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Robust Stabilization of Multiplicative/Relative Uncertain Systems and Networked Feedback Control

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***Abstract*— Stabilization of uncertain dynamic systems is investigated focusing on plant models that involve both multiplicative and relative uncertainties that are bounded by $\mathcal{H}_\infty$-norm. A new notion of stability margin is proposed to study the robustness of the feedback stability. It is shown that the largest possible stability margin is a two-disk problem. An upper and lower bounds are derived which differ by only a factor of $\sqrt{2}$. The results are then applied to networked feedback control systems where logarithmic quantization is employed at both the plant input and output. Our results show that the coarsest quantization density can be computed based on the robust stability margin for plant models involving $\mathcal{H}_\infty$-norm bounded multiplicative and relative uncertainties.**

1. INTRODUCTIUON

Stabilization of dynamic systems involving $\mathcal{H}_\infty$-norm bounded uncertainties has been extensively studied in the control literature. Many different forms of unstructured dynamic uncertainties are investigated. There is a class of uncertain systems which deserves a special attention. This is the one that involves stable perturbations described by normalized coprime factors of the plant model [7], [12]. Such a class of systems is equivalent to gap metric uncertain systems and admits many interesting robustness properties in terms of feedback stability [6]. In [8], [9] a similar class of uncertain systems is proposed that involves multiplicative and relative modeling errors described by coprime factors of the plant model, and shown to be equivalent to gap metric uncertain systems as well.

In this paper we investigate stabilization of dynamic systems that involve multiplicative and relative uncertainties. Such uncertain systems appear similar to those in [8], [9]. However there is a major difference in that the $\mathcal{H}_\infty$-norm bounded multiplicative and relative uncertainties are not related to each other, and are thus not gap metric uncertainties. For this reason the stability margin formula from [6], [7] does not apply. In fact the overall uncertainty is not even unstructured, although the dynamic uncertainty in either multiplicative or relative form is unstructured. It follows that the corresponding robust stabilization is a μ-synthesis problem. We propose a new notion of stability margin, and show that it is an equivalent two-disk problem [1]. In addition we will derive an upper and lower bounds that are similar to the one for gap matric uncertain systems, and differ by only a factor of $\sqrt{2}$.

Our research on robust stabilization of multiplicative and relative uncertain systems is motivated by networked feedback control. Specifically logarithmic quantization is employed in [3] at the plant input where the problem of coarsest quantization is raised and studied under state feedback. A Lyapunov approach was adopted to obtain the coarsest quantization above which quadratic stability holds. It was recognized in [4] that the quantization error resulted from logarithmic quantization is equivalent to the sector uncertainty, and hence the small gain theorem and $\mathcal{H}_\infty$ approach [13] are applicable. Their results make the connection between the coarsest quantization in networked control systems and the robust stability margin in $\mathcal{H}_\infty$ control. However results are unavailable in the existing literature for the same problem of coarsest quantization under output feedback. It turns out that the sector uncertainty associated with the logarithmic quantization error can be represented in either multiplicative or relative form. Thus for networked feedback control systems where logarithmic quantization is employed at both the plant input and plant output, coarsest quantization is equivalent to feedback stabilization for systems involving both multiplicative and relative uncertainties. Our robust stabilization results can be applied to compute and estimate the coarsest quantization density.

2. PROBLEM FORMULATION

Denote $\mathcal{H}_\infty$ as the collection of all stable and proper trabsfer functions. Let $G(z) \in \mathcal{H}_\infty$. Then its $\mathcal{H}_\infty$ norm is defined as

$$\|G\|_\infty = \sup_{|z|<1} |G(z)| = \operatorname{ess\,sup}_{\omega} |G(e^{j\omega})| \qquad (1)$$

The dynamic uncertain systems in consideration are of single-input/single-output, and represented by

$$P_\Delta(z) = [1 + \Delta_m(z)]P(z)[1 + \Delta_r(z)]^{-1} \qquad (2)$$

where $P(z)$ is a known rational transfer function, and $\Delta_m(z), \Delta_r(z) \in \mathcal{H}_\infty$ are unknown. Stabilization of uncertain dynamic systems is investigated focusing on plant models that involve both multiplicative and relative uncertainties that are bounded by $\mathcal{H}_\infty$-norm. A new notion of stability margin is proposed to study the robustness of the feedback stability. It is shown that the largest possible stability margin is a two-disk problem. An upper and lower bounds are derived which differ by only a factor of $\sqrt{2}$.

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The results are then applied to networked feedback control systems where logarithmic quantization is employed at both the plant input and output. Our results show that the coarsest quantization density can be computed based on the robust stability margin for plant models involving $\mathcal{H}_\infty$-norm bounded multiplicative and relative uncertainties. We aim at synthesizing a feedback controller $K(z)$ such that the feedback system in Fig. 1 is stabilized for all stable $\Delta_m(z)$ and $\Delta_r(z)$ such that $\|\Delta_m\|_\infty \leq \delta_m$ and $\|\Delta_r\|_\infty \leq \delta_r$. The dynamic uncertainties $\Delta_m(z)$ and $\Delta_r(z)$ are termed unstructured, but $\Delta(z) = \mathrm{diag}\{\Delta_r(z), \Delta_m(z)\}$ as in (6) is structured that has a diagonal structure of size 2 [2].

