ROBUST STABILIZATION FOR ℓ_p GAP PERTURBATIONS*

LI QIU[†] AND DANIEL E. MILLER[‡]

Abstract. This paper studies robust stabilization of linear feedback systems. The special features of this study are: (1) the input and output signal spaces of systems are assumed to be any ℓ_p spaces; (2) system perturbations are measured by the gap function.

Key words. gap, graph, robust control, robust stability, ℓ_p spaces.

1. Introduction.

A typical feedback system is shown in Figure 1.1, where u_1, e_1, y_1 are signals in a Banach space $\mathcal{U}$; u_2, e_2, y_2 are signals in another Banach space $\mathcal{Y}$; P is an unbounded linear operator from $\mathcal{U}$ to $\mathcal{Y}$, considered as the plant; C is an unbounded linear operator from $\mathcal{Y}$ to $\mathcal{U}$, considered as the controller. Typically, $\mathcal{U}$ and $\mathcal{Y}$ are spaces of functions of either continuous time or discrete time and, as a result, the systems P and C are physically constrained to be causal, i.e., the values of their outputs at any time instance can not depend on the values of their outputs at any future time instance. The unboundedness of systems P and C are due to their possible instability. Very often the plant P is not exactly known, or more precisely is only known to belong to certain set of plants. In this case, we say the plant is uncertain and call the set which the plant belongs to the uncertainty set. The purpose of robust control is to design the controller C so that the feedback system behaves in a desirable way for each possible plant P in the uncertainty set. Although the controller design is the ultimate goal, a typical robust control theory progresses in three stages:

1. Description of uncertainty: Construct a mathematical description of the uncertainty set.
2. Robustness analysis: Determine if the feedback system behaves in a desirable way for each P in the uncertainty set when a controller is given.
3. Robust controller design: Design a controller C to satisfy the robustness requirement.

Two commonly used methods to describe the uncertainty are the parameter uncertainty and the norm bounded uncertainty. The former method

---

* The authors would like to thank Tryphon Georgiou, Yoshito Ohta and Bo Bernhardsson for helpful discussions.

† Department of Electrical & Electronic Engineering, University of Science & Technology, Clearwater Bay, Kowloon, Hong Kong. This author was supported by the Institute for Mathematics and its Applications, University of Minnesota, with funds provided by the National Science Foundation of USA.

‡ Dept. of Elect. & Comp. Eng., University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. Supported by the Natural Sciences and Engineering Research Council of Canada.

B. A. Francis et al. (eds.), *Robust Control Theory*

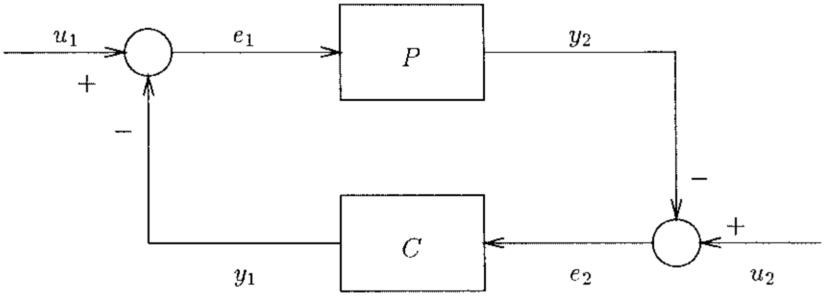

FIG. 1.1. *The Standard Feedback System*

embeds the uncertainty set into an Euclidean space by assuming that the uncertainty is caused by a collection of uncertain parameters in the system model. The latter method characterizes the uncertainty in terms of a norm bounded operator based on the assumption that the uncertainty is caused by the bounded operator perturbation that enters the system in an additive, multiplicative, or more generally linear fractional way [30]. Both methods work well and each leads to a tangible theory if one is sure that the uncertainty description matchs the real physical uncertainty. However, the limitation of these descriptions is obvious: the parameter uncertainty model can not handle unmodelled dynamics and the norm bounded uncertainty model has difficulty in dealing with uncertain unstable dynamics.

Recently, researchers in robust control have developed another uncertainty model. This is given by norm bounded perturbations on the denominator and numerator of a coprime factorization of the system [30,15,1]. This model overcomes some limitations of the two earlier uncertainty models. However a system may not have a coprime factorization, which limits the applicability of this uncertainty model, and if it does have one, the nonuniqueness of its coprime factorizations makes this model depend on a particular artificial system representation. Although this dependence might be an advantage when the same representation is used in the identification and the uncertainty results from the identification, it may not be desirable otherwise.

Since a system is considered as an unbounded linear operator between Banach spaces, the gap function, which has long established its merit in the perturbation analysis of unbounded operators, becomes a handy tool in describing the uncertainty. A ball defined from the gap gives us a natural description of an uncertainty set, and such an uncertainly set has many nice analytic properties. The pioneering work of robust control theory using the gap was done by Zames and El-Sakkary [33] in 1980. Since then, a series of works have further developed and completed the theory, see [6,11,12,24] for linear time invariant finite dimensional systems, [34,14] for linear time-invariant infinite-dimensional systems, and [8,9] for linear time-varying systems. Two variations to the gap metric were introduced in [25] and [32] and were shown to be useful in studying robust control for linear time-invariant systems; see also [7,29]. With the recognition that Hilbert space gap theory is better developed due to the tremendously helpful inner product, most of these works are based on the assumption that signal spaces $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, such as $\mathcal{L}_2$ or ℓ_2. However, in many control application, the natural signal spaces are merely Banach spaces. Prominent examples are ℓ_∞ and $\mathcal{L}_\infty$. Although the work in [34] treats systems defined as Banach space operators, the results there are qualitative rather that quantitative and causality, which is a fundamental property of a physical system, is not incorporated into the study. The rapid development in the robust control theory for systems with Hilbert signal spaces in the past few years has shed light on how it can be extended to Banach signal spaces. In this paper, we study robust feedback control for systems with ℓ_p signal spaces. The causality of systems is assumed *a priori*. A theory for $\mathcal{L}_p$ spaces is also important, but it is technically more involved and we leave it for future research.

One significant departure of our study from the gap based robust stabilization theory for systems with Hilbert signal spaces is that our analysis is primarily based on the directed gap. This is made possible by the causality assumption. It can be shown that in the Hilbert space case, the gap and the directed gap make little difference, whereas in the Banach space case, the directed gap in general produces much tighter results.

The structure of this paper is as follows. Section 2 is an introduction to the directed gap and the gap between subspaces of a Banach space. Most of the results are from [18], but a few new results are included and proved. In section 3, we define the class of systems under consideration to be the causal linear operators between spaces of real sequences. This class of systems includes all discrete time linear physically realizable systems. They can be finite dimensional or infinite dimensional, time-invariant or time varying. We then show that the graphs of such systems are always closed. Hence, we define the directed gap and the gap between such systems to be the directed gap and the gap between their graphs. Section 4 carries out the robust stability analysis assuming that system uncertainty is described by a directed gap ball. It is shown that the stability robustness of a feedback system can be given by the reciprocal of the norm of a closed loop operator. This makes the robust stabilization problem into a problem of minimizing the induced norm of some closed loop operator. Such an optimal control problem has been the subject of intensive study in the past decade and in many cases, solutions are readily available. Section 5 compares the topological properties of the directed gap balls used in the stability robustness analysis and those of the usual gap balls. It is shown that the directed gap balls and gap balls generate the same topology on the set of all stabilizable systems. This further justifies the use of directed gap balls in the robustness analysis. Section 6 gives a close look at the systems which admit right and left coprime factorizations as well as finite dimensional shift invariant systems. Some computational issues are considered. Section 7 is the conclusion.

When this paper was nearly finished, the authors received the preprint [35] in which several results recognizing the importance of the directed gap were obtained. However, it is only in the present paper that, based on causality, one type of directed gap balls is shown to have the right qualitative and quantitative properties for the robust control study.

2. Gap between subspaces. The gap function between subspaces was originally introduced in the Russian mathematical literature, first for Hilbert spaces [20] and then for Banach spaces [21]. This section reviews some results relevant to robust control problems and gives some new results. Material in this section is mainly from [18]. Only new results will be proved.

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$ and denote by $\Gamma(\mathcal{X})$ the set of all subspaces (closed linear manifolds) of $\mathcal{X}$. Let $\mathcal{S}_1$, $\mathcal{S}_2 \in \Gamma(\mathcal{X})$. The *directed gap* from $\mathcal{S}_1$ to $\mathcal{S}_2$ is defined by

$$\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) = \sup_{x \in \mathcal{S}_1, \|x\| \leq 1} \inf_{y \in \mathcal{S}_2} \|x - y\|.$$

The *gap* between $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by

$$\delta(\mathcal{S}_1, \mathcal{S}_2) = \max\{\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2), \vec{\delta}(\mathcal{S}_2, \mathcal{S}_1)\}.$$

It is well-known that in general δ is not a metric on $\Gamma(\mathcal{X})$, unless $\mathcal{X}$ is a Hilbert space, since it may not satisfy the triangle inequality. The directed gap is even further away from being a metric since it is not symmetric and not positive. Nevertheless, for $S \in \Gamma(\mathcal{X})$ and $r \geq 0$, we can define the balls:

$$\begin{aligned}
\overrightarrow{\mathbf{B}}(\mathcal{S}, r) &= \{\tilde{\mathcal{S}} \in \Gamma(\mathcal{X}) : \vec{\delta}(\mathcal{S}, \tilde{\mathcal{S}}) < r\} \\
\overleftarrow{\mathbf{B}}(\mathcal{S}, r) &= \{\tilde{\mathcal{S}} \in \Gamma(\mathcal{X}) : \vec{\delta}(\tilde{\mathcal{S}}, \mathcal{S}) < r\} \\
\mathbf{B}(\mathcal{S}, r) &= \{\tilde{\mathcal{S}} \in \Gamma(\mathcal{X}) : \delta(\tilde{\mathcal{S}}, \mathcal{S}) < r\}.
\end{aligned}$$

On varying $\mathcal{S}$ and r, the above three types of balls form the bases of three topologies $\vec{\tau}$, $\overleftarrow{\tau}$, and τ respectively[1]. In general, these three topologies are completely different. It is easy to see that τ is Hausdorff while $\vec{\tau}$ and $\overleftarrow{\tau}$ are not. If $\mathcal{S}_1 \subset \mathcal{S}_2$, then $\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) = 0$ but $\vec{\delta}(\mathcal{S}_2, \mathcal{S}_1) = 1$ unless $\mathcal{S}_1 = \mathcal{S}_2$. However, for some restricted classes of subspaces, these three topologies, or two of them, may turn out to be the same. A trivial example is the class of subspaces with equal finite dimension since, as stated in [17], for $S_1, S_2 \in \Gamma(\mathcal{X})$ with $\dim(\mathcal{S}_1) = \dim(\mathcal{S}_2) < \infty$,

$$\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) \leq \frac{\vec{\delta}(\mathcal{S}_2, \mathcal{S}_1)}{1 - \vec{\delta}(\mathcal{S}_2, \mathcal{S}_1)}.$$

[1] We leave it to the reader to verify that the three types of balls indeed form topological bases.

