Robust Stabilization Using Jury Table
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Abstract: Compressed Hankel matrix is given by using orthonormal rational functions constructed from the Jury table. The solutions to the optimal and suboptimal Nehari problems via the compressed Hankel matrix are also given. Robust stabilization problem is reduced to the Nehari problem, so it can also be solved via Jury table.

Keywords: Orthonormal function, Compressed Hankel matrix, Nehari problem, Robust stabilization

1. Introduction

In this paper, we first study Hankel operator and the Nehari problems using the Jury table. After that we reduce the robust stabilization problem to the Nehari problem, so it can also be solved via Jury table.

The motivation is to develop elementary solutions to advanced optimal control problems so to make the advanced optimal control accessible to a wider audience. These new investigation of the connection between advanced optimal and robust control problems and the classical tools yield the solution to the Nehari problems. Since the problem plays a fundamental role in $H_\infty$ optimal control theory, its elementary solution opens the door for a simple, polynomial approach to $H_\infty$ optimal control theory. Similar study for continuous time systems is also carried out by Qiu (19).

2. Jury Table and Orthonormal Functions

Consider a stable polynomial
\[ a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \]
where $a_i \in \mathbb{R}$ and $a_0 > 0$. It is said to be stable if all of its roots are inside the unit disk.

Construct the Jury table (10) as in Table 1. In the Jury table, the first row is copied from the coefficients of the polynomial,
\[ r_{00} = a_0, \quad r_{01} = a_1, \quad \ldots, \quad r_{0n} = a_n. \]

The row $r_i$, $i = 0, \ldots, n-1$, is obtained by writing the elements of the preceding row in the reverse order. The row $r_{n+1}$, $i = 0, \ldots, n-1$, is computed from its two preceding rows $r_{i+1}$ and $r_{n-1}$ as
\[ r_{i+1} = \frac{1}{r_{i0}} \begin{vmatrix} r_{i0} & \cdots & r_{i(n-1)} \\ \cdots & \cdots & \cdots \\ r_{n0} & \cdots & r_{n(n-1)} \end{vmatrix}, \]
for $i = 0, \ldots, n-1$, $j = 0, \ldots, n-i-1$.

Table 1: Jury Table

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>$r_{00}$</th>
<th>$r_{01}$</th>
<th>$\ldots$</th>
<th>$r_{0(n-1)}$</th>
<th>$r_{0n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0^2$</td>
<td>$r_{00}$</td>
<td>$r_{0(n-1)}$</td>
<td>$\ldots$</td>
<td>$r_{1(n-1)}$</td>
<td>$r_{1n}$</td>
</tr>
<tr>
<td>$r_1$</td>
<td>$r_{10}$</td>
<td>$r_{11}$</td>
<td>$\ldots$</td>
<td>$r_{1(n-1)}$</td>
<td>$r_{1n}$</td>
</tr>
<tr>
<td>$r_n$</td>
<td>$r_{n0}$</td>
<td>$r_{n1}$</td>
<td>$\ldots$</td>
<td>$r_{n(n-1)}$</td>
<td>$r_{nn}$</td>
</tr>
</tbody>
</table>

The Jury stability criterion states that $a(z)$ is stable if and only if $r_{0i} > 0$ for all $i = 1, \ldots, n$.

Consider the set of strictly proper rational functions with denominator $a(z)$
\[ \mathcal{X}_a = \left\{ \frac{b(z)}{a(z)}, \quad \deg b(z) < \deg a(z) \right\}. \]

Clearly, $\mathcal{X}_a$ is an $n$-dimensional subspace of $\mathcal{R}H_2$. In applications, as evidenced later in this paper, it is desirable to find a basis, or better an orthonormal basis of $\mathcal{X}_a$.