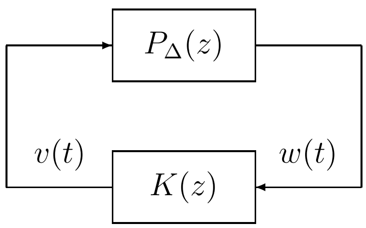

Fig. 1 Uncertainty feedback system

For feedback stability, we are often interested in computing the stability margin or the supremum of the uncertainty bound below which stabilizing feedback controllers exist. However for the feedback system in Fig. 1, there are two unstructured uncertainties. Thus the stability margin is not clearly defined. For this reason we consider $\delta = \sqrt{\delta_r^2 + \delta_m^2}$ with fixed ratios

$$r_S = \delta_r/\delta, \quad r_T = \delta_m/\delta \implies r_S^2 + r_T^2 = 1 \tag{3}$$

and search for the supremum of δ below which stabilizing feedback controllers exist. This new notion of stability margin helps to clarify the mathematical issue at hand and leads to results which have applications to computing the coarsest quantization density as defined in [3].

Because $P(z)$ is assumed to be rational, it admits normalized coprime factorization $P(z) = N(z)M(z)^{-1}$ satisfying

$$|N(e^{j\omega})|^2 + |M(e^{j\omega})|^2 = 1 \quad \forall \omega \in \mathbb{R} \tag{4}$$

By the fact that $0 < \delta_m < 1$ and $0 < \delta_r < 1$,

$$P_\Delta(z) = \{[1 + \Delta_m(z)]N(z)\}\{[1 + \Delta_r(z)]M(z)\}^{-1} \tag{5}$$

is also coprime factorization. Denote the transfer matrix consisting of the coprime factors of $P_\Delta(z)$ by

$$G_\Delta(z) = [I + \Delta(z)]\, G(z), \qquad G(z) = \begin{bmatrix} M(z) \\ N(z) \end{bmatrix} \tag{6}$$

with $\Delta(z) = \mathrm{diag}\{\Delta_r(z), \Delta_m(z)\}$. It follows from (4) that $G(z)$ is an inner. The next lemma holds. See also [6], [7].

Lemma 2.1: Denote $\mathcal{S}_\mathrm{o}(\mathcal{H}_\infty)$ as the collection of all the outer functions or the subset of $\mathcal{H}_\infty$ whose elements admit inverses in $\mathcal{H}_\infty$. Suppose that the nominal feedback system is stable, i.e., the feedback system in Fig. 1 is stable for the case $\delta_m = \delta_r = 0$. Then the uncertainty feedback system in Fig. 1 is robustly stable in the case $\delta = \sqrt{\delta_r^2 + \delta_m^2} > 0$, if and only if

$$\inf_{W_1, W_2 \in \mathcal{S}_\mathrm{o}(\mathcal{H}_\infty)} \left\| \begin{bmatrix} r_S W_2 & 0 \\ 0 & r_T W_1 \end{bmatrix} T \right\|_\infty < \delta^{-1} \tag{7}$$

where

$$T(z) = \begin{bmatrix} 1 \\ P \end{bmatrix} (1 - KP)^{-1} \begin{bmatrix} 1 & -K \end{bmatrix} \tag{8}$$

We skip the proof. It is commented that similar uncertainty feedback systems to that in Fig. 1 are investigated in the literature [8], [9] so that the results on gap-metric or ν-metric are applicable that is different from the robust stability problem as considered in this paper. Here we are interested in supremum of $\delta = \sqrt{\delta_r^2 + \delta_m^2}$ over all the stabilizing controllers $K(z)$ and $W_1(z), W_2(z) \in \mathcal{S}_\mathrm{o}(\mathcal{H}_\infty)$. It turns out that the gap-metric stability margin is instrumental to the stability margin to be studied in this paper.

Consider first the case $r_S = r_T = 1/\sqrt{2}$. Denote

$$T_W(z) = \begin{bmatrix} W_2 \\ W_1 P \end{bmatrix} (1 - KP)^{-1} \begin{bmatrix} W_2^{-1} & -KW_1^{-1} \end{bmatrix} \tag{9}$$

The following quantity is related to our stability margin:

$$\gamma_\mathrm{opt} := \inf_{W_1, W_2 \in \mathcal{S}_\mathrm{o}(\mathcal{H}_\infty), K \text{ stabilizing}} \|T_W\|_\infty \tag{10}$$

where $\delta_{\max} = \sqrt{2}\gamma_\mathrm{opt}^{-1}$ holds. It follows that the computation of the stability margin is a μ-synthesis problem. Indeed for each given pair $W_1(z), W_2(z) \in \mathcal{S}_\mathrm{o}(\mathcal{H}_\infty)$, the $\mathcal{H}_\infty$ optimization problem

$$\gamma_\mathrm{opt}(W_1, W_2) = \inf_{K \text{stabilizing}} \|T_W\|_\infty \tag{11}$$

has a solution $K(z)$ in the literature. Specifically we have the following expression from [7]:

$$\inf_{K \text{stabilizing}} \|T\|_\infty = \frac{1}{\sqrt{1 - \left\| \begin{bmatrix} M^\sim & N^\sim \end{bmatrix} \right\|_H^2}} \tag{12}$$

where $T(z)$ is the same as in (8), $\|\cdot\|_H$ denotes the Hankel operator norm, and $(\cdot)^\sim$ denotes para-hermitian. Simple calculations can be carried out for the transfer matrix $T_W(z)$ in (9) to arrive at

$$T_W(z) = \begin{bmatrix} 1 \\ P_W \end{bmatrix} (1 - K_W P_W)^{-1} \begin{bmatrix} 1 & -K_W \end{bmatrix} \tag{13}$$