We will give a nontrivial example in the following sections, which will be one of the interesting results of this paper.

In the following, our emphasis will be on the directed gap instead of the gap. In many cases, results on the gap, some of which are important in our development, can be easily obtained from the corresponding results on the directed gap. We leave such work to the reader. The symbols $\mathcal{S}$, $\mathcal{S}_1$, and $\mathcal{S}_2$ will always be reserved for members of $\Gamma(\mathcal{X})$.

Let $\mathcal{X}^*$ be the dual of $\mathcal{X}$. For $x \in \mathcal{X}$ and $\phi \in \mathcal{X}^*$, we use $\langle x, \phi \rangle$ to mean $\phi(x)$. Denote by $\mathcal{S}^\perp$ the annihilator of $\mathcal{S} \in \Gamma(\mathcal{X})$.

LEMMA 2.1. [18, p. 201]

(a) *For each* $x \in \mathcal{X}$, $\inf_{y \in \mathcal{S}} \|x - y\| = \sup_{\psi \in \mathcal{S}^\perp, \|\psi\| \leq 1} \langle x, \psi \rangle$.

(b) *For each* $\phi \in \mathcal{X}^*$, $\inf_{\psi \in \mathcal{S}^\perp} \|\phi - \psi\| = \sup_{y \in \mathcal{S}, \|y\| \leq 1} \langle y, \phi \rangle$.

Lemma 2.1 enables us to convert the sup-inf expression of the directed gap into a pure supremum expression, which potentially simplifies the computation.

PROPOSITION 2.2. [18, p. 201]

(a) $\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) = \sup_{x \in \mathcal{S}_1, \|x\| \leq 1} \sup_{\psi \in \mathcal{S}_2^\perp, \|\psi\| \leq 1} \langle x, \psi \rangle$.

(b) $\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) = \vec{\delta}(\mathcal{S}_2^\perp, \mathcal{S}_1^\perp)$.

The back annihilator of $\mathcal{T} \in \Gamma(\mathcal{X}^*)$ is defined by $\mathcal{T}^\top = \{x \in \mathcal{X} : \langle x, \phi \rangle = 0 \text{ for all } \phi \in \mathcal{T}\}$. Clearly, $\mathcal{T}^\top \in \Gamma(\mathcal{X})$. It is know that $(\mathcal{S}^\perp)^\top = \mathcal{S}$, and if $\mathcal{T}$ is weak*-closed then $(\mathcal{T}^\top)^\perp = \mathcal{T}$ [27, p. 91]. We will be working on subspaces of ℓ_p spaces. Since not every ℓ_p has an easily characterizable dual space, but every ℓ_p space is the dual space of an easily characterizable space, we will find back annihilators more convenient to use.

PROPOSITION 2.3. *If* $\mathcal{T}_1, \mathcal{T}_2 \in \Gamma(\mathcal{X}^*)$ *are weak*-closed, then*

(a) $\vec{\delta}(\mathcal{T}_1, \mathcal{T}_2) = \sup_{\phi \in \mathcal{T}_1, \|\phi\| \leq 1} \sup_{y \in \mathcal{T}_2^\top, \|y\| \leq 1} \langle y, \phi \rangle$.

(b) $\vec{\delta}(\mathcal{T}_1, \mathcal{T}_2) = \vec{\delta}(\mathcal{T}_2^\top, \mathcal{T}_1^\top)$.

Proof. Apply Proposition 2.2 by letting $\mathcal{S}_1 = \mathcal{T}_1^\top$ and $\mathcal{S}_2 = \mathcal{T}_2^\top$. □

Let A be a bounded linear operator on $\mathcal{X}$ with bounded inverse. By the open mapping theorem, this requires only that A be a bijective bounded linear operator [26, p. 195]. Define the *condition number* of A to be $\kappa(A) = \|A\|\|A^{-1}\|$.

PROPOSITION 2.4. $\vec{\delta}(A\mathcal{S}_1, A\mathcal{S}_2) \leq \kappa(A)\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2)$.

Proof. For arbitrary $u \in A\mathcal{S}_1$ with $\|u\| \leq 1$, there exists $x \in \mathcal{S}_1$ such that $u = Ax$. We have $\|x\| \leq \|A^{-1}\|$. For each $\epsilon > 0$,

$$\inf_{z \in \mathcal{S}_2} \left\|\frac{x}{\|A^{-1}\|} - z\right\| < \vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) + \epsilon .$$

Hence, there exists $y \in \mathcal{S}_2$ such that

$$\|x - y\| \leq \|A^{-1}\|[\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) + \epsilon] .$$

Set $v = Ay$; we have

$$\|u - v\| = \|A(x - y)\| \leq \|A\|\|x - y\| \leq \kappa(A)[\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) + \epsilon] ,$$

which implies that

$$\inf_{w \in A\mathcal{S}_2} \|u - w\| \leq \kappa(A)[\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) + \epsilon] .$$

Since $u \in A\mathcal{S}_1$ with $\|u\| \leq 1$ is arbitrary and $\epsilon > 0$ is arbitrary, we obtain

$$\vec{\delta}(A\mathcal{S}_1, A\mathcal{S}_2) \leq \kappa(A)\vec{\delta}(\mathcal{S}_1, \mathcal{S}_2) .$$

This complete the proof. □

Associated with the concept of the gap, there is another useful concept called the *minimum opening*. The *directed minimum opening* from $\mathcal{S}_1$ to $\mathcal{S}_2$ is defined by

$$\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) = \begin{cases} \inf\limits_{x \in \mathcal{S}_1, \|x\|=1} \inf\limits_{y \in \mathcal{S}_2} \|x - y\| & \text{if } \mathcal{S}_1 \neq \{0\} \\ 1 & \text{if } \mathcal{S}_1 = \{0\}. \end{cases}$$

The *minimum opening* between $\mathcal{S}_1$ and $\mathcal{S}_2$ is defined by

$$\rho(\mathcal{S}_1, \mathcal{S}_2) = \min\{\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2), \vec{\rho}(\mathcal{S}_2, \mathcal{S}_1)\}.$$

In the following, the notation $\mathcal{S}_1 \oplus \mathcal{S}_2$ means $\mathcal{S}_1 + \mathcal{S}_2$ and at the same time claims $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$.

PROPOSITION 2.5. [18, p. 219]

(a) $\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) \geq \dfrac{\vec{\rho}(\mathcal{S}_2, \mathcal{S}_1)}{1 + \vec{\rho}(\mathcal{S}_2, \mathcal{S}_1)}$.

(b) $\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) = 0$ *iff* $\vec{\rho}(\mathcal{S}_2, \mathcal{S}_1) = 0$.

(c) $\mathcal{S}_1 \oplus \mathcal{S}_2$ *is closed iff* $\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) \neq 0$.

Recall the definition of the gap balls and the directed gap balls.

THEOREM 2.6.

(a) $\tilde{\mathcal{S}}_1 \oplus \mathcal{S}_2$ *is closed for all* $\tilde{\mathcal{S}}_1 \in \overleftarrow{\mathbf{B}}(\mathcal{S}_1, r)$ *iff* $r \leq \vec{\rho}(\mathcal{S}_2, \mathcal{S}_1)$.

(b) $\tilde{\mathcal{S}}_1 + \mathcal{S}_2 = \mathcal{X}$ *for all* $\tilde{\mathcal{S}}_1 \in \overrightarrow{\mathbf{B}}(\mathcal{S}_1, r)$ *iff* $r \leq \vec{\rho}(\mathcal{S}_2^{\perp}, \mathcal{S}_1^{\perp})$.

Proof. The sufficiency part of (a) follows from [18, Theorem 4.4.24]. To prove the necessity part of (a), assume $r > \vec{\rho}(\mathcal{S}_2, \mathcal{S}_1)$. Then there exists $y \in \mathcal{S}_2$ with $\|y\| = 1$ and $x \in \mathcal{S}_1$ such that $\|y - x\| < r$. Let $\tilde{\mathcal{S}}_1 = \operatorname{span}\{y\}$. Then $\vec{\delta}(\tilde{\mathcal{S}}_1, \mathcal{S}_1) \leq \|y - x\| < r$, i.e., $\mathcal{S}_1 \in \overleftarrow{\mathbf{B}}(\mathcal{S}_1, r)$, and

$\tilde{\mathcal{S}}_1 \cap \mathcal{S}_2 = \operatorname{span}\{y\} \neq \{0\}$. Statement (b) follows from (a) since $\tilde{\mathcal{S}}_1 + \mathcal{S}_2 = \mathcal{X}$ iff $\tilde{\mathcal{S}}_1^\perp \oplus \mathcal{S}_2^\perp$ is closed and $\tilde{\mathcal{S}}_1 \in \overrightarrow{\mathbf{B}}(\mathcal{S}_1, r)$ iff $\tilde{\mathcal{S}}_1^\perp \in \overleftarrow{\mathbf{B}}(\mathcal{S}_1^\perp, r)$ □

If $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{X}$, then we say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *complementary*. The projection onto $\mathcal{S}_1$ along $\mathcal{S}_2$ is denoted by $\Pi_{\mathcal{S}_1\|\mathcal{S}_2}$.

PROPOSITION 2.7. *If $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{X}$, then*
(a) $\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) = \|\Pi_{\mathcal{S}_1\|\mathcal{S}_2}\|^{-1}$;
(b) $\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) = \vec{\rho}(\mathcal{S}_2^\perp, \mathcal{S}_1^\perp)$.

Proof.

$$\begin{aligned}
&\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2) \\
&\quad = \inf_{x\in\mathcal{S}_1,\, \|x\|=1}\ \inf_{y\in\mathcal{S}_2} \|x-y\| = \inf_{x\in\mathcal{S}_1,\, \|x\|\neq 0}\ \inf_{y\in\mathcal{S}_2} \tfrac{\|x-y\|}{\|x\|} \\
&\quad = \inf_{z\in\mathcal{X},\, \Pi_{\mathcal{S}_1\|\mathcal{S}_2} z\neq 0} \frac{\|z\|}{\|\Pi_{\mathcal{S}_1\|\mathcal{S}_2} z\|} = \left(\sup_{z\in\mathcal{X},\, \Pi_{\mathcal{S}_1\|\mathcal{S}_2} z\neq 0} \frac{\|\Pi_{\mathcal{S}_1\|\mathcal{S}_2} z\|}{\|z\|} \right)^{-1} \\
&\quad = \|\Pi_{\mathcal{S}_1\|\mathcal{S}_2}\|^{-1}
\end{aligned}$$

This proves (a). Statement (b) follows from [18, Theorem 4.4.8]. □

Notice that $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{X}$ is equivalent to (a) $\mathcal{S}_1 \oplus \mathcal{S}_2$ is closed and (b) $\mathcal{S}_1 + \mathcal{S}_2 = \mathcal{X}$. Combining Theorem 2.6 (a) and (b) and using Proposition 2.7(b), one can easily show the following corollary.

