbol{\Gamma}}(kT) & \hat{\boldsymbol{H}}(kT)\end{bmatrix} = \hat{\boldsymbol{\theta}}^{\mathrm{T}}(kT). \quad (25)$$

To initialize the above algorithms, we take $\boldsymbol{P}_i(0) = p_0\boldsymbol{I}$ ($i = 0, 1, 2$) with p_0 normally a large positive number (e.g., $p_0 = 10^6$), and $\hat{\boldsymbol{x}}(0)$ and $\hat{\boldsymbol{\theta}}(0)$ some small real vectors, e.g., $\hat{\boldsymbol{x}}(0) = \boldsymbol{1}/p_0$ and $\hat{\boldsymbol{\theta}}(0) = \boldsymbol{1}/p_0$ with $\boldsymbol{1}$ being an column vector/matrix, of appropriate sizes, whose elements are all 1.

The combined parameter and state estimation algorithm in (16)–(20) and (22)–(25) or (21)–(25) performs a hierarchical computation process with k increasing because the state estimates $\hat{\boldsymbol{x}}(kT+T)$ depend not only on the previous estimates $\hat{\boldsymbol{x}}(kT)$ but also on the parameter estimates $\hat{\boldsymbol{\theta}}(kT)$, and the parameter estimates $\hat{\boldsymbol{\theta}}(kT+T)$ depend not only on the previous estimates $\hat{\boldsymbol{\theta}}(kT)$ but also on the state estimates $\hat{\boldsymbol{x}}(kT)$. Thus, this algorithm is referred to as the hierarchical identification algorithm for non-uniformly sampled systems.

The state estimation is very useful for designing state feedback. This combined state and parameter estimation algorithm for non-uniformly sampled systems can be regarded as the extension of that for general dual-rate sampled-data systems (Ding & Chen, 2005d).

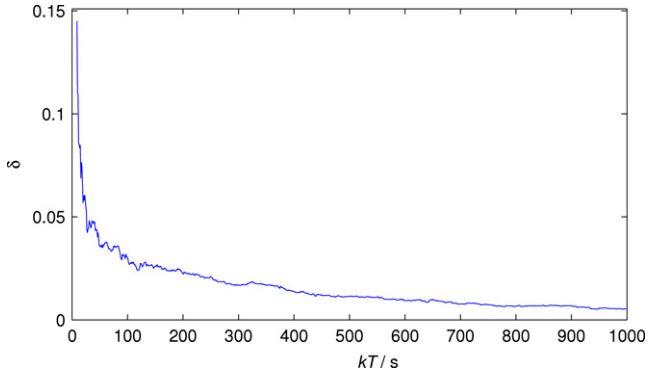


Fig. 5. The parameter estimation error δ vs. kT.

7. Example

Consider the system depicted in Fig. 4 with the process model P,

$$P(s) = \frac{s+0.8}{s^2+0.8s+0.8}$$

which has the following state space realization,

$$\begin{cases}\dot{\boldsymbol{x}}(t) = \begin{bmatrix}-0.8 & -0.8\\ 1 & 0\end{bmatrix}\boldsymbol{x}(t) + \begin{bmatrix}1\\ 0\end{bmatrix}u(t),\\ y(t) = [1, 0.8]\boldsymbol{x}(t).\end{cases}$$

Let $t_1 = \sqrt{2}-1$ s, $t_2 = 1 = T$ s, i.e., $\tau_1 = t_1$ s, $\tau_2 = 2-\sqrt{2}$ s. Discretizing this example system and introducing the noise vectors, we get

$$\boldsymbol{x}(kT+T) = \begin{bmatrix}0.22659 & -0.48086\\ 0.60107 & 0.70745\end{bmatrix}\boldsymbol{x}(kT) + \begin{bmatrix}0.15443 & 0.44665\\ 0.22129 & 0.1444\end{bmatrix}\begin{bmatrix}u(kT)\\ u(kT+t_1)\end{bmatrix} + \boldsymbol{w}(kT),$$

$$\begin{bmatrix}y(kT)\\ y(kT+t_1)\end{bmatrix} = \begin{bmatrix}1 & 0.8\\ 0.93905 & 0.47557\end{bmatrix}\boldsymbol{x}(kT) + \begin{bmatrix}0 & 0\\ 0.40553 & 0\end{bmatrix}\begin{bmatrix}u(kT)\\ u(kT+t_1)\end{bmatrix} + \boldsymbol{v}(kT).$$