Therefore $T_W(z)$ has an identical expression to $T(z)$ in (8) except that $P(z)$ is now replaced by $P_W(z) = W_1(z)P(z)W_2(z)^{-1}$ and $K(z)$ by $K_W(z) = W_2(z)K(z)W_1(z)^{-1}$. With $P_W(z) = N_W(z)M_W(z)^{-1}$ as a normalized coprime factorization for $P_W(z)$, there holds

$$\inf_{K_W \text{stabilizing}} \|T_W\|_\infty = \frac{1}{\sqrt{1 - \left\| \begin{bmatrix} M_W^\sim & N_W^\sim \end{bmatrix} \right\|_H^2}} \tag{14}$$

Once $K_W(z)$ is obtained, the controller $K(z) = W_2(z)^{-1}K_W(z)W_1(z)^{-1}$ is also available. We may then search for a new pair of $W_1(z), W_2(z) \in \mathcal{S}_\mathrm{o}(\mathcal{H}_\infty)$ in computing the left hand side of (7). This is so called W-K iteration in μ-synthesis. However such an iterative scheme may not be convergent, and can be costly in computation. An alternative approach is taken. In the next section we will present our results in computing the stability margin $\delta_{\max}$ for the general case of $r_{_S} \neq r_{_T}$.

3. Stability Margin

Because of the use of weighting functions, there is no loss of generality to assume that $P(z) = N_{\rm i}(z)M_{\rm i}(z)^{-1}$ is allpass or $|P(z)| = 1$ for all $|z| = 1$ where both $N_{\rm i}(z)$ and $M_{\rm i}(z)$ are inners. Our first result characterizes the stability margin $\delta_{\max}$.

Theorem 3.1: Suppose that $P(z) = N_{\rm i}(z)M_{\rm i}(z)^{-1}$ with $N_{\rm i}(z)$ and $M_{\rm i}(z)$ inners, and

$$V(z)M_{\rm i}(z) - U(z)N_{\rm i}(z) = \sqrt{2} \quad \forall\ |z| \geq 1 \tag{15}$$

for some $U(z), V(z) \in \mathcal{H}_\infty$. Then the stability margin for the feedback system in Fig. 1 has the following expression:

$$\frac{\delta_{\max}}{\sqrt{2}} = \left[\inf_{Q\in\mathcal{H}_\infty}\left\{\sup_{\omega\in\mathbb{R}}\left|V_Q(e^{j\omega})\right| r_{_S} + \left|U_Q(e^{j\omega})\right| r_{_T}\right\}\right]^{-1}$$

where $r_{_S} > 0$, $r_{_T} > 0$ are fixed satisfying $r_{_S}^2 + r_{_T}^2 = 1$, and $V_Q(z) = V(z) + Q(z)N_{\rm i}(z)$, $U_Q(z) = U(z) + Q(z)M_{\rm i}(z)$.

Proof: It is clear that $\{M_{\rm i}(z)/\sqrt{2}, N_{\rm i}(z)/\sqrt{2}\}$ are normalized coprime factors of $P(z)$. Thus (15) is the Bezout identity, and all stabilizing controllers are parameterized by

$$K(z) = [U(z)+Q(z)M_{\rm i}(z)/\sqrt{2}][V(z)+Q(z)N_{\rm i}(z)/\sqrt{2}]^{-1}$$

where $Q(z) \in \mathcal{H}_\infty$. Since the uncertain system is given by $P_\Delta(z) = [1 + \Delta_m(z)]P(z)[1 + \Delta_r(z)]^{-1}$, the closed-loop system is stable for all $\Delta_r(z), \Delta_m(z) \in \mathcal{H}_\infty$ satisfying $\|\Delta_r\|^2 + \|\Delta_m\|_\infty^2 < \delta_{\max}^2$, if and only if

$$|1 - P_\Delta(z)K(z)| \neq 0 \ \ \forall\ |z| \geq 1 \tag{16}$$

Hence robust stability is equivalent to $(0 \leq \varepsilon < 1)$

$$\left|1 - \varepsilon\delta_{\max}\left(|V_Q(e^{j\omega})|r_{_S} + |U_Q(e^{j\omega})|r_{_T}\right)/\sqrt{2}\right|$$

$\forall z = e^{j\omega}, \omega \in \mathbb{R}$ by taking $\Delta_m = \varepsilon\delta_m e^{-j\phi}$ and $\Delta_M = -\varepsilon\delta_r e^{-j\psi}$ with ϕ and ψ arguments of $(U + QM_{\rm i})N_{\rm i}$ and $(V + QN_{\rm i})M_{\rm i}$, respectively. Hence by using the definition of $\delta_{\max}$ leads to the expression in Theorem 3.1. ■

Theorem 3.1 shows that computation of stability margin is a two-disk problem. Results in [1] can be used to calculate $\delta_{\max}$. It is worth to pointing out that

$$\begin{aligned} S(z) &= [1 - K(z)P(z)]^{-1} = V_Q(z)/\sqrt{2} \\ S_c(z) &= K(z)P(z)[1 - K(z)P(z)]^{-1} = U_Q(z)/\sqrt{2} \end{aligned}$$

are the sensitivity, and complementary sensitivity functions, respectively. Hence the stability margin in Theorem 3.1 can alternatively be written as

$$\delta_{\max} = \left[\inf_{Q\in\mathcal{H}_\infty}\left\{\sup_{\omega\in\mathbb{R}}\left|S(e^{j\omega})\right| r_{_S} + \left|S_c(e^{j\omega})\right| r_{_T}\right\}\right]^{-1} \tag{17}$$

The following result gives an upper and lower bound for the stability margin $\delta_{\max}$.