COROLLARY 2.8. *Assume $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{X}$. Then $\tilde{\mathcal{S}}_1 \oplus \mathcal{S}_2 = \mathcal{X}$ for all $\tilde{\mathcal{S}}_1 \in \mathbf{B}(\mathcal{S}_1, r)$ if $r \leq \rho(\mathcal{S}_1, \mathcal{S}_2)$.*

Unfortunately, the condition in this corollary is not tight in general. Consider the case when $\mathcal{X} = \mathbb{R}^2$ with the Hölder ∞-norm and $\mathcal{S}_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and $\mathcal{S}_2 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. Then $\rho(\mathcal{S}_1, \mathcal{S}_2) = 0.5$. On the other hand, $\tilde{\mathcal{S}}_1 \oplus \mathcal{S}_2 \neq \mathcal{X}$ if and only if $\tilde{\mathcal{S}}_1 = \{0\}$, $\tilde{\mathcal{S}}_1 = \mathcal{S}_2$, or $\tilde{\mathcal{S}}_1 = \mathbb{R}_2$. In each of these three cases, $\delta(\tilde{\mathcal{S}}_1, \mathcal{S}_1) = 1$. Therefore, $\tilde{\mathcal{S}}_1 \oplus \mathcal{S}_2 = \mathcal{X}$ for all $\tilde{\mathcal{S}}_1 \in \mathbf{B}(\mathcal{S}_1, r)$ if and only if $r \leq 1$. This shows that the condition given by Corollary 2.8 is not tight. In fact, in the finite dimensional case, we have the following improved result.

COROLLARY 2.9. *Assume that $\mathcal{X}$ is finite dimensional and $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{X}$. Then $\tilde{\mathcal{S}}_1 \oplus \mathcal{S}_2 = \mathcal{X}$ for all $\tilde{\mathcal{S}}_1 \in \mathbf{B}(\mathcal{S}_1, r)$ if $r \leq \max\{\vec{\rho}(\mathcal{S}_1, \mathcal{S}_2), \vec{\rho}(\mathcal{S}_2, \mathcal{S}_1)\}$.*

Proof. This follows from the fact that in the case when $\mathcal{X}$ is finite dimensional, $\tilde{\mathcal{S}}_1 \cap \mathcal{S}_2 = \{0\}$ and $\tilde{\mathcal{S}}_1^\perp \cap \mathcal{S}_2^\perp = \{0\}$ are equivalent as long as the dimensions of $\tilde{\mathcal{S}}_1$ and $\mathcal{S}_1$ are equal, while the dimension equality is guaranteed by $\delta(\tilde{\mathcal{S}}_1, \mathcal{S}_1) < r \leq 1$. □

3. Gap between systems. Let the set of $\mathbb{R}^n$-valued sequences be denoted by s^n, i.e., $s^n = \{(x_0\ x_1\ x_2 \cdots) : x_i \in \mathbb{R}^n\}$. Clearly, s^n is a linear space over $\mathbb{R}$. The truncation operators Π_k, $k = 0, 1, 2, \ldots$, on s^n

are defined by $\Pi_k(x_0\ x_1 \cdots x_k\ x_{k+1} \cdots) = (x_0\ x_1 \cdots x_k\ 0 \cdots)$. A linear operator F from s^n to s^m is said to be *causal* if $\Pi_k F(I - \Pi_k) = 0$ for all $k \geq 0$. A system considered in this paper is simply a causal linear operator from s^n to s^m. We will always assume that the natural bases in s^m and s^n are used[2]. As a consequence, a vector in s^n can be represented by a semi-infinite column vector and a system from s^n to s^m can be represented by a semi-infinite lower triangular block matrix. So $y = Fu$ simply means

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} F_{00} & 0 & 0 & \cdots \\ F_{10} & F_{11} & 0 & \cdots \\ F_{20} & F_{21} & F_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

where F_{ij} are $m \times n$ real matrices. Conversely, any semi-infinite lower triangular block matrix represents a system. This is a convenient consequence of the sequence space and the causality requirement. Due to this natural homomorphism between the set of causal operators and the set of lower triangular matrices, it becomes unnecessary to distinguish the two sets, which will share the same notation $\mathcal{L}^{m\times n}$ in the sequel. Another convenient consequence of causality is that the invertibility is easy to characterize: $F \in \mathcal{L}^{n\times n}$ is invertible if and only if F_{ii}, $i \geq 0$, is nonsingular. This implies that if $F_1, F_2 \in \mathcal{L}^{n\times n}$ and $F_1 F_2 = I$, then $F_1 = F_2^{-1}$. Finally, we introduce another important operator on s^n: the shift operator S defined by $S(x_0\ x_1\ x_2 \cdots) = (0\ x_0\ x_1\ x_2 \cdots)$. A system F in $\mathcal{L}^{m\times n}$ is said to be *shift-invariant* if it commutes with S, i.e., $FS = SF$. The matrix representation of a shift-invariant system is a block lower triangular Toeplitz matrix, i.e., $F_{ij} = F_{(i+k)(j+k)}$ for $i, j, k \geq 0$. A system F in $\mathcal{L}^{m\times n}$ is said to be *finite dimensional* if there exist matrix functions $A(k) \in \mathbb{R}^{l\times l}$, $B(k) \in \mathbb{R}^{l\times n}$, $C(k) \in \mathbb{R}^{m\times l}$, $D(k) \in \mathbb{R}^{m\times n}$, $k \geq 0$ such that

$$F_{ij} = \begin{cases} C(i)[\prod_{k=j+1}^{i-1} A(k)]B(j) & \text{if } j < i-1 \\ C(i)B(j) & \text{if } j = i-1 \\ D(i) & \text{if } j = i. \end{cases}$$

For $p = [1, \infty)$, the space ℓ_p^n is the set of all $x \in s^n$ for which

$$\left(\sum_{i=0}^{\infty} \|x_i\|_p^p\right)^{1/p} < \infty, \tag{3.1}$$

[2] The special algebraic structure of sequence space s^n makes it clear to everybody what the "natural basis" ought to be without referring to any topological structure. However, this natural basis is not an algebraic basis (Hamal basis) in s^n. It is rather a Schauder basis in s^n when it is endowed with the natural topology generated by semi-norms $p_{kl}(x) = |x_{k_l}|$, where we assume $x = (x_0\ x_1 \cdots x_k \cdots)$ and $x_k = [x_{k_1} \cdots x_{k_n}]'$. See [16] for more details.

and the space ℓ_∞^n is the set of all $x \in s^n$ for which

$$\sup_{i\geq 0} \|x_i\|_\infty < \infty, \tag{3.2}$$

where $\|\cdot\|_p$ is the Hölder p-norm. We know that ℓ_p^n is a Banach space with its norm, also denoted by $\|\cdot\|_p$, defined by the left hand side of (3.1) or (3.2). The space c_0^n is the subspace of ℓ_∞^n consisting of all $x \in \ell_\infty^n$ with $\lim_{i\to\infty} x_i = 0$. The norm in c_0^n is inherited from ℓ_∞^n but will be given its own notation $\|\cdot\|_{c_0}$. It is well-known that ℓ_p^n are (isometrically isomorphic to) the dual space of ℓ_q^n if $p = (1, \infty]$, where $\frac{1}{p} + \frac{1}{q} = 1$, or c_0^n if $p = 1$, in the sense that $x(\eta) = \langle \eta, x\rangle := \eta' x$ for η in the primal space and x in its dual space. To avoid unnecessary repetition in the sequel, we will always assume $\frac{1}{p} + \frac{1}{q} = 1$ is satisfied whenever p and q appear together.

A semi-infinite (not necessarily lower triangular) matrix of the form

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots \\ A_{10} & A_{11} & A_{12} & \cdots \\ A_{20} & A_{21} & A_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with $A_{ij} \in \mathbb{R}^{m\times n}$ represents a possibly unbounded operator from ℓ_p^n into ℓ_p^m with domain

$$\mathcal{D}_p(A) = \{u \in \ell_p^n : Au \text{ converges and belongs to } \ell_p^m\}$$

and graph

$$\mathcal{G}_p(A) = \left\{ \begin{bmatrix} u \\ Au \end{bmatrix} : u \in \mathcal{D}_p(A) \right\}.$$

The graph $\mathcal{G}_p(A)$ is clearly a linear manifold in $\ell_p^n \times \ell_p^m$. We define the norm in $\ell_p^n \times \ell_p^m$ as $\left\|\begin{bmatrix} u \\ y \end{bmatrix}\right\|_p = (\|u\|_p^p + \|y\|_p^p)^{1/p}$ if $p \in [1, \infty)$ and $\left\|\begin{bmatrix} u \\ y \end{bmatrix}\right\|_\infty = \max\{\|u\|_\infty, \|y\|_\infty\}$. Thus $\ell_p^n \times \ell_p^m$ is identical to ℓ_p^{n+m}. Note the difference between our method and Kato's method [18] in norming the product space.

The matrix A can also represent a possibly unbounded operator from c_0^n into c_0^n with domain

$$\mathcal{D}_{c_0}(A) = \{u \in c_0^n : Au \text{ converges and belongs to } c_0^m\}$$

and graph

$$\mathcal{G}_{c_0}(A) = \left\{ \begin{bmatrix} u \\ Au \end{bmatrix} : u \in \mathcal{D}_{c_0}(A) \right\}.$$

The graph $\mathcal{G}_{c_0}(A)$ is clearly a linear manifold in $c_0^n \times c_0^m$. We define the norm in $c_0^n \times c_0^m$ as $\left\| \begin{bmatrix} u \\ y \end{bmatrix} \right\|_{c_0} = \max\{\|u\|_{c_0}, \|y\|_{c_0}\}$. Thus $c_0^n \times c_0^m$ is identical to c_0^{n+m}.

For the notation convenience, we also need the concept of inverse graphs of A defined by

$$\mathcal{G}_p'(A) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{G}_p(A) = \left\{ \begin{bmatrix} Au \\ u \end{bmatrix} : u \in \mathcal{D}_p(A) \right\}$$

and

$$\mathcal{G}_{c_0}'(A) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{G}_{c_0}(A) = \left\{ \begin{bmatrix} Au \\ u \end{bmatrix} : u \in \mathcal{D}_{c_0}(A) \right\}.$$

If A represents a bounded operator on ℓ_p^n to ℓ_p^m, then we say A is ℓ_p-bounded and the induced norm of A is denoted by $\|A\|_p$. Similarly, if A represents a bounded operator on c_0^n to c_0^m, then we say A is c_0-bounded and the induced norm of A is denoted by $\|A\|_{c_0}$.

