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Robust Stability of Feedback Systems on Banach Spaces*

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Abstract

This paper studies the robust stability of feedback systems. The special features of this study are: (1) the input-output signal spaces of the feedback systems are assumed to be Banach spaces, possibly $\mathcal{L}_\infty$, ℓ_∞, $\mathcal{L}_1$, and ℓ_1; (2) the perturbations in systems are measured by the gap funtion.

1 Introduction

Consider a feedback system

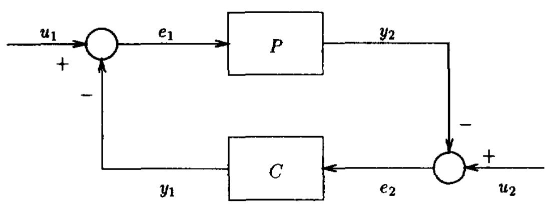

Figure 1: The Standard Feedback System

where

u_1, e_1, y_1 are signals in a Banach space $\mathcal{U}$;

u_2, e_2, y_2 are signals in another Banach space $\mathcal{Y}$;

P is an unbounded linear operator from $\mathcal{U}$ to $\mathcal{Y}$, considered usually as the plant;

C is an unbounded linear operator from $\mathcal{Y}$ to $\mathcal{U}$, considered usually as the controller.

The purpose of this paper is to study the robust stability of this feedback system when the plant is subject to uncertainties measured by the *gap* function [7].

A new robust control theory based on the gap description of uncertainties for systems on Hilbert spaces has recently emerged. Especially, many good results have been discovered for the case when $\mathcal{U}$ and $\mathcal{Y}$ are $\mathcal{L}_2$ or ℓ_2, and P and C are finite dimensional time-invariant systems. A list of major publications (by no means complete) on this theory is given in the reference section ([2]–[6], [9]–[15]). However, in many control applications, the signal spaces are not Hilbert spaces. Here, we strive to extend some existing results for Hilbert space case to Banach space case. Our ultimate goal is to study robust stability of feedback systems on some common signal spaces such as $\mathcal{L}_\infty$, ℓ_∞, $\mathcal{L}_1$, and ℓ_1.

2 General gap theory

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$ and let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two subspaces (closed linear manifolds) of $\mathcal{X}$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are both nontrivial, the *gap* between $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by

$$\gamma(\mathcal{S}_1,\mathcal{S}_2) = \max\left\{\sup_{x\in\mathcal{S}_1,\|x\|=1}\inf_{y\in\mathcal{S}_2}\|x-y\|,\ \sup_{y\in\mathcal{S}_2,\|y\|=1}\inf_{x\in\mathcal{S}_1}\|y-x\|\right\}.$$

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We also define $\gamma(\mathcal{S}_1,\mathcal{S}_2) = 0$ if $\mathcal{S}_1 = \mathcal{S}_2 = \{0\}$ and $\gamma(\mathcal{S}_1,\mathcal{S}_2) = 1$ if one of $\mathcal{S}_1$ or $\mathcal{S}_2$ is $\{0\}$ and the other one is not $\{0\}$.

It is well-known that in general γ is not a metric on the set of all subspaces of $\mathcal{X}$, but it does induce a topology on this set. A comprehensive treatment of the gap function can be found in [7].

Let A be a bounded linear operator on $\mathcal{X}$ with bounded inverse. By closed graph theorem, this requires only that A be a bijective bounded linear operator [6, Problem 52]. Define the condition number of A to be $\kappa(A) = \|A\|\|A^{-1}\|$. The following result is new.

Proposition 1 $\kappa^{-1}(A)\gamma(\mathcal{S}_1,\mathcal{S}_2) \le \gamma(A\mathcal{S}_1, A\mathcal{S}_2) \le \kappa(A)\gamma(\mathcal{S}_1,\mathcal{S}_2)$.

