ROBUST STABILIZATION OF PERIODIC AND MULTIRATE SYSTEMS WITH GAP AND ν-GAP UNCERTAINTY

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Abstract: In this paper we present a state space solution to the robust stabilization problem of general discrete-time periodic and multirate systems where the uncertainty is described in terms of the gap and $\nu-$gap metrics. This robust stability problem is converted to a constrained $\mathcal{H}_\infty$ optimal control problem by using the lifting technique. Then the optimal robust stability margin is explicitly computed and a method is provided to get the controllers satisfying the optimal robust stabilization margin. The solution amounts to solving two discrete-time algebraic Riccati equations and an extended Parrot problem.

Keywords: multirate systems, periodic systems, robust stabilization, gap metric, $\nu-$gap metric

1. INTRODUCTION

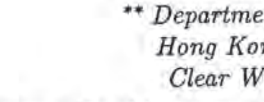

Fig. 1. A general multirate system.

A general multivariable discrete-time multirate system is depicted in Figure 1. Here the signals $u_1, \dots, u_p$ and $y_1, \dots, y_q$ are discrete-time signals with different sampling rates. Such a multirate system can result from sampling an analog system using multirate samplers and holds or can appear as it is in some special applications. In our study, we assume that this system is linear and causal, and satisfies certain periodic property. Because of this, it can be converted to an equivalent LTI system using the so-called lifting or blocking technique (Qiu and Chen, 1994; Ravi, *et al.*, 1990; Voulgaris, *et al.*, 1994). Hence the analysis and design techniques for LTI systems can be applied to such a multirate system. However, it has been known that the lifting results in a peculiar constraint on the equivalent LTI system, due to the causality of the original system. When solving a design problem, such as the robust stabilization problem considered in this paper, extra effort is needed to make sure that the designed multirate system is causal.

Our formulation of a multirate system includes a single rate periodic system as a special case. Periodic systems occur naturally in many applications and can be intentionally used to achieve something that time invariant system can not. See a recent survey paper (Bittanti and Colaneri, 2000) and the references therein. In this paper, periodic and multirate systems are treated in a unified framework by using the lifting technique. A related study was presented in (Iglesias, 2000) in which a method to design a strictly proper controller for the discrete-

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time, normalized left-coprime factorization robust stabilization was given.

The first issue in robust control is the description of the uncertainty. The most natural way to describe system uncertainty is by using a metric in the set of all systems under consideration and an uncertain system is then simply a ball defined by this metric centered at a nominal system with certain radius. There are several metrics in the literature for this very purpose: gap metric (Georgiou and Khargonekar, 1987), pointwise gap metric (Qiu and Davison, 1992), ν-gap metric (Vinnicombe, 1993). In this paper, both the gap and ν-gap metrics are studied for multirate and periodic systems. We will see that the gap metric has an easy generization. However, it is not easy to extend the ν-gap metric to multirate and periodic systems following the method in (Vinnicombe, 1993). Here we generalize the treatment in (Wan and Huang, 2000) to define the ν-gap metric for multirate systems.

2. PRELIMINARY

Consider the multirate system P_{mr} showing in Figure 1. Assume that P_{mr} is linear and causal. Also assume that the signals $u_i, i = 1, 2, \cdots, p$, and $y_j, j = 1, 2, \cdots, q$, are synchronized at time zero. Let the sampling interval of u_i be $m_i h$ and that of y_j be $n_j h$, where h is a real number giving a time unit and m_i, n_j are integers. The overall input and output are denoted by

$$u = \begin{bmatrix} u_1 \cdots u_p \end{bmatrix}^T, y = \begin{bmatrix} y_1 \cdots y_q \end{bmatrix}^T,$$

respectively. Here u and y are vectors of signals with different sampling rates. Let l be a common multiple of m_i and n_j, $i = 1, \cdots, p$, $j = 1, \cdots, q$. Let $\bar{m}_i = l/m_i$ and $\bar{n}_j = l/n_j$. Denote the sets $\{m_i\}$ and $\{n_j\}$ by M and N respectively and the sets $\{\bar{m}_i\}$ and $\{\bar{n}_j\}$ by $\bar{M}$ and $\bar{N}$ respectively. Let U be the unit delay operator and let

$$U_{\bar{M}} = \text{diag}\{U^{\bar{m}_1}, \dots, U^{\bar{m}_p}\},$$
$$U_{\bar{N}} = \text{diag}\{U^{\bar{n}_1}, \dots, U^{\bar{n}_q}\}.$$