In the following, a system $F \in \mathcal{L}^{m\times n}$ will always be considered as an unbounded operator from ℓ_p^n into ℓ_p^m for some fixed $p \in [1, \infty]$. The transpose of F:

$$F' = \begin{bmatrix} F_{00}' & F_{10}' & F_{20}' & \cdots \\ 0 & F_{11}' & F_{21}' & \cdots \\ 0 & 0 & F_{22}' & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

will be considered as an unbounded operator from ℓ_q^n into ℓ_q^m if $p = (1, \infty]$ or from c_0^n into c_0^m if $p = 1$.

PROPOSITION 3.1. *Let $F = \mathcal{L}^{m\times n}$. Then*
(a) *$\mathcal{G}_p(F)$ is weak*-closed;*
(b) *for $p \in (1, \infty]$, F is ℓ_p-bounded iff F' is ℓ_q-bounded;*
(c) *F is ℓ_1-bounded iff F' is c_0-bounded.*

Proof. It is easy to see that F' and F form an adjoint pair in the sense that

$$\langle \eta, Fx \rangle = \langle F'\eta, x \rangle \qquad \text{for all } x \in \mathcal{D}_p(F) \text{ and } \eta \in \mathcal{D}_q(F')$$

if $p \in (1, \infty]$ or

$$\langle \eta, Fx \rangle = \langle F'\eta, x \rangle \qquad \text{for all } x \in \mathcal{D}_1(F) \text{ and } \eta \in \mathcal{D}_{c_0}(F').$$

Since F' is upper triangular, each of $\mathcal{D}_q(F')$, $q \in [1, \infty)$, and $\mathcal{D}_{c_0}(F')$ contains all finitely nonzero sequences and hence is dense. Consequently,

F is the adjoint operator of F'. (See [18, pp. 167-168] for justification.) This means

$$\mathcal{G}'_p(-F) = \mathcal{G}_q(F')^\perp$$

for $p \in (1, \infty]$ and

$$\mathcal{G}'_1(-F) = \mathcal{G}_{c_0}(F')^\perp$$

(See [18, pp. 167–168].) Since an annihilator is always weak*-closed, it follows that $\mathcal{G}'_p(-F)$ is weak*-closed and so is $\mathcal{G}_p(F)$. This proves (a).

The "if" parts of (b) and (c) are standard since the adjoint operator of a bounded operator is bounded.

If F is ℓ_p-bounded, then each of its columns is in ℓ_p^m. The Hölder inequality then implies that $\eta' F$ is well defined for each $\eta \in \ell_q^m$. Now let x be an arbitrary element in ℓ_p^n. We have

$$|\eta' F x| \le \|\eta\|_q \|Fx\|_p \le (\|\eta\|_q \|F\|_p)\|x\|_p \,.$$

It then follows from [19, Lemma 10.4] that $F'\eta$ belongs to ℓ_q^n and $\|F'\eta\|_q \le \|F\|_p \|\eta\|_q$. This proves the "only if" part of (b) and something extra: F' is ℓ_∞-bounded if F is ℓ_1-bounded. Notice that we have not used the causality of F so far in this paragraph. Assume now that F is ℓ_1-bounded. Then we know that F' is ℓ_∞-bounded. Let $\eta \in c_0^m$. By using the causality of F, we obtain

$$(I - \Pi_k)F'\eta = (I - \Pi_k)F'(I - \Pi_k)\eta \to 0$$

as $k \to \infty$ since $(I - \Pi_k)\eta \to 0$. This implies that $F'\eta \in c_0^n$. □

In particular, Proposition 3.1 implies that $\mathcal{G}_p(F)$ is always closed for each system $F \in \mathcal{L}^{m\times n}$, so we can define the directed gap and the gap between systems to be the directed gap and the gap between their graphs, i.e.,

$$\begin{aligned} \vec{\delta}_p(F_1, F_2) &= \vec{\delta}[\mathcal{G}_p(F_1), \mathcal{G}_p(F_2)] \\ \delta_p(F_1, F_2) &= \delta[\mathcal{G}_p(F_1), \mathcal{G}_p(F_2)] \,. \end{aligned}$$

Similar to the subspace case, we can define balls $\overrightarrow{\mathbf{B}}_p(F, r)$, $\overleftarrow{\mathbf{B}}_p(F, r)$, $\mathbf{B}_p(F, r)$ and topologies $\vec{\tau}_p$, $\overleftarrow{\tau}_p$, τ_p.

The subset of $\mathcal{L}^{m\times n}$ consisting ℓ_p-bounded operators will be denoted by $\mathcal{B}_p^{m\times n}$. Systems in $\mathcal{B}_p^{m\times n}$ are also said to be ℓ_p-stable.

An important observation at this point is that the map from systems to their graphs is not injective in general. For example, both

$$(3.3)\quad F_1 = \begin{bmatrix} 2^{0^2} & 0 & 0 & \cdots \\ 2^{1^2} & 2^{0^2} & 0 & \cdots \\ 2^{2^2} & 2^{1^2} & 2^{0^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 2^{0^2}+1 & 0 & 0 & \cdots \\ 2^{1^2}+1 & 2^{0^2}+1 & 0 & \cdots \\ 2^{2^2}+1 & 2^{1^2}+1 & 2^{0^2}+1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

have trivial graphs, i.e., $\mathcal{G}_p(F_1) = \mathcal{G}_p(F_2) = \{0\}$, but $F_1 \neq F_2$. This implies that the space $(\mathcal{L}^{m\times n}, \tau_p)$ is not Hausdorff even though $(\Gamma(\mathcal{X}), \tau)$ is for any Banach space $\mathcal{X}$. Nevertheless, it can be easily seen that $(\mathcal{B}^{m\times n}, \tau_p)$ is Hausdorff. In Section 5, we will show τ_p, as well as $\vec{\tau}_p$, is Hausdorff in a larger set.

4. Robust stabilization. Now consider the feedback system shown in Figure 1.1. We will simply call it (P, C). Assume $P \in \mathcal{L}^{m\times n}$, $C \in \mathcal{L}^{n\times m}$. The equations governing the system variables are

$$\begin{aligned} e_1 + Ce_2 &= u_1 \\ Pe_1 + e_2 &= u_2. \end{aligned}$$

Consider the linear manifold

$$\mathcal{S}_p = \left\{ \begin{bmatrix} e_1 + Ce_2 \\ Pe_1 + e_2 \\ e_1 \\ e_2 \end{bmatrix} : e_1 \in \mathcal{D}_p(P), e_2 \in \mathcal{D}_p(C) \right\} \subset \ell_p^{(n+m)+(n+m)}. \tag{4.1}$$

Since $\mathcal{S}_p = \mathcal{G}'_p\left(\begin{bmatrix} I & C \\ P & I \end{bmatrix}\right)$, it follows that $\mathcal{S}_p$ is closed. If there exists $F \in \mathcal{L}^{(n+m)\times(n+m)}$ such that $\mathcal{G}_p(F) = \mathcal{S}_p$, i.e. $\begin{bmatrix} I & C \\ P & I \end{bmatrix}$ is invertible, then (P, C) is said to be *well-posed*. A necessary and sufficient condition for the existence of such F is that $\begin{bmatrix} 0 \\ 0 \\ e_1 \\ e_2 \end{bmatrix} \notin \mathcal{S}_p$ if either e_1 or e_2 is nonzero. This is also equivalent to the kernel of $\begin{bmatrix} I & C \\ P & I \end{bmatrix}$ being trivial. In this case, F is said to be the closed loop operator of (P, C). If $F \in \mathcal{B}_p^{(n+m)\times(n+m)}$, i.e., $\begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1}$ is ℓ_p-bounded, then (P, C) is said to be ℓ_p-stable.

PROPOSITION 4.1. *The following three statements are equivalent:*
(a) *(P, C) is ℓ_p-stable;*
(b) *$\mathcal{G}_p(P) \oplus \mathcal{G}'_p(C) = \ell_p^{n+m}$;*
(c) *$\mathcal{G}_p(P) + \mathcal{G}'_p(C) = \ell_p^{n+m}$.*

Proof. Suppose (P, C) is ℓ_p-stable. Then $\begin{bmatrix} 0 \\ 0 \\ e_1 \\ e_2 \end{bmatrix} \notin \mathcal{S}_p$ if either e_1 or e_2 is nonzero, and $\left\{ \begin{bmatrix} e_1 + Ce_2 \\ Pe_1 + e_2 \end{bmatrix} : e_1 \in \mathcal{D}_p(P), e_2 \in \mathcal{D}_p(C) \right\} = \ell_p^{n+m}$. The former implies $\mathcal{G}_p(P) \cap \mathcal{G}_p'(C) = \{0\}$ and the latter implies $\mathcal{G}_p(P) + \mathcal{G}_p'(C) = \ell_p^{n+m}$. This proves (a) $\Rightarrow$ (b). If $\mathcal{G}_p(P) \oplus \mathcal{G}_p'(C) = \ell_p^{n+m}$, then we have $\begin{bmatrix} I \\ P \end{bmatrix} e_1 + \begin{bmatrix} C \\ I \end{bmatrix} e_2 = 0$ with $e_1 \in \mathcal{D}_p(P)$ and $e_2 \in \mathcal{D}_p(C)$ if and only if $e_1 = 0$ and $e_2 = 0$, and we also have $\left\{ \begin{bmatrix} e_1 + Ce_2 \\ Pe_1 + e_2 \end{bmatrix} : e_1 \in \mathcal{D}_p(P), e_2 \in \mathcal{D}_p(C) \right\} = \ell_p^{n+m}$. This means that there exists $F \in \mathcal{L}^{(n+m)\times(n+m)}$ such that $\mathcal{G}_p(F) = \mathcal{S}_p$ and the domain of F is ℓ_p^{n+m}. By the closed graph theorem, F is ℓ_p-bounded. This proves (b) $\Rightarrow$ (a). It is trivial that (b) $\Rightarrow$ (c). Notice that $\mathcal{G}_p(P) \cap \mathcal{G}_p'(C) \neq \{0\}$ means that the kernel of $\begin{bmatrix} I & C \\ P & I \end{bmatrix}$ is nontrivial. Since $\begin{bmatrix} I & C \\ P & I \end{bmatrix}$ is a block lower triangular matrix, at least one of the block diagonal elements must be singular. Then $\begin{bmatrix} I & C \\ P & I \end{bmatrix} [\mathcal{D}_p(P) \times \mathcal{D}_p(C)]$ can not be ℓ_p^{n+m} Hence $\mathcal{G}_p(P) + \mathcal{G}_p'(C) \neq \ell_p^{n+m}$. This proves (c) $\Rightarrow$ (b). □

The following theorem follows from Theorem 2.6(b), Proposition 2.7(b), and Proposition 4.1.