Associated with the concept of the gap, there is another useful concept called the *minimum opening.* The minimum opening between $\mathcal{S}_1$ and $\mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are two nontrivial subspaces of $\mathcal{X}$, is defined by

$$\rho(\mathcal{S}_1,\mathcal{S}_2) = \min\left\{\inf_{x\in\mathcal{S}_1,\|x\|=1}\inf_{y\in\mathcal{S}_2}\|x-y\|,\ \inf_{y\in\mathcal{S}_2,\|y\|=1}\inf_{x\in\mathcal{S}_1}\|y-x\|\right\}.$$

It can be shown that $\mathcal{S}_1\cap\mathcal{S}_2 = \{0\}$ and $\mathcal{S}_1+\mathcal{S}_2$ is closed if and only if $\rho(\mathcal{S}_1,\mathcal{S}_2) \neq 0$. In the following, we write $\mathcal{S}_1\oplus\mathcal{S}_2 = \mathcal{X}$ to mean $\mathcal{S}_1\cap\mathcal{S}_2 = \{0\}$ and $\mathcal{S}_1+\mathcal{S}_2 = \mathcal{X}$. The following result follows from [7, Theorem 4.24].

Proposition 2 *Assume* $\mathcal{S}_1\oplus\mathcal{S}_2 = \mathcal{X}$. *Then* $\tilde{\mathcal{S}}_1\oplus\mathcal{S}_2 = \mathcal{X}$ *if* $\gamma(\tilde{\mathcal{S}}_1,\mathcal{S}_1) < \rho(\mathcal{S}_1,\mathcal{S}_2)$.

3 Robust stability

Here we need the concept of the graph of an operator. Let F be an unbounded linear operator from $\mathcal{X}$ to $\mathcal{Z}$. The domain of F is defined by

$$\mathcal{D}_F = \{x\in\mathcal{X} : Fx\in\mathcal{Z}\},$$

and the graph of F is defined by

$$\mathcal{G}_F = \left\{\begin{bmatrix} x \\ Fx\end{bmatrix} : x\in\mathcal{D}_F\right\}.$$

Clearly, $\mathcal{G}_F$ is a linear manifold of $\mathcal{X}\times\mathcal{Z}$. The product space $\mathcal{X}\times\mathcal{Z}$ is a Banach space if we assume that the norm in $\mathcal{X}\times\mathcal{Z}$ is defined by

$$\left\|\begin{bmatrix} x \\ z\end{bmatrix}\right\| = \phi(\|x\|,\|z\|)$$

where ϕ is a norming function. We leave the particular choice of ϕ to the user. A natural choice usually depends on the nature of $\mathcal{X}$ and $\mathcal{Z}$. If $\mathcal{G}_F$ is closed, then F is said to be a closed operator. Define the gap between two closed operators, denoted by δ, to be the gap between their graphs, i.e.,

$$\delta(F_1,F_2) = \gamma(\mathcal{G}_{F_1},\mathcal{G}_{F_2}).$$

Then a ball of closed operators is a set of the form

$$\mathcal{B}(F,r) = \{\tilde{F} : \delta(\tilde{F},F) < r\}.$$

The collection of all balls thus gives a base of a topology in the set of all closed operators from $\mathcal{X}$ to $\mathcal{Z}$.

In the following, we also need the so-called inverse graph of an operator which is given by

$$\mathcal{G}'_F = \left\{ \begin{bmatrix} Fx \\ x \end{bmatrix} : x \in \mathcal{D}_F \right\}.$$

Now consider the feedback system shown in Figure 1. We will simply call it (P, C). Assume P and C are closed operators. We will first look at the qualitative properties. The equations governing the system variables are

$$\begin{aligned} e_1 + Ce_2 &= u_1 \\ Pe_1 + e_2 &= u_2. \end{aligned}$$

Consider the linear manifold

$$\mathcal{S} = \left\{ \begin{bmatrix} e_1 + Ce_2 \\ Pe_1 + e_2 \\ e_1 \\ e_2 \end{bmatrix} : e_1 \in \mathcal{D}_P, e_2 \in \mathcal{D}_C \right\} \subset \mathcal{U} \times \mathcal{Y} \times \mathcal{U} \times \mathcal{Y}.$$

It is easy to show that S is closed. If there exists a closed operator F from $\mathcal{U} \times \mathcal{Y}$ to $\mathcal{U} \times \mathcal{Y}$ such that $\mathcal{G}_F = S$, then (P, C) is said to be well-posed. A necessary and sufficient condition for the existence of such F is that $\begin{bmatrix} 0 \\ 0 \\ e_1 \\ e_2 \end{bmatrix} \notin S$ if either e_1 or e_2 is nonzero. In this case, F is said to be the closed loop operator of (P, C). If F is bounded, then (P, C) is said to be stable. Denote by H the map which maps a well-posed (P, C) to its closed loop operator F. Then the domain of H is the set of all well-posed (P, C) and its range is a subset of the set of all closed operators from $\mathcal{U} \times \mathcal{Y}$ to $\mathcal{U} \times \mathcal{Y}$.