Corollary 3.2: Under the same hypotheses of Thereom 3.1, the stability margin is bounded as

$$\begin{aligned} \delta_{\max} &\geq \frac{\sqrt{1 - \left\|\left[\ r_{_S}N_{\rm i}^\sim \quad r_{_T}M_{\rm i}^\sim\ \right]\right\|_H^2}}{\sqrt{2}r_{_S}r_{_T}} \\ \delta_{\max} &\leq \frac{\sqrt{1 - \left\|\left[\ r_{_S}N_{\rm i}^\sim \quad r_{_T}M_{\rm i}^\sim\ \right]\right\|_H^2}}{r_{_S}r_{_T}} \end{aligned}$$

Proof: Upper and lower bounds can be obtained by using

$$\left\|\left[\ r_{_S}S \quad r_{_T}S_c\ \right]\right\|_\infty \leq \frac{\sqrt{2}}{\delta_{\max}} \leq \sqrt{2}\left\|\left[\ r_{_S}S \quad r_{_T}S_c\ \right]\right\|_\infty \tag{18}$$

As $P(z) = N_{\rm i}(z)M_{\rm i}(z)^{-1}$ is allpass, there holds

$$\left\|\left[\ r_{_S}S \quad r_{_T}S_c\ \right]\right\|_\infty = r_{_S}r_{_T}\left\|T_r\right\|_\infty$$

where $T_r(z) = T_W(z)$ with $W_1 = r_{_S}$ and $W_2 = r_{_T}$. Since $P_r(z) = [r_{_S}N_{\rm i}(z)][r_{_T}M_{\rm i}(z)]^{-1}$ is a normalized coprime factorization of $P_r(z)$ by $r_{_S}^2 + r_{_T}^2 = 1$, we have

$$\begin{aligned} \gamma_{\rm opt} &= \inf_{K\ \text{stabilizing}}\left\|\left[\ r_{_S}S \quad r_{_T}S_c\ \right]\right\|_\infty \\ &= \frac{r_{_S}r_{_T}}{\sqrt{1 - \left\|\left[\ r_{_S}N_{\rm i}^\sim \quad r_{_T}M_{\rm i}^\sim\ \right]\right\|_H^2}} \end{aligned} \tag{19}$$

in light of (12) by noting that $|r_{_S}N_{\rm i}(z)|^2 + |r_{_T}M_{\rm i}(z)|^2 = r_{_S}^2 + r_{_T}^2 = 1$ for all $|z| = 1$. We thus obtain the upper and lower bounds for $\delta_{\max}$. ■

Although Corollary 3.2 gives an upper and lower bound for the stability margin that differ by a factor of $\sqrt{2}$, Hankel operator norm is involved giving rise of difficulty in optimizing $\delta_{\max}$ by searching for the values of $r_{_S} > 0$ and $r_{_T} > 0$ satisfying $r_{_S}^2 + r_{_T}^2 = 1$. In the following theorem we present an explicit expression for the upper and lower bounds of $\delta_{\max}$.

Theorem 3.3: Suppose that the plant model $P(z) = N_{\rm i}(z)M_{\rm i}(z)^{-1}$ where $M_{\rm i}(z)$ and $N_{\rm i}(z)$ are coprime and non-trivial inners. Let $\{A_M, B_M, C_M, D_M\}$ and $\{A_N, B_N, C_N, D_N\}$ be balanced realizations of $M_{\rm i}(z)$ and $N_{\rm i}(z)$, respectively. Then

$$\left\|\left[\ r_{_S}N_{\rm i}^\sim \quad r_{_T}M_{\rm i}^\sim\ \right]\right\|_H^2 = \frac{1}{2} + \sqrt{\frac{1}{4} - r_{_S}^2 r_{_T}^2[1 - \overline{\sigma}(Z)^2]} \tag{20}$$

where Z satisfies $Z = A_M Z A_N + B_M B_N'$, $\overline{\sigma}(\cdot)$ denotes the maximum singular value, and $r_{_S}^2 + r_{_T}^2 = 1$. In addition there holds the maximum stability margin

$$\sqrt{[1 - \overline{\sigma}(Z)]} \leq \sup_{r_{_S}>0, r_{_T}>0} \delta_{\max} \leq \sqrt{2[1 - \overline{\sigma}(Z)]} \tag{21}$$

Proof: The hypotheses on balanced realizations for $M_{\rm i}(z)$ and $N_{\rm i}(z)$, and on inner transfer functions for $M_{\rm i}(z)$ and