With the above notion, we finally assume that the system P_{mr} satisfies

$$P_{mr} U_{\bar{M}} = U_{\bar{N}} P_{mr}.$$

This property will be called the $(\bar{M}, \bar{N})$-shift-invariance. This multirate system is more general than what one can get from sampling an LTI analog system using a multirate sampling scheme. As an extreme case, if all m_i and n_j are equal to 1 and $l > 1$, then the multirate system is actually an l-periodic single-rate discrete-time system.

Define a lifting operator

$$L_m : \{\cdots | x(0), x(1), \cdots\} \mapsto$$
$$\left\{ \cdots \left| \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(m-1) \end{bmatrix}, \begin{bmatrix} x(m) \\ x(m+1) \\ \vdots \\ x(2m-1) \end{bmatrix}, \cdots \right. \right\}.$$

Its inverse L_m^{-1} is called a delifting operator. Let

$$L_{\bar{M}} = \text{diag}\{L_{\bar{m}_1}, \dots, L_{\bar{m}_p}\}$$
$$L_{\bar{N}} = \text{diag}\{L_{\bar{n}_1}, \dots, L_{\bar{n}_q}\}.$$

Then the lifted system $P = L_{\bar{N}} P_{mr} L_{\bar{M}}^{-1}$ is an LTI system in the sense that $PU = UP$ since

$$PU = L_{\bar{N}} P_{mr} L_{\bar{M}}^{-1} U = L_{\bar{N}} P_{mr} U_{\bar{M}} L_{\bar{M}}^{-1}$$
$$= L_{\bar{N}} U_{\bar{N}} P_{mr} L_{\bar{M}}^{-1} = U L_{\bar{N}} P_{mr} L_{\bar{M}}^{-1} = UP.$$

Hence it has transfer function $\hat{P}$ in λ-transform, i.e. replacing z in z-transform by $\frac{1}{\lambda}$. However, P is not an arbitrary LTI system, instead it is subject to a constraint that is resulted from the causality of P_{mr}. This constraint is best described using nest operators.

Let $\mathcal{X}$ be a finite dimensional vector space. A nest in $\mathcal{X}$, denoted $\{\mathcal{X}_i\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the non-increasing ordering:

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{n-1} \supseteq \mathcal{X}_n = \{0\}.$$

Denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the set of linear operators $\mathcal{X} \to \mathcal{Y}$ and abbreviate it as $\mathcal{L}(\mathcal{X})$ if $\mathcal{X} = \mathcal{Y}$. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are equipped respectively with nest $\{\mathcal{X}_i\}$ and $\{\mathcal{Y}_i\}$ which have the same number of subspaces, say, $n+1$ as above. A linear map $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is said to be a nest operator if

$$T\mathcal{X}_i \subseteq \mathcal{Y}_i, \ i = 0, 1, \cdots, n.$$

The set of all nest operators (with given nests) is denoted $\mathcal{N}(\{\mathcal{X}_i\}, \{\mathcal{Y}_i\})$ and abbreviated $\mathcal{N}(\{\mathcal{X}_i\})$ if $\{\mathcal{X}_i\} = \{\mathcal{Y}_i\}$.