The Jury table can be used to construct such an orthonormal basis of $\mathcal{X}_a$, see 5), 4) and 17). Recall the Jury table of $a(z)$ and for the rows $r_i$, $i = 1, 2, \ldots, n$, define polynomials
\[ r_1(z) = r_{10} z^{n-1} + r_{11} z^{n-2} + \cdots + r_{1(n-1)} \]
\[ \vdots \]
\[ r_{n-1}(z) = r_{n-10} z^{n-1} + r_{n-11} \]
\[ r_n(z) = r_{n0}. \]

Since $a(z)$ is stable, $r_{0i} > 0$, for $i = 1, 2, \ldots, n$. We can define
\[ \alpha_i = \sqrt{\frac{r_{00}}{r_{0i}}}, \quad i = 0, 1, 2, \ldots, n. \]

**Theorem 1** The functions $E_i(z) = \frac{r_{0i}(z)}{a(z)}$, $i = 1, 2, \ldots, n$, form orthonormal basis of $\mathcal{X}_a$. 

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3. Hankel Operator and Compressed Hankel Matrix

Hankel operators find various applications in engineering problems such as in model reduction and optimal control. The computation of the Hankel singular values and Schmidt pairs is the key for these applications and is studied in [1], [9], and [7]. Let \( P^+ : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \) and \( P^- : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \) denote the orthogonal projections such that
\[
P^+ \left( \sum_{k=-\infty}^{\infty} f(k) z^{-k} \right) = \sum_{k=0}^{\infty} f(k) z^{-k},
\]
\[
P^- \left( \sum_{k=-\infty}^{\infty} f(k) z^{-k} \right) = \sum_{k=-\infty}^{0} f(k) z^{-k}.
\]

Let \( J : \mathcal{L}_2 \rightarrow \mathcal{L}_2 \) denote the reversal operator and \( S : \mathcal{L}_2 \rightarrow \mathcal{L}_2 \) denote the backward shift operator such that
\[
J F(z) = F(z^{-1}), \quad SF(z) = zF(z).
\]

**Definition** Given a stable system with strictly proper transfer function \( G(z) \), the associated Hankel operator \( \Gamma_G : \mathcal{H}_2 \rightarrow \mathcal{H}_2 \) is defined by
\[
\Gamma_G U(z) = P^+ (G(z) U(z)), \quad U(z) \in \mathcal{H}_2.
\]

It is well-known that \( \Gamma_G \) is a finite rank operator when \( G(z) \) is rational.

**Lemma 1** \(^7\) Let \( G(z) = \frac{b(z)}{a(z)} \) be a strictly proper stable transfer function. Then
\[
\text{Im } \Gamma_G = SX_a, \quad (\text{Ker } \Gamma_G)^\perp = JX_a.
\]

The Hankel operator \( \Gamma_G \) is the orthogonal direct sum of a zero operator and a compression of \( \Gamma_G \) mapping \( JX_a \) into \( SX_a \). Everything interesting about it is contained in the compression.

This compressed Hankel operator can be represented by a matrix if we choose a basis in \( (\text{Ker } H_G)^\perp \) and a basis in \( \text{Im } H_G \). Note that both \( (\text{Ker } H_G)^\perp \) and \( \text{Im } H_G \) are isomorphic to \( X_a \). Hence we can use the orthonormal basis of \( X_a \)
\[
E(z) := \begin{bmatrix} E_1(z) & E_2(z) & \cdots & E_n(z) \end{bmatrix}
\]
defined in Theorem 1 to form an orthonormal basis in \( (\text{Ker } H_G)^\perp \)
\[
E(z^{-1}) := \begin{bmatrix} E_1(z^{-1}) & E_2(z^{-1}) & \cdots & E_n(z^{-1}) \end{bmatrix}
\]
and one in \( \text{Im } H_G \)
\[
zF(z) = \begin{bmatrix} zE_1(z) & zE_2(z) & \cdots & zE_n(z) \end{bmatrix}.
\]