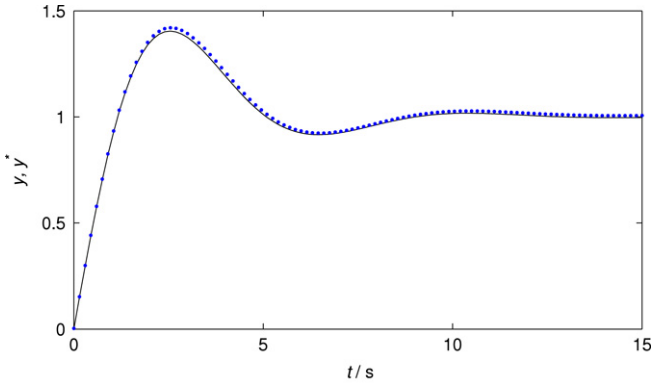
The identification procedure is summarized as follows: First, we use the *idinput* function in Matlab to generate a random signal sequence with zero mean and unit variance as the input signal and two uncorrelated noise sequences with zero mean and variances $\sigma^2 = 0.1^2$ as $\boldsymbol{w}(kT)$ and $\boldsymbol{v}(kT)$, and then compute the states and outputs. Second, based on the input–output and state data with corrupted noises, we apply the identification algorithm in the preceding section to estimate the parameters of the non-uniformly sampled system. The parameter estimation error δ versus kT is shown in Fig. 5, where $\delta := \|\hat{\boldsymbol{\theta}}(kT) - \boldsymbol{\theta}\|/\|\boldsymbol{\theta}\|(\|\boldsymbol{X}\|^2 := \mathrm{tr}[\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}])$, $\boldsymbol{\theta}$ represents the true parameter matrix, $\hat{\boldsymbol{\theta}}(kT)$ the estimate of $\boldsymbol{\theta}$. The identification results are as follows. The estimated system matrices:

$$\hat{\boldsymbol{G}} = \begin{bmatrix}0.23336 & -0.47827\\ 0.60267 & 0.71027\end{bmatrix}, \qquad \hat{\boldsymbol{F}} = \begin{bmatrix}0.15635 & 0.44578\\ 0.22443 & 0.14320\end{bmatrix},$$

$$\hat{\boldsymbol{\Gamma}} = \begin{bmatrix}1.00020 & 0.79904\\ 0.93456 & 0.47766\end{bmatrix}, \qquad \hat{\boldsymbol{H}} = \begin{bmatrix}0.00466 & 0.00037\\ 0.40294 & 0.00125\end{bmatrix}.$$

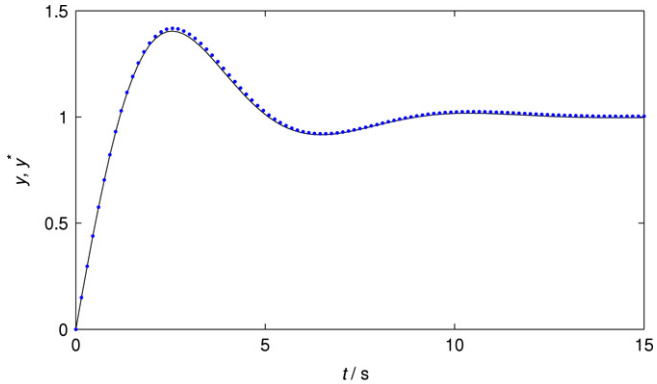
The eigenvalues of $\hat{\boldsymbol{G}}$:

$$\lambda[\hat{\boldsymbol{G}}] = \{0.79877 + \mathrm{j}0.26836, 0.79877 - \mathrm{j}0.26836\}.$$



Solid line: The output of $P(s)$;
Dots: The outputs of the estimated model $\hat{P}(s)$

Fig. 6. The step responses of the original system and the estimated model.



Solid line: The output of $P(s)$;
Dots: The outputs of the estimated model $\hat{P}_0(s)$

Fig. 7. The step responses of the original system and the estimated model with $D = 0$.