$N_\mathrm{i}(z)$ imply that their respective reachability and observability gramians are identity matrices. It follows that

$$G_r(z) = \begin{bmatrix} r_S M_\mathrm{i}(z) \\ r_T N_\mathrm{i}(z) \end{bmatrix} = \left[\begin{array}{cc|c} A_M & 0 & B_M \\ 0 & A_N & B_N \\ \hline r_S C_M & 0 & r_S D_M \\ 0 & r_T C_N & r_T D_N \end{array}\right]$$

The assumption on balanced realizations for $M_\mathrm{i}(z)$ and $N_\mathrm{i}(z)$ lead to

$$P = A_G P A_G' + B_G B_G', \qquad Q = A_G' Q A_G + C_G' C_G$$
$$\implies \quad P = \begin{bmatrix} I & Z \\ Z' & I \end{bmatrix}, \qquad Q = \begin{bmatrix} r_S^2 I & 0 \\ 0 & r_T^2 I \end{bmatrix}$$

where Z solves $Z = A_M Z A_N + B_M B_N'$. Denote $\mathcal{R}_M$ and $\mathcal{R}_N$ as the infinity size reachability matrix of $M_\mathrm{i}(z)$ and $N_\mathrm{i}(z)$, respectively. Then

$$Z = \mathcal{R}_M \mathcal{R}_N', \quad \mathcal{R}_M \mathcal{R}_M = I, \quad \mathcal{R}_N \mathcal{R}_N = I \qquad (22)$$

It follows that $\overline{\sigma}(Z) \leq 1$, and

$$\|G_r^\sim\|_H^2 = \overline{\lambda}(PQ) = \overline{\lambda}\left(\begin{bmatrix} r_S^2 I & r_S r_T Z \\ r_S r_T Z' & r_T^2 I \end{bmatrix}\right)$$

Furthermore $\lambda_{\max} = \overline{\lambda}(PQ)$ is the maximum root of

$$\det\left(\begin{bmatrix} (\lambda_{\max} - r_S^2) I & -r_S r_T Z \\ -r_S r_T Z' & (\lambda_{\max} - r_T^2) I \end{bmatrix}\right) = 0 \iff$$
$$\lambda_{\max} = \frac{1}{2} + \sqrt{\frac{1}{4} - r_S^2 r_T^2 + r_S^2 r_T^2 \overline{\sigma}(Z)^2} = \|G_r^\sim\|_H^2$$

that verifies (20). To prove the bounds for the maximum stability margin in (21), we need to show

$$\sup_{r_S>0, r_T>0} \frac{\sqrt{1 - \|\left[\, r_S N_\mathrm{i}^\sim \quad r_T M_\mathrm{i}^\sim \,\right]\|_H^2}}{\sqrt{2} r_S r_T} = \sqrt{1 - \overline{\sigma}(Z)} \qquad (23)$$

Denote $x = r_S^2 r_T^2$. Then $0 < x \leq 1/4$ by $r_S > 0$, $r_T > 0$, and $r_S^2 + r_T^2 = 1$. We aim at computing the supremum of

$$f(x) = \frac{\frac{1}{2} - \sqrt{\frac{1}{4} - x[1 - \overline{\sigma}(Z)^2]}}{x} \qquad (24)$$

over the interval $(0,\ 1/4]$, as $\sqrt{f(x)/2} \leq \delta_{\max} \leq \sqrt{f(x)}$. It is easy to verify that $f(x)$ is an increasing function of x for $x \in [0,\ 1/4]$, and thus $f(x)$ achieves the maximum at $x = 1/4$ that verifies (23). ∎

Remark 3.4: It is interesting to observe that the upper and lower bounds of $\delta_{\max}$ are maximized by taking $\delta_r = \delta_m = \delta_{\max}/\sqrt{2}$, even though we may have more unstable poles than unstable zeros, or vice versus. It is also observed that in light of the expressions in (22), $\overline{\sigma}(Z)$ represents the gap between the reachable subspaces of $M_\mathrm{i}(z)$ and of $N_\mathrm{i}(z)$. Thus if the two reachable subspaces are orthogonal to each other, then $1 \leq \delta_{\max} \leq \sqrt{2}$ implying that robust stability is attained even if each of the multiplicative and relative uncertainty bounds is close to 1; On the other hand if the gap between the two reachable subspaces is zero, then $\delta_{\max} = 0$. ∎

4. Networked Stabilization

We are motivated to study stabilization of networked feedback systems where logarithmic quantization is employed at both the input and output of the plant. As shown in [4], feedback stability associated with the logarithmic quantizer is equivalent to robust stability associated with the sector bounded uncertainty in the multiplicative form. We would like to point out that such a multiplicative uncertainty is not unique in representation of the sector bounded nonlinearity. In fact it can be equivalently converted to the form of relative uncertainty. Indeed the quantization values at the plant ichoosnput are given by

$$\mathcal{U} = \left\{\pm u_{(i)} = \rho^i u_{(0)} : \quad i = \pm1, \pm2, \cdots,\ 0 < \rho < 1\right\}$$

Mathematically the logarithmic quantization function can be defined by ($u_{(0)} > 0$ is assumed)

$$f(v) := \begin{cases} u_{(i)}, & \text{if } 1 - \delta_u < \frac{v}{u_{(i)}} \leq 1 + \delta_u \\ 0, & \text{if } v = 0 \\ -f(-v), & \text{if } v < 0 \end{cases} \qquad (25)$$

The relationship between ρ_u and δ_u is thus found to be

$$\delta_u = \frac{1 - \rho_u}{1 + \rho_u} \iff \rho_u = \frac{1 - \delta_u}{1 + \delta_u} \qquad (26)$$

It follows that the quantized input $u(t)$ can be represented by

$$u(t) = [1 + \Delta_r(t)]^{-1} v(t), \qquad |\Delta_r(t)| \leq \delta_u \qquad (27)$$

for which the sector uncertainty is in the form of relative error.