Let us see how to characterize the causality constraint on P by using nest operators. Write $\underline{u} = L_{\bar{M}} u$, $\underline{y} = L_{\bar{N}} y$. Then

$$\underline{u}(0) = [u_1(0) \cdots u_1((\bar{m}_1 - 1)m_1 h) \cdots u_p(0) \cdots u_p((\bar{m}_p - 1)m_p h)]^T,$$
$$\underline{y}(0) = [y_1(0) \cdots y_1((\bar{n}_1 - 1)n_1 h) \cdots y_q(0) \cdots y_q((\bar{n}_q - 1)n_q h)]^T.$$

Define for $k = 0, 1, \dots, l$,

$$\mathcal{U}_k = \{\underline{u}(0) : u_i(rm_i h) = 0 \text{ if } rm_i h < kh\}$$
$$\mathcal{Y}_k = \{\underline{y}(0) : y_j(rn_j h) = 0 \text{ if } rn_j h < kh\}.$$

Then the lifted plant P will have

$$\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}).$$

Now consider a linear causal $(\bar{M},\bar{N})$-shift-invariant multirate system P_{mr}. The graph of P_{mr} is defined as $\mathcal{G}(P_{mr}) =$

$$\left\{ \begin{bmatrix} u \\ P_{mr}u \end{bmatrix} ; u \in \ell_+^2, P_{mr}u \in \ell_+^2 \text{ converges} \right\}.$$

Here ℓ_+^2 means possibly the direct sum of a collection of ℓ_+^2 signal spaces with different sampling rates. Clearly $\mathcal{G}(P_{mr})$ is a subspace of $\ell_+^2 \oplus \ell_+^2$. A subspace $\mathcal{G}$ of $\ell_+^2 \oplus \ell_+^2$ is called $(\bar{M},\bar{N})$-shift-invariant if

$$\begin{bmatrix} U_{\bar{M}} & 0 \\ 0 & U_{\bar{N}} \end{bmatrix} \mathcal{G} \subset \mathcal{G}.$$

It is easy to see that the graph of P_{mr} is $(\bar{M},\bar{N})$-shift-invariant. The gap between P_{mr} and $\tilde{P}_{mr}$ is defined by

$$\delta(P_{mr},\tilde{P}_{mr}) = \|\Pi_{\mathcal{G}(P_{mr})} - \Pi_{\mathcal{G}(\tilde{P}_{mr})}\|,$$

where $\Pi_{\mathcal{G}(P_{mr})}$ and $\Pi_{\mathcal{G}(\tilde{P}_{mr})}$ are the orthogonal projections from $\ell_+^2 \oplus \ell_+^2$ onto $\mathcal{G}(P_{mr})$ and $\mathcal{G}(\tilde{P}_{mr})$ respectively. Since the lifting operators $L_{\bar{M}}$ and $L_{\bar{N}}$ are unitary operators, it is clear that

$$\delta(P_{mr},\tilde{P}_{mr}) = \delta(P,\tilde{P}),$$

where $\tilde{P} = L_{\bar{N}}\tilde{P}_{mr}L_{\bar{M}}^{-1}$.

A subgraph of a linear causal $(\bar{M},\bar{N})$-shift-invariant multirate system is defined as an $(\bar{M},\bar{N})$-shift-invariant subspace of its graph. We denote the set of all subgraphs as $\mathcal{S}_\mathcal{G}(P_{mr})$. To define the ν-gap between two multirate systems, we need the notation of the index of a subgraph $\mathcal{V}$ with respective to $\mathcal{G}(P_{mr})$, defined as (Wan and Huang, 2000)

$$\mathrm{ind}(\mathcal{V}) := \dim\left(\mathcal{G}(P_{mr}) \ominus \mathcal{V}\right).$$

The ν-gap between two plants P_{mr} and $\tilde{P}_{mr}$ is then defined by

$$\delta_\nu(P_{mr},\tilde{P}_{mr}) = \inf_{\substack{\mathcal{V} \in \mathcal{S}_\mathcal{G}(P_{mr}) \\ \tilde{\mathcal{V}} \in \mathcal{S}_\mathcal{G}(\tilde{P}_{mr}) \\ \mathrm{ind}(\mathcal{V}) = \mathrm{ind}(\tilde{\mathcal{V}})}} \|\Pi_\mathcal{V} - \Pi_{\tilde{\mathcal{V}}}\|$$