We call the matrix representation under this basis \textit{Compressed Hankel Matrix} and denote it by \( H_G \). The singular values of \( H_G \) are the Hankel singular values of \( G(z) \) and are denoted by \( \sigma_1, \sigma_2, \ldots, \sigma_n \). We assume that \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \). The largest singular value is the Hankel norm of \( G(z) \) and is denoted by \( \|G(z)\|_H \). Let \( (u_i, v_i) \) be a left and right singular vectors of \( H_G \) corresponding to \( \sigma_i \) and let
\[
U_i(z) = E(z^{-1}) u_i, \quad V_i(z) = zE(z^{-1}) v_i.
\]

Then \( (U_i(z), V_i(z)) \) is a Schmidt pair of \( \Gamma_G \) corresponding to \( \sigma_i \).

We are interested in computing the Hankel singular values and Schmidt pairs of \( \Gamma_G \), the key is to find \( H_G \) from \( G(z) = \frac{b(z)}{a(z)} \). The following result can be found in \([17]\).

**Theorem 2** Construct the Jury table of \( a(z) \). Define matrix \( A \) as in (7) and \( M \) as:
\[
M = \begin{bmatrix}
\alpha_1 r_{10} & 0 & \cdots & 0 \\
\alpha_1 r_{11} & \alpha_2 r_{20} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 r_{(n-1)} & \alpha_2 r_{2(n-2)} & \cdots & \alpha_n r_{n0}
\end{bmatrix}.
\]

where \( k_i = \frac{a_i(z)}{r_{i0}}, \quad i = 0, 1, \ldots, n \). Then
\[
H_G = a^{-1}(A)^{-1} b(A) M^{-1}.
\]

The adjoint Hankel operator \( \Gamma^*_G : \mathcal{H}_2 \rightarrow \mathcal{H}_2^\perp \) is given by
\[
\Gamma^*_G U(z) = P_- (G(z^{-1}) U(z)), \quad U(z) \in \mathcal{H}_2
\]
and
\[
\text{Im } \Gamma^*_G = JX_a, \quad (\text{Ker } \Gamma^*_G)^\perp = SX_a.
\]

**Corollary 1** The adjoint Hankel operator \( \Gamma^*_G \) satisfies
\[
\Gamma^*_G = SJ \Gamma G SJ.
\]
Remark 1: Corollary 1 implies that the compressed matrix representation of $F_G$ is also $H_G$. By definition, the matrix representation of $F_G$ is $H_G$. Hence $H_G$ must be symmetric and

$$U_i(z) = \pm V_i(z^{-1}) = \pm SJV_i(z). \quad (8)$$

This fact may offer some simplification in the computation.

4. Solution to Nehari Problem

In this section, we apply the materials in the last section to the solutions of the optimal and suboptimal Nehari problem. The Nehari problem plays an important role in robust and optimal control; it is an approximation problem with respect to the $L_\infty$ norm. Given a stable strictly proper system $G(z) = \frac{b(z)}{a(z)}$, find $Q(z) \in H_\infty$ to minimize $\|G(z^{-1}) - Q(z)\|_\infty$. The following theorem is well-known, see also 7).

**Theorem 3** Let $(U_1(z), V_1(z))$ be the Schmidt pair of $H_G$ corresponding to the largest Hankel singular value $\sigma_1$. Then

$$\min_{Q(z) \in H_\infty} \|G(z^{-1}) - Q(z)\|_\infty = \sigma_1,$$

and the unique minimizing $Q(z)$ is given by

$$Q(z) = G(z^{-1}) - \frac{V_1(z^{-1})}{U_1(z^{-1})}.$$

Since the Hankel singular values and Schmidt pairs can be obtained using the orthonormal basis constructed from the Jury table, a computational method for solving the Nehari problem is thus obtained.