For the quantizer at the output of the plant, it is defined in the same way as in [4] via

$$g(y) := \begin{cases} w_{(i)}, & \text{if } \frac{w_{(i)}}{1+\delta_y} < y \leq \frac{w_{(i)}}{1-\delta_y} \\ 0, & \text{if } y = 0 \\ -g(-y), & \text{if } y < 0 \end{cases} \qquad (28)$$

where the set of quantized outputs has the same logarithmic form ($w_{(0)} > 0$ is assumed):

$$\mathcal{W} = \left\{\pm w_{(i)} = \rho_y^i w_{(0)} : \quad i = \pm1, \pm2, \cdots,\ 0 < \rho_y < 1\right\}$$

A similar relation between ρ_y and δ_y to that in (26) is

$$\delta_y = \frac{1 - \rho_y}{1 + \rho_y} \iff \rho_y = \frac{1 - \delta_y}{1 + \delta_y} \qquad (29)$$

With the quantizer in (28), the corresponding sector uncertainty has the multiplicative form:

$$w(t) = [1 + \Delta_m(t)] y(t), \qquad |\Delta_m(t)| \leq \delta_y \qquad (30)$$

It is interesting to observe that even though the two logarithmic quantization functions are defined differently and correspond to different forms of (multiplicative/relative) uncertainties, their quantization densities [3] have the same expression:

$$\eta_f = 2/\log(\rho_u^{-1}), \qquad \eta_g = 2/\log(\rho_y^{-1}) \qquad (31)$$

In the existing literature, minimization of the quantization density is an important problem area for feedback control over the networks. The number of bits represents resource of the network. For logarithmic quantization, the resource is measured by quantization density. It is clear that minimization of the quantization densities in (31) over all stabilizing feedback controllers is equivalent to minimization of ρ_u and ρ_y which is in turn equivalent to maximization of δ_u and δ_y, respectively over all stabilizing feedback controllers.

A possible doubt is whether or not the relative sector uncertainty is equivalent to the multiplicative sector uncertainty. The answer is firmative. Let us first consider the case when the logarithmic quantization is employed at only the plant output. With the multiplicative form for the quantization error as in (30), a sufficient condition for feedback stability is $\delta_y < \delta_{\max}^{(\mathrm{mul})}$ with

$$\delta_{\max}^{(\mathrm{mul})} = \min\left\{ \left(\inf_{K \text{ stabilizing}} \left\| \frac{KP}{1-PK} \right\|_\infty \right)^{-1}, 1 \right\} \quad (32)$$

This condition is also necessary if quadratic stability is of the interest [3], [4]. On the other hand if the relative form of the quantization error is employed, the dual condition for feedback stability is $\delta_y < \delta_{\max}^{(\mathrm{rel})}$ with

$$\delta_{\max}^{(\mathrm{rel})} = \min\left\{ \left(\inf_{K \text{ stabilizing}} \left\| \frac{1}{1-PK} \right\|_\infty \right)^{-1}, 1 \right\} \quad (33)$$

The next result shows that the coarsest quantization density computed from either $\delta_{\max}^{(\mathrm{mul})}$ or $\delta_{\max}^{(\mathrm{rel})}$ remains the same by proving $\delta_{\max}^{(\mathrm{mul})} = \delta_{\max}^{(\mathrm{rel})}$.

Theorem 4.1: Suppose that $P(z)$ has n poles and m zeros strictly outside the unit circle, denoted by $\{p_i^{(\mathrm{u})}\}_{i=1}^n$ and $\{z_k^{(\mathrm{u})}\}_{k=1}^m$, respectively with $n \geq 1$ and $m \geq 1$. Then $\delta_{\max}^{(\mathrm{mul})} = \delta_{\max}^{(\mathrm{rel})}$ which are defined in (32) and (33), respectively.

Proof: Nevanlinna-Pick interpolation [5] can be used to derive the stability margins for multiplicative and relative uncertainties. In the case of multiplicative uncertainty, the existence of robustly stabilizing controllers is equivalent to the existence of an $\mathcal{H}_\infty$ function $T(z) = \gamma_{\mathrm{mul}}^{-1} P(z)K(z)[1-P(z)K(z)]^{-1}$ such that $T(p_i^{(\mathrm{u})}) = \gamma_{\mathrm{mul}}^{-1}$ and $T(z_i^{(\mathrm{u})}) = 0$ for $1 \leq i \leq n$ and $1 \leq k \leq m$ such that $\|T\|_\infty \leq 1$ at each given $\gamma_{\mathrm{mul}} > 1/\delta_{\max}^{(\mathrm{mul})}$. This is equivalent to the problem of Nevanlinna-Pick interpolation that has an equivalent condition:

$$P_{\mathrm{mul}} = \begin{bmatrix} Z & \Omega \\ \Omega' & (1-\gamma_{\mathrm{mul}}^{-2})P \end{bmatrix} \geq 0 \quad (34)$$