where $\Pi_\mathcal{V}$ and $\Pi_{\tilde{\mathcal{V}}}$ are the orthogonal projections from $\ell_+^2 \oplus \ell_+^2$ onto $\mathcal{G}_s(P_{mr})$ and $\mathcal{G}_s(\tilde{P}_{mr})$ respectively. The ν-gap between two multirate systems can be computed from that between their equivalent LTI systems, where many efficient methods are available (Vinnicombe, 1993; Wan and Huang, 2000). Note that $\mathcal{V}$ is a subgraph of a multirate system P_{mr} if and only if $\begin{bmatrix} L_{\bar{M}} & 0 \\ 0 & L_{\bar{N}} \end{bmatrix} \mathcal{V}$ is a subgraph of $\mathcal{G}(P)$, where $P = L_{\bar{N}}P_{mr}L_{\bar{M}}^{-1}$. Then we have the following result:

Lemma 1. Let P_{mr} and $\tilde{P}_{mr}$ be two linear causal $(\bar{M},\bar{N})$-shift-invariant multirate systems and their equivalent LTI systems are P and $\tilde{P}$ respectively, that is

$$P = L_{\bar{N}}P_{mr}L_{\bar{M}}^{-1}, \quad \tilde{P} = L_{\bar{N}}\tilde{P}_{mr}L_{\bar{M}}^{-1}.$$

Then we have $\delta_\nu(P_{mr},\tilde{P}_{mr}) = \delta_\nu(P,\tilde{P})$.

The gap metric ball and ν-gap metric ball centered at P_{mr} with radius r are respectively given by

$$\mathcal{B}(P_{mr},r) = \{\tilde{P}_{mr} : \delta(P_{mr},\tilde{P}_{mr}) < r\}$$
$$\mathcal{B}_\nu(P_{mr},r) = \{\tilde{P}_{mr} : \delta_\nu(P_{mr},\tilde{P}_{mr}) < r\}.$$

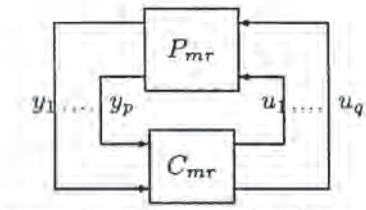

Fig. 2. A general multirate feedback control system.

Now consider the feedback system shown in Figure 2. Here we assume that P_{mr} is linear causal $(\bar{M},\bar{N})$-shift-invariant and C_{mr} is linear causal $(\bar{N},\bar{M})$-shift-invariant. An interesting problem is then to design C_{mr} for given P_{mr} so that the feedback system is stable and has optimal robust stability in the sense that it can tolerate the maximum amount of gap or ν-gap metric uncertainty in both P_{mr} and C_{mr}. Let $P = L_{\bar{N}}P_{mr}L_{\bar{M}}^{-1}$ and $C = L_{\bar{M}}C_{mr}L_{\bar{N}}^{-1}$. Then P and C are LTI and hence have transfer functions $\hat{P}$ and $\hat{C}$ respectively. For fixed P_{mr} and C_{mr}, the stability robustness of the feedback system is given by the following lemma (Vinnicombe, 1999; Qiu and Davison, 1992)

Lemma 2. Given a nominal plant P_{mr} and a stabilizing controller C_{mr}, assume that r_1 and r_2 are positive real numbers, then the feedback system with plant $\tilde{P}_{mr}$ and controller $\tilde{C}_{mr}$ is stable for all $\tilde{P}_{mr} \in \mathcal{B}(P_{mr},r_1)$ (or $\tilde{P}_{mr} \in \mathcal{B}_\nu(P_{mr},r_1)$) and all $\tilde{C}_{mr} \in \mathcal{B}(C_{mr},r_2)$ (or $\tilde{C}_{mr} \in \mathcal{B}_\nu(C_{mr},r_2)$)if and only if

$$\arcsin r_1 + \arcsin r_2 + \arccos b_{P,C} \leq \frac{1}{2}\pi,$$

where

$$b_{P,C} = \| \begin{bmatrix} I \\ \hat{P} \end{bmatrix} (I - \hat{C}\hat{P})^{-1} \begin{bmatrix} I & -\hat{C} \end{bmatrix} \|_\infty^{-1}.$$

The quantity $b_{P,C}$ is defined as the robust stability margin. For a given P, the optimal robust stabilization problem is to maximize $b_{P,C}$ by choosing a stabilizing controller C.