The suboptimal Nehari problem is to characterize all $Q(z) \in H_\infty$ such that $\|G(z^{-1}) - Q(z)\|_\infty \leq \gamma$ with $\|G(z)\|_H < \gamma$. It is studied in 6, 8, 9, and 10, the methods in these papers are all related to the state space system theory. Our approach to the solution will be based on the orthonormal basis and the compressed Hankel matrix $H_G$ in Theorem 2.

We also define the entropy of $F(z)$ as

$$I[F(z)] = \frac{\gamma^2}{2\pi} \int_{-\pi}^{\pi} \ln|1 - \gamma^{-2} F(e^{-j\omega})F(e^{j\omega})|d\omega.$$  

Given a strictly proper transfer function $G(z) = \frac{b(z)}{a(z)}$, we can expand $G(z)$ as

$$G(z) = \beta_1 E_1(z) + \ldots + \beta_n E_n(z) = E(z)\beta, \quad (9)$$

where $E(z) = \begin{bmatrix} E_1(z) & E_2(z) & \cdots & E_n(z) \end{bmatrix}$ are the orthonormal functions constructed from Jury table.

Finding $\beta_i, i = 1, \ldots, n$, is simple. One only need to compare the coefficients in (9) and solve a set of linear equations. It turns out that these equations have special structure and we can obtain the orthonormal basis and these coefficients $\beta_i$ simultaneously by using the augmented Jury table, see details in 17).

**Theorem 4** Let $G(z) = \frac{b(z)}{a(z)} \in H_\infty$ be rational, strictly proper and $\|G(z)\|_H \leq \gamma$. Expand $G(z)$ as

$$G(z) = E(z)\beta$$

and let

$$\alpha = \sqrt{1 + \beta\gamma^2 I - H_G^{-1}\beta^{-1}}, \quad \beta_1 = \frac{\gamma E(z) \left(\beta^2 I - H_G^{-1}\right) \beta^{-1}}{\alpha},$$

$$\gamma = |1 + \gamma E(z) H_G^{-1}\beta^{-1}|/\alpha.$$  

(1) Define

$$V(z) = \begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix}, \quad (13)$$

where

$$V_{11}(z) = Y(z^{-1}) - \gamma^{-1}G(z^{-1})X(z),$$

$$V_{12}(z) = X(z^{-1}) - \gamma^{-1}G(z^{-1})Y(z),$$

$$V_{21}(z) = X(z),$$

$$V_{22}(z) = Y(z).$$

Then the set of all $Q(z)$ such that $\|G(z^{-1}) - Q(z)\|_\infty \leq \gamma$ is given by

$$\{Q(z) = -\gamma L[V(z), R(z)] : R(z) \in H_\infty, \|R(z)\|_\infty \leq 1\},$$

where

$$L[V(z), R(z)] = \frac{V_{11}(z)R(z) + V_{12}(z)}{V_{21}(z)R(z) + V_{22}(z)}.$$  

(2) Define

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \frac{1}{Y(z)} \begin{bmatrix} U(z) & 1 \\ 1 & -X(z) \end{bmatrix}, \quad (15)$$

with

$$U(z) = X(z^{-1}) - \gamma^{-1}G(z^{-1})Y(z).$$

Then the set of all $Q(z)$ such that $\|G(z^{-1}) - Q(z)\|_\infty \leq \gamma$ is given by

$$\{Q(z) = -\gamma F[P(z), R(z)] : R(z) \in H_\infty, \|R(z)\|_\infty \leq 1\}$$

where

$$F[P(z), R(z)] = P_{11}(z) + P_{12}(z)R(z)(I - P_{22}(z)R(z)^{-1})P_{21}(z).$$

(3) By setting $R(z) = 0$, the central $Q(z)$ satisfying

$$\|G(z^{-1}) - Q(z)\|_\infty \leq \gamma$$

which minimizes $I[G(z^{-1}) - Q(z)]$ is given by

$$Q(z) = -\gamma V_{12}(z)V_{22}(z)^{-1} = -\gamma P_{11}(z).$$


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5. Robust Stabilization

In this section, we will study a typical robust stabilization problem [10, 14]. In this problem, we design a controller $K$, for a given plant $P$, such that the following quantity is maximized.