where Z, P, and Ω are given respectively as

$$P = \left[\frac{p_i^{(\mathrm{u})} \bar{p}_k^{(\mathrm{u})}}{p_i^{(\mathrm{u})} \bar{p}_k^{(\mathrm{u})} - 1} \right]_{i,k=1,1}^{n,n}$$

$$Z = \left[\frac{z_i^{(\mathrm{u})} \bar{z}_k^{(\mathrm{u})}}{z_i^{(\mathrm{u})} \bar{z}_k^{(\mathrm{u})} - 1} \right]_{i,k=1,1}^{m,m}$$

$$\Omega = \left[\frac{z_i^{(\mathrm{u})} \bar{p}_k^{(\mathrm{u})}}{z_i^{(\mathrm{u})} \bar{p}_k^{(\mathrm{u})} - 1} \right]_{i,k=1,1}^{m,n}$$

On the other hand, the existence of robustly stabilizing controllers for relative uncertainty is equivalent to the existence of an $\mathcal{H}_\infty$ function $S(z) = \gamma_{\mathrm{rel}}^{-1}[1-P(z)K(z)]^{-1}$ such that $S(z_i^{(\mathrm{u})}) = \gamma_{\mathrm{rel}}^{-1}$ and $S(p_k^{(\mathrm{u})}) = 0$ for $1 \leq i \leq m$ and $1 \leq k \leq n$ such that $\|S\|_\infty \leq 1$ at each given $\gamma_{\mathrm{rel}} > 1/\delta_{\max}^{(\mathrm{rel})}$. This is equivalent to

$$P_{\mathrm{rel}} = \begin{bmatrix} (1-\gamma_{\mathrm{rel}}^{-2})Z & \Omega \\ \Omega' & P \end{bmatrix} \geq 0 \quad (35)$$

Because $Z > 0$ and $P > 0$, $P_{\mathrm{mul}} \geq 0$ and $P_{\mathrm{rel}} \geq 0$ are equivalent to

$$Z - \frac{\Omega P^{-1} \Omega'}{(1-\gamma_{\mathrm{mul}}^{-2})} \geq 0, \qquad (1-\gamma_{\mathrm{rel}}^{-2})Z - \Omega P^{-1}\Omega' \geq 0 \quad (36)$$

respectively. It follows that the infimum of γ_{mul} under the condition $P_{\mathrm{mul}} \geq 0$ is the same as the infimum of γ_{rel} under the condition $P_{\mathrm{mul}} \geq 0$ that concludes the proof. ∎

The proof of Theorem 4.1 shows that $\gamma_{\mathrm{opt}} = 1/\delta_{\max}^{(\mathrm{mul})} = 1/\delta_{\max}^{(\mathrm{rel})}$ has an alternative expression:

$$\gamma_{\mathrm{opt}} = \frac{1}{\sqrt{1-\overline{\sigma}^2(\Gamma_p^{-1}\Omega\Gamma_z^{-1})}} \quad (37)$$

where $P = \Gamma_p\Gamma_p'$ and $Z = \Gamma_z\Gamma_z'$ are Cholesky factorizations. The above resembles to the formula in (19), and both depend on only unstable zeros and poles of $P(z)$.

With the doubt cleared on relative form description for the quantization error, the feedback system consisting of the logarithmic quantization at both the input and output of the plant model can be represented by the feedback system in Fig. 1. The only difference from Section 2 is that the uncertainties are now nonlinear and time-varying instead of linear and time-invariant dynamic uncertainties. In order for our stabilization results to apply, we make an assumption on the plant model:

$$P(z) = F_1(z)P_0(z)F_2(z) = \mathrm{N_i}(z)\mathrm{M_i}(z)^{-1} \quad (38)$$

where $F_1(z), F_2(z) \in \mathcal{S}_{\mathrm{o}}(\mathcal{H}_\infty)$ are filters employed at the input and output of the original plant $P_0(z)$, and $P(z)$ is allpass. In addition we are interested in searching for $\delta_{\max}^{(\mathrm{u,y})}$, the supremum of $\delta_u = \delta_y$ below which stabilizing feedback controller exists.

Theorem 4.2: Suppose that the composite plant model $P(z)$ given in (38) is allpass for some filters $F_1(z), F_2(z) \in \mathcal{S}_{\mathrm{o}}(\mathcal{H}_\infty)$. Let $V(z), U(z) \in \mathcal{H}_\infty$ satisfy Bezout identity (15). There exists a stabilizing controller for the feedback system in Fig. 1 with nonlinear and time-varying uncertainties $|\Delta_r(t)| \leq \delta_u$ and $|\Delta_m(t)| \leq \delta_y$ satisfying $\delta_u = \delta_y$, if $\delta_u = \delta_y < \delta_{\max}^{(\mathrm{u,y})}$ that is bounded as

$$\delta_{\max}^{(\mathrm{u,y})} \geq \sqrt{1 - \frac{1}{2}\left\| \begin{bmatrix} M_{\mathrm{i}}^{\sim} & N_{\mathrm{i}}^{\sim} \end{bmatrix} \right\|_H^2}$$

$$\delta_{\max}^{(\mathrm{u,y})} \leq \sqrt{2 - \left\| \begin{bmatrix} M_{\mathrm{i}}^{\sim} & N_{\mathrm{i}}^{\sim} \end{bmatrix} \right\|_H^2}$$