Define

$$\hat{G} = \begin{bmatrix} \begin{bmatrix} I & 0 \\ \hat{P} & 0 \end{bmatrix} & \begin{bmatrix} I \\ \hat{P} \end{bmatrix} \\ \begin{bmatrix} \hat{P} & I \end{bmatrix} & \hat{P} \end{bmatrix}. \quad (1)$$

Then it is easy to see that

$$\begin{bmatrix} I \\ \hat{P} \end{bmatrix} (I - \hat{C}\hat{P})^{-1} \begin{bmatrix} I & -\hat{C} \end{bmatrix} = \mathcal{F}(\hat{G},\hat{C})$$
$$:= \hat{G}_{11} + \hat{G}_{12}\hat{C}(I - \hat{G}_{22}\hat{C})^{-1}\hat{G}_{21} \quad (2)$$

Hence our optimal robust stabilization problem becomes a special discrete-time $\mathcal{H}_\infty$ optimal control problem. Since the causality of P_{mr} and C_{mr} is equivalent to

$$\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\}) \tag{3}$$

and

$$\hat{C}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\}), \tag{4}$$

we should consider the causality constraint (4) and possibly utilize the causality constraint (3) in solving the special discrete-time $\mathcal{H}_\infty$ optimal control problem. The continuous-time counterpart of such an $\mathcal{H}_\infty$ optimal control problem (without causality constraint) has been explicitly solved in (Georgiou and Smith, 1990; Glover and McFarlane, 1989).

3. THE MAIN RESULTS

Suppose that P has a stabilizable and detectable state space realization $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Then G in (1) has a state space realization

$$\hat{G} = \left[\begin{array}{c|cc} A & \begin{bmatrix} B & 0 \end{bmatrix} & B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} I & 0 \\ D & 0 \end{bmatrix} & \begin{bmatrix} I \\ D \end{bmatrix} \\ C & \begin{bmatrix} D & I \end{bmatrix} & D \end{array}\right]. \tag{5}$$

Let X and Y be the stabilizing solutions of Riccati equations

$$\begin{aligned} X = {} & A'XA + C'C - (A'XB + C'D) \\ & (B'XB + I + D'D)^{-1}(B'XA + D'C), \quad (6) \\ Y = {} & AYA' + BB' - (AYC' + BD') \\ & (CYC' + I + DD')^{-1}(CYA' + DB'). \quad (7) \end{aligned}$$

Denote

$$\begin{aligned} F &= -(B'XB + I + D'D)^{-1}(B'XA + D'C), \quad (8) \\ L &= -(AYC' + BD')(CYC' + I + DD')^{-1}. \quad (9) \end{aligned}$$

Here $(A + BF)$ and $(A + LC)$ are stable since X and Y are stablilizing solutions. Choose constant matrices $R \in \mathcal{N}(\{\mathcal{U}_r\}), S \in \mathcal{N}(\{\mathcal{Y}_r\})$ satisfying (Chen and Qiu,1994),

$$\begin{aligned} R'R &= B'XB + I + D'D, \quad (10) \\ SS' &= CYC' + I + DD'. \quad (11) \end{aligned}$$

Theorem 1. Given a nominal plant $\hat{P} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{N}(\{\mathcal{U}_r\}, \{\mathcal{Y}_r\})$, let X and Y be the stabilizing solutions of Riccati equations (6) and (7), and let F, L, R, S be defined as in (8)-(11). Then the optimal robust stabilization margin $\sup_C b_{P,C}$ is equal to

$$\left\{(\max_r \| \begin{bmatrix} I - \Pi_{\mathcal{U}_r} & 0 \\ 0 & I \end{bmatrix} H_T|_{(\mathcal{Y}_r \oplus \mathcal{R}^n)})\|^2 + 1\right\}^{-\frac{1}{2}},$$

where $H_T =$

$$\begin{bmatrix} R'^{-1}D'S & R'^{-1}B'(X + XYX)^{\frac{1}{2}} \\ (Y^{-1} + X)^{-\frac{1}{2}}[(C + DF)'S - (A + BF)'XLS] & (Y^{-1} + X)^{-\frac{1}{2}}(A + BF)'(X + XYX)^{\frac{1}{2}} \end{bmatrix}. \tag{12}$$

Theorem 1 follows easily from Propositions 1, 2 and Lemma 3 which are to be established in the following.