THEOREM 4.2. *Assume* (P, C) *is* ℓ_p*-stable. Then* $(\tilde{P}, C)$ *is* ℓ_p*-stable for all* $\tilde{P} \in \vec{\mathbf{B}}_p(P, r)$ *if* $r \le \vec{\rho}[\mathcal{G}_p(P), \mathcal{G}_p'(C)]$.

Theorem 4.2 shows that $\vec{\rho}[\mathcal{G}_p(P), \mathcal{G}_p'(C)]$ gives a robustness measure of the feedback system (P, C). It is desirable to have a closer relation between this measure and the systems P and C. Fortunately, we have one.

THEOREM 4.3. *Assume* (P, C) *is* ℓ_p*-stable. Then* $\vec{\rho}[\mathcal{G}_p(P), \mathcal{G}_p'(C)] = \left\| \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \; C] \right\|_p^{-1}$.

Proof. We know from Proposition 4.1 and Proposition 2.7(a) that

$$\vec{\rho}[\mathcal{G}_p(P), \mathcal{G}_p'(C)] = \|\Pi_{\mathcal{G}_p(P) \| \mathcal{G}_p'(C)}\|_p^{-1}.$$

For each $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \ell_p^{n+m}$, let

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Then $e_1 \in \mathcal{D}_p(P)$ and $e_2 \in \mathcal{D}_p(C)$, and

$$\begin{aligned} \Pi_{\mathcal{G}_p(P) \| \mathcal{G}_p'(C)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} I \\ P \end{bmatrix} e_1 \\ &= \begin{bmatrix} I \\ P \end{bmatrix} [I \; 0] \begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \;\; -C] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

This shows

$$\Pi_{\mathcal{G}_P \| \mathcal{G}'_p(C)} = \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \;\; -C]$$

This completes the proof. □

Since $\mathbf{B}_p(P,r) \subset \overrightarrow{\mathbf{B}}_p(P,r)$, the statement in Theorem 4.2 still holds if we replace $\overrightarrow{\mathbf{B}}_p(P,r)$ by $\mathbf{B}_p(P,r)$ but the result will be more conservative in general. The statement in Theorem 4.2 would no longer be true if we replace $\overrightarrow{\mathbf{B}}_p(P,r)$ by $\overleftarrow{\mathbf{B}}_p(P,r)$.

Finally, in this section let us comment on the optimal robust controller design problem: Given $P \in \mathcal{L}^{m\times n}$, design $C \in \mathcal{L}^{n \times m}$ such that the robustness measure $\vec{\rho}[\mathcal{G}_p(P), \mathcal{G}'_p(C)]$ is maximized. From Theorem 4.3, it is seen that this design problem is a minimum norm optimal control problem. If $p = 2$ and P is shift-invariant, then this optimal control problem is an $\mathcal{H}_\infty$ optimal control problem [15,12]. If $p = \infty$ or $p = 1$ and P is shift-invariant, then this optimal control problem is an ℓ_1 optimal control problem [2,3,22].

5. Topological properties. Note that our robust stability condition is given in terms of the directed gap ball $\overrightarrow{\mathbf{B}}(P,r)$. A consequence of this is that the feedback stability is a robust property under the topology $\vec{\tau}_p$. As we have argued in Section 2, the directed gap and gap have completely different topological properties. The topology $\vec{\tau}$ in $\Gamma(\mathcal{X})$ is strictly weaker than τ. A similar situation occurs in $\mathcal{L}^{m\times n}$. As an extreme example, take F to be either F_1 or F_2 in (3.3); we know that $\mathcal{G}_p(F) = \{0\}$. Therefore $\vec{\delta}_p(F, \tilde{F}) = 0$ for all $\tilde{F} \in \mathcal{L}^{1\times 1}$ but $\delta_p(F, \tilde{F}) = 1$ for all $\tilde{F} \in \mathcal{L}^{1\times 1}$ with a nontrivial graph. It has been established in [33,34] that "the gap topology is the weakest topology such that the feedback stability is a robust property". Firstly this statement needs further clarification in our case due to the complication caused by τ_p being non-Hausdorff. Secondly, if this statement is true, then there seems to be a paradox since the weaker topology $\vec{\tau}_p$ also makes the feedback stability a robust property according to Theorem 4.2. In this section, we show that on the set of stabilizable systems τ_p is a Hausdorff topology (easy!) and $\vec{\tau}_p$ is actually the same as τ_p (hard!). Since we are only concerned with stabilizable systems, this result justifies the use of the directed gap.

A system $P \in \mathcal{L}^{m\times n}$ is said to be *ℓ_p-stabilizable* if there exists a controller $C \in \mathcal{L}^{n\times m}$ such that (P, C) is ℓ_p-stable. Not all systems are ℓ_p-stabilizable; examples of unstabilizable systems are given in (3.3). Denote the set of ℓ_p-stabilizable systems in $\mathcal{L}^{m\times n}$ by $\mathcal{P}_p^{m\times n}$. Our easy task is fulfilled by showing the following result.

PROPOSITION 5.1. *For $P_1,\ P_2 \in \mathcal{P}_p^{m\times n}$, $\mathcal{G}_p(P_1) = \mathcal{G}_p(P_2)$ implies $P_1 = P_2$.*

Proof. Let $C \in \mathcal{P}^{n\times m}$ be a stabilizing compensator for P_1. By Proposition 4.1, this is equivalent to $\mathcal{G}_p(P_1)\oplus\mathcal{G}'_p(C) = \ell_p^{n+m}$. If $\mathcal{G}_p(P_1) = \mathcal{G}_p(P_2)$, then (P_2, C) is also stable. Since

$$\mathcal{G}_p\left(\begin{bmatrix} I & C \\ P_i & I \end{bmatrix}^{-1}\right) = \left\{\begin{bmatrix} e_1 + Ce_2 \\ P_ie_1 + e_2 \\ e_1 \\ e_2 \end{bmatrix} : e_1 \in \mathcal{D}_p(P), e_2 \in \mathcal{D}_p(C)\right\}$$
$$= \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \mathcal{G}_p(P_i)\times\mathcal{G}'_p(C)$$

for $i = 1, 2$, it is clear that

$$\mathcal{G}_p\left(\begin{bmatrix} I & C \\ P_1 & I \end{bmatrix}^{-1}\right) = \mathcal{G}_p\left(\begin{bmatrix} I & C \\ P_2 & I \end{bmatrix}^{-1}\right).$$

Since we know that different stable systems have different graphs, it follows that

$$\begin{bmatrix} I & C \\ P_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & C \\ P_2 & I \end{bmatrix}^{-1}.$$

This implies that

$$P_1(I - CP_1)^{-1} = P_2(I - CP_2)^{-1} \quad \text{and} \quad (I - CP_1)^{-1} = (I - CP_2)^{-1}.$$

Therefore, we must have $P_1 = P_2$. $\square$

The rest of this section is dedicated to our hard task. We proceed in several steps. First, let us partially order the elements of $\mathcal{L}^{m\times n}$ in the following way: $F_1 \preceq_p F_2$ if $\mathcal{G}_p(F_1) \subset \mathcal{G}_p(F_2)$. By definition, a system $F \in \mathcal{L}^{m\times n}$ is ℓ_p-maximal if $\tilde{F} \in \mathcal{L}^{m\times n}$ with $F \preceq_p \tilde{F}$ implies $\tilde{F} \neq F$. The following result was first established for $p = 2$ and shift-invariant systems in [13]

PROPOSITION 5.2. *A system is ℓ_p-stabilizable only if it is ℓ_p-maximal.*

Proof. Suppose that P is ℓ_p-stabilizable and C is a stabilizing compensator for P. By Proposition 4.1 $\mathcal{G}_p(P)\oplus\mathcal{G}'_p(C) = \ell_p^{m+n}$. If there exists a $\tilde{P}$ such that $P \preceq_p \tilde{P}$, then $\mathcal{G}_p(\tilde{P}) + \mathcal{G}'_p(C) = \ell_p^{m+n}$, so by Proposition 4.1 again, $\mathcal{G}_p(\tilde{P}) \cap \mathcal{G}'_p(C) = \{0\}$. This forces $\mathcal{G}_p(\tilde{P}) = \mathcal{G}_p(P)$, i.e., $\tilde{P} = P$. $\square$

A consequence of this proposition is that if F_1 and F_2 are stabilizable, then $\vec{\delta}_p(F_1, F_2) = 0$ implies $F_1 = F_2$. This makes $\vec{\delta}_p$ a step closer to a distance function but it is still not enough to establish the equivalence between $\vec{\tau}_p$ and τ_p.

PROPOSITION 5.3.
(a) *$\mathcal{B}_p^{m\times n}$ and $\mathcal{P}_p^{m\times n}$ are open subsets of $(\mathcal{L}^{m\times n}, \vec{\tau}_p)$.*
(b) *$\mathcal{B}_p^{m\times n}$ and $\mathcal{P}_p^{m\times n}$ are open subsets of $(\mathcal{L}^{m\times n}, \tau_p)$.*
(c) *$\vec{\mathbf{B}}_p(0,1) \subset \mathcal{B}_p^{m\times n}$.*
(d) *For $p \in (1,\infty]$, $\vec{\mathbf{B}}_p(0,1) = \mathcal{B}_p^{m\times n}$.*

Proof. Statement (a) follows from Theorem 4.2. Statement (b) follows from (a) since τ_p is stronger that $\vec{\tau}_p$. Statement (c) follows from Theorems 4.2 and 4.3 by setting $P = 0$ and $C = 0$. The only nontrivial part is (d). We only need to prove $\mathcal{B}_p^{m\times n} \subset \vec{\mathbf{B}}_p(0,1)$. Let $F \in \mathcal{B}_p^{m\times n}$. For $p \in (1,\infty)$,

$$\begin{aligned}
\vec{\delta}_p(0,F) &= \sup_{u\in\ell_p^n, \|u\|_p\le 1} \inf_{v\in\ell_p^n} \left\| \begin{bmatrix} u \\ 0 \end{bmatrix} - \begin{bmatrix} v \\ Fv \end{bmatrix} \right\|_p \\
&\le \sup_{u\in\ell_p^n, \|u\|_p\le 1} \left\| \begin{bmatrix} u \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{u}{1+\|F\|_p^q} \\ \frac{Fu}{1+\|F\|_p^q} \end{bmatrix} \right\|_p \\
&= \sup_{u\in\ell_p^n, \|u\|_p\le 1} \left\| \begin{bmatrix} \frac{\|F\|_p^q u}{1+\|F\|_p^q} \\ \frac{-Fu}{1+\|F\|_p^q} \end{bmatrix} \right\|_p \\
&= \left[\left(\frac{\|F\|_p^q}{1+\|F\|_p^q} \right)^p + \left(\frac{\|F\|_p}{1+\|F\|_p^q} \right)^p \right]^{1/p} \\
&= \frac{\|F\|_p}{(1+\|F\|_p^q)^{1/q}} \\
&< 1;
\end{aligned}$$

the case of $p = \infty$ is similar. This shows $F \in \vec{\mathbf{B}}_p(0,1)$. □

For $p = 1$, the containment in Proposition 5.3(c) is strict. For example, if

$$F = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

then $F \in \mathcal{B}_1$ but $\vec{\delta}_1(0,F) = 1$. For the same F as above, $\vec{\delta}_\infty(0,F) = \frac{1}{2}$ but $\delta_\infty(0,F) = 1$. This provides evidence that Theorem 4.2 becomes more conservative if $\vec{\mathbf{B}}_p(P,r)$ is replaced by $\mathbf{B}_p(P,r)$.