The transform matrix:

$$\boldsymbol{T} = \begin{bmatrix} -0.29544 + \mathrm{j}0.59596 & -0.29544 - \mathrm{j}0.59596 \\ 0.74669 & 0.74669 \end{bmatrix}.$$

The eigenvalues of $\hat{\boldsymbol{A}}$:

$$\{\lambda_1[\hat{\boldsymbol{A}}], \lambda_2[\hat{\boldsymbol{A}}]\} = \{-0.39484 + \mathrm{j}0.79505, -0.39484 - \mathrm{j}0.79505\}.$$

The estimated parameter matrices of the continuous-time system:

$$\begin{aligned} \hat{\boldsymbol{A}} &= \boldsymbol{T} \begin{bmatrix} \lambda_1[\hat{\boldsymbol{A}}] & \\ & \lambda_2[\hat{\boldsymbol{A}}] \end{bmatrix} \boldsymbol{T}^{-1} \\ &= \begin{bmatrix} -0.78897 & -0.79050 \\ 0.99613 & -0.00071 \end{bmatrix}. \end{aligned}$$

$$\hat{\boldsymbol{B}} = \left[\int_0^T \exp(\hat{\boldsymbol{A}}t)\mathrm{d}t\right]^{-1} [\hat{\boldsymbol{F}}(:, 1) + \hat{\boldsymbol{F}}(:, 2)] = \begin{bmatrix} 0.99652 \\ 0.00358 \end{bmatrix},$$

$$\hat{\boldsymbol{C}} = \hat{\boldsymbol{\Gamma}}(1, :) = [1.00020, 0.79904].$$

$$\hat{D} = [\hat{\boldsymbol{H}}(1, 1) + \hat{\boldsymbol{H}}(2, 2)]/2 = 0.00296.$$

The estimated transfer function:

$$\begin{aligned} \hat{P}(s) &= \hat{\boldsymbol{C}}[s\boldsymbol{I} - \hat{\boldsymbol{A}}]^{-1}\hat{\boldsymbol{B}} + \hat{D} \\ &= \frac{0.00296s^2 + 1.002s + 0.7956}{s^2 + 0.7897s + 0.788}. \end{aligned}$$

Fig. 6 compares step responses of the two systems $P(s)$ and $\hat{P}(s)$. From Figs. 5 and 6, we can see that the parameter estimation error δ is becoming smaller (in general) as the number of iterations (k) increase, and the step response of the estimated model $\hat{P}(s)$ is very close to that of the original system $P(s)$. This indicates that the estimated model can capture the process dynamics very well and can achieve satisfactory results.

For the system with unknown parameters, it is impossible to know beforehand that D equals zero, so we must estimate D. Otherwise, D does not require identifying and the estimated transfer function will be:

$$\hat{P}_0(s) = \frac{s + 0.7933}{s^2 + 0.7897s + 0.788}.$$

The step responses of $P(s)$ and $\hat{P}_0(s)$ in Fig. 7 are also very close.

8. Conclusions

This paper addresses some related issues of non-uniformly sampled systems, including computation of single-rate models with different sampling periods and recovery of the continuous-time systems from their non-uniformly sampled systems. It is shown that a continuous-time system can be reconstructed uniquely from its non-uniformly sampled discrete-time model. Finally, parameter identification algorithms for non-uniformly sampled discrete-time systems are developed.