Let

$$\hat{T} = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right] = \left[\begin{array}{c|c} A + BF & BR^{-1} \\ \hline S'(C + DF) - A'L'X(A + BF) & S'DR^{-1} \end{array}\right] \tag{13}$$

Denote $\hat{T}^{\sim}(\lambda) = \hat{T}(\frac{1}{\lambda})'$ as the adjoint of $\hat{T}(\lambda)$.

Proposition 1.

$$\inf_{\hat{C}} \|\mathcal{F}(\hat{G}, \hat{C})\|_\infty^2 = \inf_{\hat{Q} \in \mathcal{RH}_\infty, \hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})} \|\hat{T}^{\sim} + \hat{Q}\|_\infty^2 + 1. \tag{14}$$

Remark: This result is also given in (Georgiou and Smith, 1990) through an operator approach. The minimization problem in the right hand side of (14), if the causality constraint is removed, is a standard Nehari problem, which has an explicit solution (Al-Husari, *et. al.*, 1997).

Proof: Let X and Y be the stabilizing solutions of Riccati equations (6)-(7), then all stabilizing controllers of P satisfying the causality constraint are characterized by an LFT (Ravi, *et. al.*, 1990):

$$\hat{C} = \mathcal{F}(\hat{J}, \hat{Q}), \tag{15}$$

where $\hat{Q} \in \mathcal{RH}_\infty, \hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\}, \{\mathcal{U}_r\})$,

$$\hat{J} = \left[\begin{array}{c|cc} A + BF + LC + LDF & -L & B + LD \\ \hline F & \begin{bmatrix} 0 & I \\ I & -D \end{bmatrix} \\ -C - DF & & \end{array}\right]. \tag{16}$$

Under this characterization, the closed loop transfer function is

$$\mathcal{F}(\hat{G}, \hat{C}) = \mathcal{F}[\hat{G}, \mathcal{F}(\hat{J}, \hat{Q})] = \hat{T}_{11} + \hat{T}_{12}\hat{Q}\hat{T}_{21},$$

where

$$\hat{T}_{11} = \left[\begin{array}{cc|c} A+BF & BF & \begin{bmatrix} B & 0 \end{bmatrix} \\ 0 & A+LC & -\begin{bmatrix} B+LD & L \end{bmatrix} \\ \hline \begin{bmatrix} F \\ C+DF \end{bmatrix} & \begin{bmatrix} F \\ DF \end{bmatrix} & \begin{bmatrix} I & 0 \\ D & 0 \end{bmatrix} \end{array} \right],$$

$$\hat{T}_{12} = \left[\begin{array}{c|c} A+BF & B \\ \hline \begin{bmatrix} F \\ C+DF \end{bmatrix} & \begin{bmatrix} I \\ D \end{bmatrix} \end{array} \right],$$

$$\hat{T}_{21} = \left[\begin{array}{c|c} A+LC & \begin{bmatrix} B+LD & L \end{bmatrix} \\ \hline C & \begin{bmatrix} D & I \end{bmatrix} \end{array} \right].$$

It follows (Chen and Francis, 1995) that

$$\hat{T}_{12}^{\sim}\hat{T}_{12} = B'XB + I + D'D, \tag{17}$$

$$\hat{T}_{21}\hat{T}_{21}^{\sim} = CYC' + I + DD'. \tag{18}$$

Now carry out matrix factorizations in (10) and (11) to get $R \in \mathcal{N}(\{\mathcal{U}_r\})$ and $S \in \mathcal{N}(\{\mathcal{Y}_r\})$. Define

$$U = \begin{bmatrix} R'^{-1}\hat{T}_{12}^{\sim} \\ I - \hat{T}_{12}R^{-1}R'^{-1}\hat{T}_{12}^{\sim} \end{bmatrix},$$

$$V = \begin{bmatrix} \hat{T}_{21}^{\sim}S'^{-1} & I - \hat{T}_{21}^{\sim}S'^{-1}S^{-1}\hat{T}_{21} \end{bmatrix}.$$