PROPOSITION 5.4. *For $F_1, F_2 \in \mathcal{B}_p^{m\times n}$,*

$$\frac{\|F_1 - F_2\|_p}{(1+\|F_1\|_p)(1+\|F_2\|_p)} \le \vec{\delta}_p(F_1,F_2) \le \|F_1 - F_2\|_p\,. \tag{5.1}$$

Proof. Let us prove the second inequality first.

$$\begin{aligned}
\vec{\delta}_p(F_1,F_2) &= \sup_{x\in\mathcal{G}_p(F_1),\|x\|_p\le 1}\ \inf_{y\in\mathcal{G}_p(F_2)} \|x-y\|_p \\
&= \sup_{u\in\ell_p^n,u\neq 0}\ \inf_{v\in\ell_p^n} \frac{\left\|\begin{bmatrix} u \\ F_1u\end{bmatrix} - \begin{bmatrix} v \\ F_2v\end{bmatrix}\right\|_p}{\left\|\begin{bmatrix} u \\ F_1u\end{bmatrix}\right\|_p} \\
&\le \sup_{u\in\ell_p^n,u\neq 0} \frac{\left\|\begin{bmatrix} u \\ F_1u\end{bmatrix} - \begin{bmatrix} u \\ F_2u\end{bmatrix}\right\|_p}{\left\|\begin{bmatrix} u \\ F_1u\end{bmatrix}\right\|_p} \\
&\le \sup_{u\in\ell_p^n,u\neq 0} \frac{\|(F_1-F_2)u\|_p}{\|u\|_p} \\
&= \|F_1-F_2\|_p\,.
\end{aligned}$$

Now we show the first inequality.

$$\begin{aligned}
\vec{\delta}_p(F_1,F_2) &= \sup_{u\in\ell_p^n,u\neq 0}\ \inf_{v\in\ell_p^n} \frac{\left\|\begin{bmatrix} u \\ F_1u\end{bmatrix} - \begin{bmatrix} v \\ F_2v\end{bmatrix}\right\|_p}{\left\|\begin{bmatrix} u \\ F_1u\end{bmatrix}\right\|_p} \\
&\ge \frac{1}{1+\|F_1\|_p} \sup_{u\in\ell_p^n,\|u\|_p=1}\ \inf_{v\in\ell_p^n} \left\|\begin{bmatrix} u \\ F_1u\end{bmatrix} - \begin{bmatrix} v \\ F_2v\end{bmatrix}\right\|_p .
\end{aligned}$$

For arbitrary $\epsilon > 0$, choose $\bar{u} \in \ell_p^n$ with $\|\bar{u}\|_p = 1$ and $\|(F_1-F_2)\bar{u}\|_p > \|F_1-F_2\|_p - \epsilon$. Then

$$\begin{aligned}
\vec{\delta}_p(F_1,F_2) &\ge \frac{1}{1+\|F_1\|_p} \inf_{v\in\ell_p^n} \left\|\begin{bmatrix} \bar{u} \\ F_1\bar{u}\end{bmatrix} - \begin{bmatrix} v \\ F_2v\end{bmatrix}\right\|_p \\
&= \frac{1}{1+\|F_1\|_p} \inf_{v\in\ell_p^n} \left\|\begin{bmatrix} \bar{u}-v \\ (F_1-F_2)\bar{u}+F_2(\bar{u}-v)\end{bmatrix}\right\|_p \\
&\ge \frac{1}{1+\|F_1\|_p} \inf_{v\in\ell_p^n} \max\{\|\bar{u}-v\|_p, \|(F_1-F_2)\bar{u}+F_2(\bar{u}-v)\|_p\}.
\end{aligned}$$

We claim that

$$\inf_{v\in\ell_p^n} \max\{\|\bar{u}-v\|_p, \|(F_1-F_2)\bar{u}+F_2(\bar{u}-v)\|_p\} \ge \frac{\|(F_1-F_2)\bar{u}\|_p}{1+\|F_2\|_p}\,.$$

Suppose this is not the case, then there exist $v \in \ell_p^n$ such that

$$\|F_2\|_p\|\bar{u}-v\|_p < \frac{\|F_2\|_p\|(F_1-F_2)\bar{u}\|_p}{1+\|F_2\|_p}$$

and

$$\|(F_1 - F_2)\bar{u}\| - \|F_2\|_p\|\bar{u} - v\|_p < \frac{\|(F_1 - F_2)\bar{u}\|_p}{1 + \|F_2\|_p}.$$

This means that

$$\|(F_1 - F_2)\bar{u}\|_p < \frac{\|(F_1 - F_2)\bar{u}\|_p}{1 + \|F_2\|_p} + \frac{\|F_2\|_p\|(F_1 - F_2)\bar{u}\|_p}{1 + \|F_2\|_p} = \|(F_1 - F_2)\bar{u}\|_p,$$

which is impossible. Hence we must have

$$\vec{\delta}_p(F_1, F_2) \geq \frac{\|(F_1 - F_2)\bar{u}\|_p}{(1 + \|F_1\|_p)(1 + \|F_2\|_p)} > \frac{\|F_1 - F_2\|_p - \epsilon}{(1 + \|F_1\|_p)(1 + \|F_2\|_p)}.$$

Since ϵ is arbitrary, the first inequality has to be true. □

We remark here that the inequalities in (5.1) are not the tightest possible. It can be shown that $1 + \|F_1\|_p$ in the far left of (5.1) can be replaced by $(1 + \|F_1\|_p^p)^{1/p}$. It is conjectured that $1 + \|F_2\|_p$ can also be replaced by $(1 + \|F_2\|_p^p)^{1/p}$.

Inequalities in (5.1) are symmetric with respect to F_1 and F_2. Hence relative topologies of $\vec{\tau}_p$, $\overleftarrow{\tau}_p$, and τ_p in $\mathcal{B}_p^{m\times n}$ are the same and this topology is the same as the one induced by the norm.

Let us denote the set of well-posed pairs $(P, C) \in \mathcal{P}_p^{m\times n} \times \mathcal{P}_p^{n\times m}$ by $\mathcal{W}_p(m, n)$, and the map which maps $(P, C) \in \mathcal{W}_p(m, n)$ to $\begin{bmatrix} I & C \\ P & I \end{bmatrix}^{-1}$ by $\mathbf{H}$.

PROPOSITION 5.5. *Under topologies $\vec{\tau}_p$, $\overleftarrow{\tau}_p$, and τ_p, the map $\mathbf{H}$ is a homeomorphism between $\mathcal{W}_p(m, n)$ and its image.*

Proof. First consider the map from (P, C) to

$$\mathcal{G}_p(P) \times \mathcal{G}_p'(C) = \left\{ \begin{bmatrix} e_1 \\ Pe_1 \\ Ce_2 \\ e_2 \end{bmatrix} : e_1 \in \mathcal{D}_p(P), e_2 \in \mathcal{D}_p(C) \right\}.$$

By Proposition 5.1 and the definition of the topologies in $\mathcal{P}^{m\times n}$ and $\mathcal{P}^{n\times m}$, this map is a homeomorphism between $\mathcal{P}^{m\times n} \times \mathcal{P}^{n\times m}$ and its image in $\Gamma(\ell^{(n+m)+(n+m)})$. Secondly, consider the map on $\Gamma(\ell^{(n+m)+(n+m)})$ defined by

$$\mathcal{T} \to \begin{bmatrix} I & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \mathcal{T}.$$

Since the big operator matrix is bounded and has a bounded inverse, this map is a homeomorphism by Proposition 2.4. This second map maps $\mathcal{G}_p(P) \times \mathcal{G}'_p(C)$ to $\mathcal{S}_p$ defined in (4.1). For those elements $\mathcal{T}$ of $\Gamma(\ell^{(n+m)+(n+m)})$ satisfying $\mathcal{T} = \mathcal{G}_p(F)$ for some $F \in \mathcal{L}^{(n+m)\times(n+m)}$, the map from $\mathcal{T}$ to F is by definition a homeomorphism. Therefore $\mathbf{H}$ is the composition of three homeomorphisms and hence itself is a homeomorphism. □

PROPOSITION 5.6. *There is a unique topology in $\mathcal{P}_p^{m\times n}$ which has the following properties:*
(a) *$\mathcal{B}_p^{m\times n}$ is open and the relative topology in $\mathcal{B}_p^{m\times n}$ is the norm topology;*
(b) *the map $\mathbf{H}$ is a homeomorphism between $\mathcal{W}_p(m,n)$ and its image.*

Proof. Let τ' and τ'' be two topologies having properties (a) and (b). Let $\mathcal{O} \in \tau'$ and $P \in \mathcal{O}$. Then there exists a $C \in \mathcal{L}^{n\times m}$ such that $\mathbf{H}(P,C)$ is ℓ_p-stable. Since $\mathcal{B}_p^{m\times n}$ is τ'-open and $\mathbf{H}$ is a τ'-homeomorphism, there exist $\mathcal{O}_1, \mathcal{O}_2 \in \tau'$ with $P \in \mathcal{O}_1 \subset \mathcal{O}$ and $C \in \mathcal{O}_2$ such that $\mathbf{H}(\tilde{P},\tilde{C})$ is stable for all $\tilde{P} \in \mathcal{O}_1$ and $\tilde{C} \in \mathcal{O}_2$. The set $\mathbf{H}(\mathcal{O}_1,\mathcal{O}_2)$ is in τ'. Since the relative topologies of τ' and τ'' in $\mathcal{B}_p^{m\times n}$ are the same, the set $\mathbf{H}(\mathcal{O}_1,\mathcal{O}_2)$ is in τ'' too. By the fact that $\mathbf{H}$ is bijective and τ''-continuous, the set $\mathcal{O}_1$ must be in τ''. Since P is arbitrarily chosen in $\mathcal{O}$, this implies that $\mathcal{O}$ is also in τ''. This proves $\tau' \subset \tau''$. In exactly the same way, we can show $\tau'' \subset \tau'$. This completes the proof. □

Note that in the above proof, we only used the fact that the $\mathbf{H}$ is τ''-continuous to show that $\tau' \subset \tau''$. Hence the proof of Proposition 5.6 also shows the following result.