$$b_{PK} := \left\| \begin{bmatrix} I & K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_{\infty}^{-1}.$$  

This quantity gives a measure of the robustness of the feedback system under the gap metric or $\nu$-gap metric uncertainty. Hence the robust stabilization problem is a special discrete-time $\mathcal{H}_\infty$ optimal control problem.

$$\inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} I & K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_{\infty}.$$  

We are interested in finding suboptimal controllers.

Let us first recall the Youla parameterization of all stabilizing controllers. For a proper system $P(z) = \frac{b(z)}{a(z)}$ where $a(z)$ and $b(z)$ are coprime polynomials with degree $n$. We first find the spectral factor $d(z)$ such that

$$z^n[a(z)a(z^{-1}) + b(z)b(z^{-1})] = z^n d(z) d(z^{-1}).$$

Then, we solve the following Diophantine equation

$$a(z)x(z) + b(z)y(z) = d^2(z).$$

Define

$$M(z) = \frac{a(z)}{d(z)}, \quad N(z) = \frac{b(z)}{d(z)}.$$  

Then the set of all controller $K(z)$ that internally stabilize $P(z)$ is given by

$$K(z) = \frac{\tilde{M}(z) - M(z)Q(z)}{\tilde{N}(z) + N(z)Q(z)}$$  

for $Q(z) \in \mathbb{R} \mathcal{H}_\infty$. Apply the parameterized controller to the above $\mathcal{H}_\infty$ problem, we can get

$$\left\| \begin{bmatrix} I & K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_{\infty} = \left\| \begin{bmatrix} x(z) b(z^{-1}) - y(z) a(z^{-1}) & d(z) d(z^{-1})^{-1} Q(z) \end{bmatrix} \right\|_{\infty} = \left\| \begin{bmatrix} w(z^{-1}) - Q(z) \end{bmatrix} \right\|_{\infty} + 1$$

for some polynomial $w(z)$ that satisfies

$$z^n[y(z)a(z^{-1}) - x(z)b(z^{-1})] = z^n d(z) w(z^{-1}).$$  

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Example 1

For

$$G(z) = \frac{b(z)}{a(z)} = \frac{\sqrt{2}z + 0.5}{z^2 + \sqrt{2}z + 0.5},$$

we wish to find all $Q(z) \in \mathcal{H}_\infty$ such that $\|G(z^{-1}) - Q(z)\|_{\infty} \leq \gamma$ with $\gamma = 8$.

Construct the Jury table, we can get

$$a_0 = 1, \quad a_1 = \frac{2\sqrt{2}}{3}, \quad a_2 = \frac{2\sqrt{2}}{3}, \quad b_0 = 0.5, \quad b_1 = \frac{2\sqrt{2}}{3}, \quad b_2 = 1.$$

Then, we find the spectral factor $d(z)$ such that

$$z^n[a(z)a(z^{-1}) + b(z)b(z^{-1})] = z^n d(z) d(z^{-1}).$$

Then, we solve the following Diophantine equation

$$a(z)x(z) + b(z)y(z) = d^2(z).$$

Define

$$M(z) = \frac{a(z)}{d(z)}, \quad N(z) = \frac{b(z)}{d(z)}.$$  

Then the set of all controller $K(z)$ that internally stabilize $P(z)$ is given by

$$K(z) = \frac{\tilde{M}(z) - M(z)Q(z)}{\tilde{N}(z) + N(z)Q(z)}$$  

for $Q(z) \in \mathbb{R} \mathcal{H}_\infty$. Apply the parameterized controller to the above $\mathcal{H}_\infty$ problem, we can get