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References

Albertos, P., & Crespo, A. (1999). Real-time control of non-uniformly sampled systems. *Control Engineering Practice*, *7*, 445–458.
Bingulac, S., & Cooper, D. L. (1990). Derivation of discrete and continuous time ramp invariant representations. *Electronics Letters*, *26*(10), 664–666.
Chen, T., & Francis, B. (1995). *Optimal sampled-data control systems*. London: Springer-Verlag.
Chen, T., & Miller, D. (2000). Reconstruction of continuous-time systems from their discretizations. *IEEE Transactions on Automatic Control*, *45*(10), 1914–1917.
Chen, T., & Qiu, L. (1994). H_∞ design of general multirate sampled-data control systems. *Automatica*, *30*(7), 1139–1152.
Ding, F., & Chen, T. (2004a). Least squares based self-tuning control of dual-rate systems. *International Journal of Adaptive Control and Signal Processing*, *18*(8), 697–714.
Ding, F., & Chen, T. (2004b). Identification of dual-rate systems based on finite impulse response models. *International Journal of Adaptive Control and Signal Processing*, *18*(7), 589–598.
Ding, F., & Chen, T. (2004c). Combined parameter and output estimation of dual-rate systems using an auxiliary model. *Automatica*, *40*(10), 1739–1748.
Ding, F., & Chen, T. (2005a). Hierarchical gradient-based identification of multivariable discrete-time systems. *Automatica*, *41*(2), 315–325.
Ding, F., & Chen, T. (2005b). Hierarchical least squares identification methods for multivariable systems. *IEEE Transactions on Automatic Control*, *50*(3), 397–402.
Ding, F., & Chen, T. (2005c). Parameter estimation of dual-rate stochastic systems by using an output error method. *IEEE Transactions on Automatic Control*, *50*(9), 1436–1441.
Ding, F., & Chen, T. (2005d). Hierarchical identification of lifted state-space models for general dual-rate systems. *IEEE Transactions on Circuits and Systems–I: Regular Papers*, *52*(6), 1179–1187.
Ding, F., & Chen, T. (2007). Performance analysis of multi-innovation gradient type identification methods. *Automatica*, *43*(1), 1–14.
Ding, F., Chen, T., & Iwai, Z. (2007). Adaptive digital control of Hammerstein nonlinear systems with limited output sampling. *SIAM Journal on Control and Optimization*, *45*(6), 2257–2276.
Ding, J., & Ding, F. (2008). The residual based extended least squares identification method for dual-rate systems. *Computers Mathematics with Applications*, *56*(6), 1479–1487.
Francis, B. A., & Georgiou, T. T. (1988). Stability theory for linear time-invariant plants with periodic digital controllers. *IEEE Transactions on Automatic Control*, *33*(9), 820–832.

Goodwin, G. C., & Sin, K. S. (1984). *Adaptive filtering, prediction and control*. Englewood Cliffs, NJ: Prentice-Hall.
Gudi, R. D., Shah, S. L., & Gray, M. R. (1994). Multirate state and parameter estimation in an antibiotic fermentation with delayed measurements. *Biotechnology and Bioengineering*, *44*(11), 1271–1278.
Gudi, R. D., Shah, S. L., & Gray, M. R. (1995). Adaptive multirate state and parameter estimation strategies with application to a bioreactor. *AIChE Journal*, *41*(11), 2451–2464.
Kreisselmeier, G. (1999). On sampling without loss of observability/controllability. *IEEE Transactions on Automatic Control*, *44*(5), 1021–1025.
Li, W., Han, Z., & Shah, S. L. (2006). Subspace identification for FDI in systems with non-uniformly sampled multirate data. *Automatica*, *42*(4), 619–627.
Ljung, L. (1999). *System identification: Theory for the user* (2nd Ed.). Englewood Cliffs, NJ: Prentice-Hall.
Qiu, L., & Chen, T. (1999). Multirate sampled-data systems: All H_∞ suboptimal controllers and the minimum entropy controller. *IEEE Transactions on Automatic Control*, *44*(3), 537–550.
Scattolini, R., & Schiavoni, N. (1995). A multirate model-based predictive controller. *IEEE Transactions on Automatic Control*, *40*, 1093–1097.
Sheng, J., Chen, T., & Shah, S. L. (2002). Generalized predictive control for non-uniformly sampled systems. *Journal of Process Control*, *12*(8), 875–885.
Sinha, N. K., & Lastman, G. J. (1982). Identification of continuous-time multivariable systems from sampled data. *International Journal of Control*, *35*(1), 117–126.
Tatiraju, S., Soroush, M., & Mutharasan, R. (1999). Multirate nonlinear state and parameter estimation in a bioreactor. *Biotechnology and Bioengineering*, *63*(1), 22–32.
Zhang, C., Middleton, R. H., & Evans, R. J. (1989). An algorithm for multirate sampling adaptive control. *IEEE Transactions on Automatic Control*, *34*, 792–795.

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