PROPOSITION 5.7. *The unique topology in $\mathcal{L}^{m\times n}$ determined by Proposition 5.6 is the weakest topology with the following properties:*
(a) *$\mathcal{B}_p^{m\times n}$ is open and the relative topology in $\mathcal{B}_p^{m\times n}$ is the norm topology;*
(b) *the map $\mathbf{H}$ is continuous on $\mathcal{W}_p(m,n)$.*

An immediate consequence of Propositions 5.3–5.7 is as follows.

THEOREM 5.8. *$\overrightarrow{\tau}_p$ and τ_p are the same topology on $\mathcal{P}_p^{m\times n}$. This topology is the weakest among all the topologies satisfying the two properties in Proposition 5.7.*

The topology $\overleftarrow{\tau}_p$ is different from τ_p on $\mathcal{P}_p^{m\times n}$. What goes wrong is that $\mathcal{B}_p^{m\times n}$ is not open under topology $\overleftarrow{\tau}_p$.

6. Systems with coprime factorizations. In this section, we consider systems with coprime factorizations as well as finite-dimensional shift-invariant systems. By using coprime factorizations, we will be able to address some computational problems involved in the robust stabilization.

An operator F in $\mathcal{L}^{m\times n}$ is said to have a *right fractional representation* over $\mathcal{B}_p$ if there exist $M \in \mathcal{B}_p^{n\times n}$ and $N \in \mathcal{B}_p^{m\times n}$ such that $F = NM^{-1}$. Such a fractional representation is said to be *coprime* if there exist $\tilde{X} \in$

$\mathcal{B}_p^{n\times n}$ and $\tilde{Y} \in \mathcal{B}_p^{n\times m}$ such that

$$\tilde{X}M + \tilde{Y}N = I. \tag{6.1}$$

Similarly, F is said to have a *left fractional representation* over $\mathcal{B}_p$ if there exist $\tilde{M} \in \mathcal{B}_p^{m\times m}$ and $\tilde{N} \in \mathcal{B}_p^{m\times n}$ such that $F = \tilde{M}^{-1}\tilde{N}$. Such a fractional representation is said to be *coprime* if there exist $X \in \mathcal{B}_p^{m\times m}$ and $Y \in \mathcal{B}_p^{n\times m}$ such that

$$\tilde{M}X + \tilde{N}Y = I. \tag{6.2}$$

Again we emphasize that the factors $M, N, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y}$ depend on p. Coprime fractional representations are also called *coprime factorizations*. The study of fractional representations and its relation to feedback stabilization constitutes an active research area. See, e.g., [5,28,31,23,4]. Denote the class of systems in $\mathcal{L}^{m\times n}$ which admit both right and left coprime factorizations over $\mathcal{B}_p$ by $\mathcal{C}_p^{m\times n}$. It is well-known that $\mathcal{C}_p^{m\times n} \subset \mathcal{P}_p^{m\times n}$. However, it has been a long standing open question if we actually have $\mathcal{C}_p^{m\times n} = \mathcal{P}_p^{m\times n}$. An affirmative answer has been given only for the cases of $p = 2$ [4] and for finite-dimensional systems [23].

PROPOSITION 6.1.

(a) *Let $F = NM^{-1}$ be a right coprime factorization of F over $\mathcal{B}_p$. Then*
$$\mathcal{G}_p(F) = \begin{bmatrix} M \\ N \end{bmatrix} \ell_p^n.$$

(b) *Let $F = \tilde{M}^{-1}\tilde{N}$ be a left fractional representation of F over $\mathcal{B}_p$. Then*
$$\mathcal{G}_p(F) = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \ell_p^{n+m} : [-\tilde{N}\ \tilde{M}] \begin{bmatrix} u \\ y \end{bmatrix} = 0 \right\}.$$

We leave the proof of this proposition to the reader. For the idea, see [30, p. 234].

PROPOSITION 6.2. *Let $F = \tilde{M}^{-1}\tilde{N}$ be a left coprime factorization of $F \in \mathcal{C}_p^{m\times n}$. Then $[-\tilde{N}\ \tilde{M}]'\ell_q^m = \mathcal{G}_p(F)^\top$ for $p \in (1,\infty]$ and $[-\tilde{N}\ \tilde{M}]'c_0^m = \mathcal{G}_1(F)^\top$.*

Proof. From Proposition 3.1, we know that $[-\tilde{N}\ \tilde{M}]'$ is bounded and its adjoint is $[-\tilde{N}\ \tilde{M}]$. It follows from [27, Theorem 4.12] and Proposition 6.1 that $([-\tilde{N}\ \tilde{M}]'\ell_q^m)^\perp = \mathcal{G}_p(F)$ for $p \in (1,\infty]$ and $([-\tilde{N}\ \tilde{M}]'c_0^m)^\perp = \mathcal{G}_1(F)$. By taking back annihilator in both sides, we obtain the desired result. □

The following theorem, which is intended for the computation of the directed gap between two systems, follows from Propositions 2.3 and 6.2 immediately.

THEOREM 6.3. *Let $F_1 = N_1M_1^{-1}$ be a right coprime factorization and*

$F_2 = \tilde{M_2}^{-1} \tilde{N}_2$ *be a left coprime factorization. Then*

$$\begin{aligned} \vec{\delta}_p(F_1, F_2) &= sup\{\eta'[-\tilde{N}_2\ \tilde{M}_2] \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x : \\ &\qquad x \in \ell_p^n,\ \eta \in \ell_q^m,\ \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x\|_p \le 1,\ \|[-\tilde{N}_2\ \tilde{M}_2]'\eta\|_q \le 1\} \end{aligned} \tag{6.3}$$

for $p \in (1, \infty]$ *and*

$$\begin{aligned} \vec{\delta}_1(F_1, F_2) &= sup\{\eta'[-\tilde{N}_2\ \tilde{M}_2] \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x : \\ &\qquad x \in \ell_1^1,\ \eta \in c_0^m,\ \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x\|_1 \le 1,\ \|[-\tilde{N}_2\ \tilde{M}_2]'\eta\|_{c_0} \le 1\}. \end{aligned} \tag{6.4}$$

Proof. Notice that $\mathcal{G}_p(F)$ is weak*-closed and use Proposition 2.3(a). □

The formula for the directed gap given in Theorem 6.3 corresponds to an infinite dimensional bilinear programming problem [10]. In the following, we will show that for the case when F_1 is a finite dimensional shift-invariant system, we can approximate this bilinear programming problem arbitrarily well by solving a finite dimensional counterpart. Unfortunately, at the present we can not do the same for the general case. However, the case for finite-dimensional shift invariant F_1 is an important special case since the nominal system F in the directed gap ball $\vec{\mathbf{B}}(F, r) = \{\tilde{F} : \vec{\delta}(F, \tilde{F}) < r\}$ is very often a finite-dimensional shift-invariant system in practical analysis and design.

We define

$$\begin{aligned} \vec{\delta}_p^k(F_1, F_2) &= \sup\{\eta'[-\tilde{N}_2\ \tilde{M}_2] \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x : \\ &\qquad x \in \Pi_k \ell_p^n,\ \eta \in \Pi_k \ell_q^m,\ \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x\|_p \le 1,\ \|[-\tilde{N}_2\ \tilde{M}_2]'\eta\|_q \le 1\} \end{aligned} \tag{6.5}$$

for $p \in (1, \infty]$ and

$$\begin{aligned} \vec{\delta}_1^k(F_1, F_2) &= \sup\{\eta'[-\tilde{N}_2\ \tilde{M}_2] \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x : \\ &\qquad x \in \Pi_k \ell_1^n,\ \eta \in \Pi_k c_0^m,\ \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x\|_1 \le 1,\ \|[-\tilde{N}_2\ \tilde{M}_2]'\eta\|_{c_0} \le 1\}. \end{aligned} \tag{6.6}$$

The fact that (M_1, N_1) are right coprime and $(\tilde{M}_2, \tilde{N}_2)$ are left coprime means that the feasible x and η in (6.5) and (6.6) lie in a compact set in $\mathbb{R}^{(k+1)\times(n+m)}$. Since the quantity being optimized depends continuously on

x and η, we conclude the "sup" can be replaced by a "max". For fixed k, computing $\vec{\delta}_p^k(F_1, F_2)$ is a bilinear programming problem with $(k+1)\times(n+m)$ variables. However, the number of constraints due to $\|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x\|_p \leq 1$ is infinite. If F_1 is a finite-dimensional shift invariant system, then it is well known that M_1 and N_1 can be chosen to be lower triangular Toeplitz band matrices, i.e., there exist $l > 0$ (the order of F_1) such that $M_{1ij} = 0$ and $N_{1ij} = 0$ if $i - j > l$. In this case, it is easily seen that the number of constraints in (6.5) and (6.6) becomes finite.

PROPOSITION 6.4. *$\vec{\delta}_p^k(F_1, F_2)$ is monotonically increasing and $\lim_{k\to\infty} \vec{\delta}_p^k(F_1, F_2) = \vec{\delta}_p(F_1, F_2)$ if*
(a) *$p \in [1, \infty)$, or*
(b) *$p = \infty$ and M_1, N_1 are shift-invariant.*

Proof. The monotonicity of $\vec{\delta}_p^k(F_1, F_2)$ follows from the fact that the constraint set for the $k+1$ case in (6.5) and (6.6) contains that for the k case. For the same reason, $\vec{\delta}_p^k(F_1, F_2) \leq \vec{\delta}_p(F_1, F_2)$ for all $k \geq 0$. Hence $\vec{\delta}_p^k(F_1, F_2)$ converges.

Now assume $p \in (1, \infty)$. Let $\epsilon > 0$, choose $x \in \ell_p^m$ and $\eta \in \ell_q^m$ so that

$$\begin{aligned} \eta'[-\tilde{N}_2\ \tilde{M}_2]\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x &\geq \vec{\delta}_p(F_1, F_2) - \epsilon, \\ \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} x\|_p &= 1, \\ \|[-\tilde{N}_2\ \tilde{M}_2]'\eta\|_q &= 1. \end{aligned}$$

First, observe that

$$\lim_{k\to\infty} \|\Pi_k x - x\|_p = 0$$

$$\lim_{k\to\infty} \|\Pi_k \eta - \eta\|_q = 0,$$

so

$$\lim_{k\to\infty} \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} \Pi_k x\|_p = 1$$

$$\lim_{k\to\infty} \|[-\tilde{N}_2\ \tilde{M}_2]'\Pi_k \eta\|_q = 1.$$

Hence,

$$x^k := \Pi_k x / \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} \Pi_k x\|_p$$

$$\eta^k := \Pi_k\eta/\|[-\tilde{N}_2\ \tilde{M}_2]'\Pi_k\eta\|_q$$

satisfy all of the constraints in the optimization problem (6.5) for k sufficiently large that x^k and η^k are well-defined. Since

$$\eta^{k\prime}[-\tilde{N}_2\ \tilde{M}_2]\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}x^k \to \eta'[-\tilde{N}_2\ \tilde{M}_2]\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}x$$

as $k \to \infty$, we conclude that $\liminf_{k\to\infty} \vec{\delta}_p^k(F_1, F_2) \geq \vec{\delta}_p(F_1, F_2) - \epsilon$. Since $\epsilon > 0$ is arbitrary, we have our desired result for the case when $p \in (1, \infty)$.