$$\left\| \begin{bmatrix} I & K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_{\infty} = \left\| \begin{bmatrix} x(z) b(z^{-1}) - y(z) a(z^{-1}) & d(z) d(z^{-1})^{-1} Q(z) \end{bmatrix} \right\|_{\infty} = \left\| \begin{bmatrix} w(z^{-1}) - Q(z) \end{bmatrix} \right\|_{\infty} + 1$$

for some polynomial $w(z)$ that satisfies

$$z^n[y(z)a(z^{-1}) - x(z)b(z^{-1})] = z^n d(z) w(z^{-1}).$$
Let $G(z) = \frac{w(z)}{d(z)}$, the original robust stabilization problem reduces to find $Q(z) \in \mathcal{RH}_\infty$ to minimize
$$
\|G(z^{-1}) - Q(z)\|_\infty.
$$
(20)

Let
$$
G(z) = G_s(z) + G(\infty)
$$
where $G_s(z)$ is a strictly proper transfer function. Also let $Q_s(z) = Q(z) - G_m$, then equation (20) becomes
$$
\|G_s(z^{-1}) - Q_s(z)\|_\infty
$$
which is a Nehari problem solved in Section 4.

Example 2 Consider
$$
P(z) = \frac{1.5}{z^2 + 1}.
$$
We wish to find the suboptimal controller $K(z)$.  

**Step 1:** (Spectral factorization) From
$$(z^2 + 1)(z^2 + 1) + 1.5z^2 = z^2d(z)d(z^{-1}),$$
we can get
$$d(z) = 2z^2 + 0.5.$$

**Step 2** (Diophantine equation) From
$$(z^2 + 1)x(z) + 1.5y(z) = (2z^2 + 0.5)^2,$$
we can get
$$x(z) = 4z^2 - 2, \quad y(z) = 1.5.$$

From
$$(1.5(z^2 + 1) - 4z^2 - 2)1.5z^2 = (2z^2 + 0.5)(3z^2 - 3),$$
we get
$$w(z) = 3z^2 - 3.$$

Hence
$$G(z) = \frac{3z^2 - 3}{2z^2 + 0.5}.$$

**Step 3** (Suboptimal Nehari problem) Let
$$G_s(z) = G(z) - G(\infty) = \frac{-3.75}{2z^2 + 0.5}, \gamma = 3.$$

Solve
$$\|G_s(z^{-1}) - Q_s(z)\|_\infty < 3, \quad Q_s(z) \in \mathcal{RH}_\infty.$$  
We get
$$V(z) = \begin{bmatrix}
1.069z^2 + 0.4677 & 0.1336z^2 \\
2z^2 + 0.5 & 2z^2 + 0.5 \\
1.203 & 1.871z^2 + 0.2673 \\
2z^2 + 0.5 & 2z^2 + 0.5
\end{bmatrix}.$$  
Hence, the central solution such that
$$\|G(z^{-1}) - Q(z)\|_\infty < 3, \quad Q(z) \in \mathcal{RH}_\infty$$
is given by
$$Q(z) = 1.5 + \frac{0.1336z^2}{1.871z^2 + 0.2673} = \frac{2.94z^2 + 0.4009}{1.871z^2 + 0.2673}.$$

The central controller $K$ is given by
$$K = \frac{1.5(1.871z^2 + 0.2673) - (z^2 + 1)(2.94z^2 + 0.4009)}{(4z^2 - 2)(1.871z^2 + 0.2673) + 1.5(2.94z^2 + 0.4009)} = \frac{-2.94z^2}{7.84z^4 + 1.737z^2 + 2.55z^2 + 0.59}.$$

6. Conclusion

Compressed Hankel matrix is given by using orthonormal rational functions constructed from the Jury table. The solutions to the optimal and suboptimal Nehari problems via the compressed Hankel matrix are also given. Robust stabilization problem is reduced to the Nehari problem, so it can also be solved via Jury table.

References