Theorem 1 *H is a homeomorphism between its domain and range.*

The proof of Theorem 1 follows from Proposition 1 easily by noticing that

$$\begin{aligned} \mathcal{S} &= \left\{ \begin{bmatrix} I & 0 & 0 & I \\ 0 & I & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ Pe_1 \\ e_2 \\ Ce_2 \end{bmatrix} : e_1 \in \mathcal{D}_P, e_2 \in \mathcal{D}_C \right\} \\ &= \begin{bmatrix} I & 0 & 0 & I \\ 0 & I & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \mathcal{G}_P \times \mathcal{G}_C \end{aligned}$$

and the 4×4 big matrix is a bijective operator.

Now let us look at the quantitative properties.

Proposition 3 *(P, C) is stable if and only if $\mathcal{G}_P \oplus \mathcal{G}'_C = \mathcal{U} \times \mathcal{Y}$.*

The proof of Proposition 3 goes as follows: The system equations define an operator from $\mathcal{D}_P \times \mathcal{D}_C$ to $\mathcal{U} \times \mathcal{Y}$ which maps $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & C \\ P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. This operator has a set-theoretic inverse if and only if it is bijective. Since we have already known that the graph of the inverse, i.e., the set $\mathcal{S}$, is closed, the inverse is always bounded by the closed graph theorem. A necessary and sufficient condition for the operator $\begin{bmatrix} I & C \\ P & I \end{bmatrix}$ to be bijective is the condition given in Proposition 3.

Theorem 2 *$(\tilde{P}, C)$ is stable for all $\tilde{P} \in \mathcal{B}(P, r)$ if $r \leq \rho(\mathcal{G}_P, \mathcal{G}'_C)$.*

Theorem 2 is simply a combined consequence of Propositions 2 and 3.

4 Concluding remarks

One might want to know if the condition in Theorem 2 is tight. In the Hilbert space case, it is tight, i.e., the condition is also necessary. However, in the Banach space case, it is not tight in general. To see this, let us first show that the condition in Proposition 2 is not tight. Let $\mathcal{X} = \mathbf{R}^2$ be equipped with the Hölder ∞-norm and let $\mathcal{S}_1$ be spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathcal{S}_2$ be spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. It is easy to compute that $\rho(\mathcal{S}_1, \mathcal{S}_2) = \frac{1}{2}$. In order for $\tilde{\mathcal{S}}_1 \cap \mathcal{S}_2 \neq \{0\}$ or $\tilde{\mathcal{S}}_1 + \mathcal{S}_2 \neq \mathcal{X}$, we either have $\dim\tilde{\mathcal{S}}_1 \neq 1$ or $\tilde{\mathcal{S}}_1 = \mathcal{S}_2$. In either case, $\gamma(\tilde{\mathcal{S}}_1, \mathcal{S}_1) = 1$. This shows that $\tilde{\mathcal{S}}_1 \oplus \mathcal{S}_2 = \mathcal{X}$ if $\gamma(\tilde{\mathcal{S}}_1, \mathcal{S}_1) < 1$ and therefore the condition in Proposition 2 is not tight. Now if we take P and C to be static systems on ℓ_∞ with gain 0 and 1 respectively, a similar argument shows that $(\tilde{P}, C)$ is stable for all $\tilde{P} \in \mathcal{B}(P, r)$ if $r \leq 1$, whereas $\rho(\mathcal{G}_P, \mathcal{G}'_C) = \frac{1}{2}$. How to improve the condition in Theorem 2 constitutes a recent research topic.

As having been said in the introduction, our ultimate goal is to study the robust stability for feedback systems on some specific signal spaces. In the specific spaces, the computation of the quantities such as $\delta(\mathcal{G}_{F_1}, \mathcal{G}_{F_2})$ and $\rho(\mathcal{G}_P, \mathcal{G}'_C)$ will be our concern. An immediate robust control problem is to find a controller C for a given plant P such that $\rho(\mathcal{G}_P, \mathcal{G}'_C)$ is minimized. These problems are also our current research topics.

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