Now suppose that $p = 1$. Here we choose $\eta \in c_0^m$ (not l_∞^m), so it is easy to see that the above analysis works in this case as well.

Now suppose that $p = \infty$, so that $q = 1$, and choose $x \in l_\infty^m$ and $\eta \in l_1^m$ as above. Here the analysis is more difficult, since we are not assured that

$$\lim_{k\to\infty} \|\Pi_k x - x\|_\infty = 0$$

unless $x \in c_0^m$. We'll replace x by $\bar{x}/\|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}x\|_\infty \in c_0^m$ with $\bar{x}$ defined by

$$\bar{x}_i := \alpha^i x_i, \quad \alpha \in (0,1).$$

Suppose that F_1 is shift-invariant; we'll show that $\bar{x}$ is a good approximation to x in the following sense:

$$\lim_{\alpha \nearrow 1} \eta'[-\tilde{N}_2\ \tilde{M}_2]\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}\bar{x} = \eta'[-\tilde{N}_2\ \tilde{M}_2]\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}x,$$

$$\lim_{\alpha \nearrow 1} \|\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}\bar{x}\|_\infty = 1.$$

The first equality is obvious since $\lim_{\alpha\nearrow 1} \bar{x} = x$ in the weak* topology. To prove the second equality, let $T = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}$. Since F_1 is shift-invariant, T_{ij} depends only on $i - j$, so we can adopt the notation $T_{i-j} := T_{ij}$. Let $y = Tx$ and $\bar{y} = T\bar{x}$; then

$$\begin{aligned} \|\bar{y}_k - \alpha^k y_k\|_\infty &= \|\sum_{i=0}^{k} T_i[\bar{x}_{k-i} - \alpha^k x_{k-i}]\|_\infty \\ &= \|\sum_{i=0}^{k} T_i[\alpha^{k-i} x_{k-i} - \alpha^k x_{k-i}]\|_\infty \\ &\leq \sum_{i=0}^{k} \|T_i\|_\infty[\alpha^{k-i} - \alpha^k]\|x_{k-i}\|_\infty \end{aligned}$$

$$\leq \left[\sum_{i=0}^{j} \|T_i\|_\infty [\alpha^{k-i} - \alpha^k] + \sum_{i=j+1}^{k} \|T_i\|\right] \|x\|_\infty$$

$$\leq \left[(\alpha^{-j} - 1)\|T\|_\infty + \sum_{i=j+1}^{\infty} \|T_i\|_\infty\right] \|x\|_\infty.$$

For any $\epsilon > 0$, we can make the second term less than $\epsilon/2$ by choosing j sufficiently large; we can then make the first term less than $\epsilon/2$ by choosing $\alpha \in (0,1)$ sufficiently close to 1. Hence, we conclude that

$$\lim_{\alpha \nearrow 1} \|T\bar{x}\|_\infty \leq 1.$$

Hence, we replace x by $\bar{x}/\|T\bar{x}\|$ with α close to 1; the remainder of the proof is the same as for the $p \in (1,\infty)$ case. □

7. Conclusion. In the robust control literature, there has been success in using the gap to study robust stabilizability for systems with ℓ_2 and $\mathcal{L}_2$ signal spaces, and attempts have also been made to generalize the ℓ_2 and $\mathcal{L}_2$ results to systems with general Banach signal spaces. In this paper we take a middle route; we consider systems with ℓ_p signal spaces. As far as we know, this is the first attempt of its kind. Some interesting results, which are on one hand nontrivial extensions of ℓ_2 results and on the other hand use special structures of ℓ_p spaces, are obtained. Since the theory is yet in its infancy, we feel that there are more questions yet to be answered than the questions answered in this paper. Other than the results obtained, an important contribution of this paper is that it has laid the foundation for further investigation. Among our major concerns are the following:

1. The tightness of the robustness condition given in Theorem 4.2. We know that it is tight, i.e., the condition is also necessary, for the case when $p = 2$ and P, C are shift-invariant. For the general case, we believe it is also tight in some sense and we are trying to prove it.
2. The method to carry out the optimal robust controller design, at least for the practically interesting cases: $p = 1$, 2, or ∞ and the given plant is finite dimensional shift-invariant. The case when $p = 2$ has been previously nicely solved. The cases $p = 1$ and $p = \infty$ are currently studied by the authors.
3. The equivalence between the stabilizability and the existence of coprime factorizations. Since our optimal robust controller design problem will likely depend on coprime factorizations, its solution can not be considered complete if we can not give a complete answer to this equivalence problem.

REFERENCES

[1] M. A. DAHLEH, *BIBO stability robustness in the presence of coprime factor perturbation*, IEEE Trans. Automat. Contr. **37** (1992), pp. 352–355.

[2] M. A. DAHLEH AND J. B. PEARSON, *ℓ^1 optimal feedback controller for MIMO discrete time systems*, IEEE Trans. Automat. Contr. **AC-32** (1987), pp. 314–322.

[3] M. A. DAHLEH AND J. B. PEARSON, *Optimal rejection of persistent disturbances, robust stability and mixed sensitivity minimization*, IEEE Trans. Automat. Contr. **33** (1988), pp. 722–731.

[4] W. N. DALE AND M. C. SMITH, *Stabilizability and existence of system representations for discrete-time time-varing systems*, Proc. IEEE Conf. Decision and Control (1990), pp. 2737–2742.

[5] C. A. DESOER, R. W. LIU, J. MURRAY, AND R. SAEKS, *Feedback system design: the fractional representation approach to analysis and synthesis*, IEEE Trans. Automat. Contr. **AC-25** (1980), pp. 399–412.

[6] A. K. EL-SAKKARY, *The gap metric: robustness of stabilization of feedback systems*, IEEE Trans. Automat. Contr. **AC-30** (1985), pp. 240–247.

[7] A. K. EL-SAKKARY, *Estimating robustness on the Riemann sphere*, Int. J. Control **49** (1989), pp. 561–567.

[8] A. FEINTUCH, *The gap metric for time-varying systems*, Systems & Control Letters **16** (1991), pp. 277–279.

[9] C. FOIAS, T. T. GEORGIOU, AND M. C. SMITH, *Geometric techniques for robust stabilization of linear time-varying systems*, Proc. IEEE Conf. Decision and Control (1990), pp. 2868–2873.

[10] F. FORGO, *Nonconvex Programming*, Akademiai Kiado, Budapest, 1988.

[11] T. T. GEORGIOU, *On the computation of the gap metric*, Systems & Control Letters **11** (1988), pp. 253–257.

[12] T. T. GEORGIOU AND M. C. SMITH, *Optimal robustness in the gap metric*, IEEE Trans. Automat. Contr. **35** (1990), pp. 673–685.

[13] T. T. GEORGIOU AND M. C. SMITH, *Graphs, causality and stabilizability: linear, shift-invariant systems on $\mathcal{L}_2[0,\infty)$*, Proc. IEEE Conf. Decision and Control (1992), pp. 1024–1029.

[14] T. T. GEORGIOU AND M. C. SMITH, *Robust stabilization in the gap metric: controller design for distributed systems*, IEEE Trans. Automat. Contr. **37** (1992), pp. 1133–1143.

[15] K. GLOVER AND D. C. MCFARLANE, *Robust stabilization of normalized coprime factor plant description with $\mathcal{H}_\infty$-bounded uncertainty*, IEEE Trans. Automat. Contr. **AC-34** (1989), pp. 821–830.

[16] P. K. KAMTHAN AND M. GUPTA, *Sequence Spaces and Series*, Marcel Dekker, Inc., New York, 1981.

[17] T. KATO, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Analyse Math. **6** (1958), pp. 261–322.

[18] T. KATO, *Perturbation Theory for Linear Operator*, Springer-Verlag, New York, 1966.

[19] J. L. KELLEY AND T. P. SRINIVASAN, *Measure and Integral*, Volume 1, Springer-Verlag, New York, 1988.

[20] M. G. KREĬN AND M. A. KRASNOSEL'SKIĬ, *Fundamental theorems concerning the extension of Hermitian operators and some of their applications to the theory of orthogonal polynomials and the moment problem*, (in Russian) Uspekhi Mat. Nauk **2** (1947), pp. 60–106.

[21] M. G. KREĬN, M. A. KRASNOSEL'SKIĬ, AND D. P. MIL'MAN, *Concerning the deficiency numbers of linear operators in Banach space and some geometric questions*, (in Russian) Sbornik Trudov Inst. Mat. Akad. Nauk Ukr. SSR **11** 1948.

[22] H. ÖZBAY, *On L^1 optimal control*, IEEE Trans. Automat. Contr. **34** (1989), pp. 884–885.

[23] K. Poolla and P. Khargonekar, *Stabilizability and stable-proper factorizations for linear time-varing systems*, SIAM J. Control & Optimization **25** (1987), pp. 723–736.

[24] L. Qiu and E. J. Davison, *Feedback stability under simultaneous gap metric uncertainties in plant and controller*, Systems & Control Letters **18** (1992), pp. 9–22.

[25] L. Qiu and E. J. Davison, *Pointwise gap metrics on transfer matrices*, IEEE Trans. Automat. Contr. **37** (1992), pp. 741–758.

[26] H. L. Royden, *Real Analysis*, MacMillan Publishing Co., Inc., New York, 1968.

[27] W. Rudin, *Functional Analysis*, McGraw-Hill Book Company, New York, 1973.

[28] R. Saeks and J. Murray, *Fractional representation, algebraic geometry, and the simultaneous stabilizability problem*, IEEE Trans. Automat. Contr. **AC-27** (1982), pp. 895–903.

[29] J. M. Schumacher, *A pointwise criterion for controller robustness*, Systems & Control Letters **18** (1992), pp. 1–8.

[30] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, The MIT Press, Cambridge, Massachusetts, 1985.

[31] M. Vidyasagar, H. Schneider, and B. A. Francis, *Algebraic and topological aspects of feedback stabilization*, IEEE Trans. Automat. Contr. **AC-27** (1982), pp. 880–894.

[32] G. Vinnicombe, *Structured uncertainty and the graph topology*, IEEE Trans. Automat. Contr. **38** (1993), pp. 1371–1383.

[33] G. Zames and A. K. El-Sakkary, *Unstable systems and feedback: the gap metric*, Proc. 16th Allerton Conf. (1980), pp. 380–385.

[34] S. Q. Zhu, *Graph topology and gap topology for unstable systems*, IEEE Trans. Automat. Contr. **AC-34** (1989), pp. 848–855.

[35] S. Q. Zhu, *Robustness of complementarity and feedback stabilization*, (preprint) 1992.