Proof: We note that the multiplicative and relative uncertainties induced by the quantization errors are nonlinear and time-varying. Hence the results in the previous section do not apply directly, because now the weightings need to be restricted to nonzero constants. On the other hand suppose that $K(z)$ stabilizes $P(z)$ in absence of the logarithmic quantization. Then the inequality

$$\rho_W = \inf_{W_1, W_2 \in \mathbb{R}} \|T_W\|_\infty < \delta^{-1} \tag{39}$$

with $\delta = \delta_u = \delta_y$ ensures stability of the quantized feedback system in Fig. 1. The above can be shown with the same proof for Lemma 2.1. We first prove the upper bound. By the assumption on the allpass plant model, $W_1, W_2 \in \mathbb{R}$ can be scaled to satisfy the normalization condition $W_1^2 + W_2^2 = 1$ implying that $P_W(z) = [W_1 N_{\rm i}(z)][W_2 D_{\rm i}(z)]^{-1}$ is a normalized coprime factorization. In addition

$$\begin{aligned} \varphi(\omega) &:= |W_2|^{-2} + |W_1|^{-2}|K(e^{j\omega})|^2 \\ &= \big(1 + |K(e^{j\omega})|\big)^2 + \left(\left|\frac{W_1}{W_2}\right| - \left|\frac{W_2 K(e^{j\omega})}{W_1}\right|\right)^2 \\ &\geq \big[1 + |K(e^{j\omega})|\big]^2 \quad \forall \omega \in \mathbb{R} \end{aligned}$$

Using again the hypotheses on $P(z)$ and normalization condition for W_1 and W_2, we have

$$\begin{aligned} \|T_W\|_\infty &= \sup_{\omega \in \mathbb{R}} \sqrt{|[1 - K(e^{j\omega})P(e^{j\omega})]|^{-2}\, \varphi(\omega)} \\ &\geq \sup_{\omega \in \mathbb{R}} \sqrt{[\frac{1 + |K(e^{j\omega})]^2|}{|1 - K(e^{j\omega})P(e^{j\omega})|^2}} \\ &\geq \left\|(1 - KP)^{-1} \begin{bmatrix} 1 & -K \end{bmatrix}\right\|_\infty \\ &= \frac{1}{\sqrt{2}} \left\| \begin{bmatrix} 1 \\ P \end{bmatrix} (1 - KP)^{-1} \begin{bmatrix} 1 & -K \end{bmatrix} \right\|_\infty \\ &\geq \frac{1}{\sqrt{2 - \left\|\begin{bmatrix} M_{\rm i}^\sim & N_{\rm i}^\sim \end{bmatrix}\right\|_H^2}} \end{aligned}$$

over all stabilizing controllers $K(z)$. Because the lower bound for $\|T_W\|_\infty$ is independent of weighting functions W_1 and W_2, and is true for all stabilizing controllers $K(z)$, we have

$$\delta_{\max}^{(\rm u,y)} \leq \sqrt{2 - \left\|\begin{bmatrix} M_{\rm i}^\sim & N_{\rm i}^\sim \end{bmatrix}\right\|_H^2} \tag{40}$$

On the other hand by taking $W_1 = W_2 = 1/\sqrt{2}$ shows that

$$\delta_{\max}^{(\rm u,y)} \geq \sqrt{1 - \frac{1}{2}\left\|\begin{bmatrix} M_{\rm i}^\sim & N_{\rm i}^\sim \end{bmatrix}\right\|_H^2} \tag{41}$$

in light of the proof of Corollary 3.2. Hence the upper and lower bounds hold. ■

Several comments are in order. First, by the results in the previous section, $\left\|\begin{bmatrix} M_{\rm i}^\sim & N_{\rm i}^\sim \end{bmatrix}\right\|_H^2 = 1 + \overline{\sigma}(Z)$ is true with Z the same as in Thereom 3.3. Second the coarsest quantization density at both the input and output is given by $2\left[\log\left(\frac{1-\delta_{\max}^{(\rm u,y)}}{1+\delta_{\max}^{(\rm u,y)}}\right)\right]^{-1}$, although at present we have only an estimate for the upper and lower bounds of $\delta_{\max}^{(\rm u,y)}$. However it remains unknow if the computation of $\delta_{\max}^{(\rm u,y)}$ is a two-disk problem as in the previous section. Finally we comment that we have assumed that in the setup of Fig. 1, feedback controller has no access to the quantized control input $u(t)$ due to the consideration of transmission error, coding error, or modeling error. This is a worst-case formulation consistent with [3], [4].

5. Conclusion

In this paper robust stabilization is investigated for dynamic systems involving $\mathcal{H}_\infty$-norm bounded multiplicative and relative uncertainties. For the linear and time-invariant dynamic uncertainties, computation of the stability margin is an equivalent two-disk problem. Upper and lower bounds are derived for the proposed stability margin that differ by a factor of $\sqrt{2}$. On the other hand when logarithmic quantization is employed at the plant input and output, it induces similar multiplicative and relative uncertainties which are nonlinear and time-varying. Similar lower and upper bounds for the associated stability margin are derived to aid computation of the coarsest quantization density. In addition the stability margin induced by logarithmic quantization at only the plant output is obtained via Nevanlinna-Pick interpolation, and shows the equivalence of the multiplicative and relative uncertainties. Our results shed some new lights to networked stabilization in the case of output feedback complementing the existing work reported in [3], [4].

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