Then (17) and (18) imply $U^{\sim}U = I$ and $VV^{\sim} = I$. Hence we have

$$\begin{aligned} \|\mathcal{F}(\hat{G},\hat{C})\|_\infty &= \|\hat{T}_{11} + \hat{T}_{12}\hat{Q}\hat{T}_{21}\|_\infty \\ &= \|\hat{U}(\hat{T}_{11} + \hat{T}_{12}\hat{Q}\hat{T}_{21})\hat{V}\|_\infty \\ &= \left\| \begin{bmatrix} \hat{T}^{\sim} + R\hat{Q}S & -I \\ 0 & 0 \end{bmatrix} \right\|_\infty \\ &= (\|\hat{T}^{\sim} + \hat{Q}_R\|_\infty^2 + 1)^{\frac{1}{2}}, \end{aligned}$$

where $\hat{Q}_R = R\hat{Q}S$.
Notice that $\hat{Q}_R(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$ if and only if $\hat{Q}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$. □

The constrained optimal distance problem in the right hand side of (14) is solved in (Georgiou and Khargonekar, 1987). The controllability Grammian Ψ and the observability Grammian Φ of system T are required. Ψ and Φ are the solutions of the following Lyapunov equations

$$\Psi = \tilde{A}\Psi\tilde{A}' + \tilde{B}\tilde{B}', \tag{19}$$

$$\Phi = \tilde{A}'\Phi\tilde{A} + \tilde{C}'\tilde{C}. \tag{20}$$

Lemma 3.

$$\begin{aligned} &\inf_{\hat{Q}_R \in \mathcal{RH}_\infty, \hat{Q}_R(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})} \|\hat{T}^{\sim} + \hat{Q}_R\|_\infty \\ &= \inf_{\hat{Q}_R(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})} \left\| \begin{bmatrix} \tilde{D}' + \hat{Q}_R(0) & \tilde{B}'\Phi^{\frac{1}{2}} \\ \Psi^{\frac{1}{2}}\tilde{C}' & \Psi^{\frac{1}{2}}\tilde{A}'\Phi^{\frac{1}{2}} \end{bmatrix} \right\| \\ &= \max_r \left\| \begin{bmatrix} I - \Pi_{\mathcal{U}_r} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{D}' & \tilde{B}'\Phi^{\frac{1}{2}} \\ \Psi^{\frac{1}{2}}\tilde{C}' & \Psi^{\frac{1}{2}}\tilde{A}'\Phi^{\frac{1}{2}} \end{bmatrix} |_{(\mathcal{Y}_r \oplus \mathcal{R}^n)} \right\|. \end{aligned}$$

We will see that Ψ and Φ are simple functions of X and Y.

Proposition 2.

$$\Psi = Y(I + XY)^{-1}, \tag{21}$$

$$\Phi = (I + XY)X. \tag{22}$$

Proof: The proof is parallel to the continuous-time case (Georgiou and McFarlane, 1989) and hence omitted here.

To characterize all controllers satisfying the robust stabilization margin given by Theorem 1, one way is to characterize all Q_R by solving the optimal distance problem in Lemma 3. This can be done by using the method proposed in (Georgiou and Khargonekar, 1987). Then Q is simply $R^{-1}Q_RS^{-1}$. We can also follow the coprime factorization approach proposed in (Georgiou and McFarlane, 1989).

Theorem 2. Given a nominal plant $\hat{P} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ with $D \in \mathcal{N}(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$, let X and Y be the stabilizing solutions of Riccati equations (6) and (7), and let F, L, R, S be defined as in (8)-(11). Then the optimal robust stabilization margin is

$$\sup_C b_{P,C}^2 =$$

$$1 - \max_r \left\| \begin{bmatrix} I - \Pi_{\mathcal{U}_r} & 0 & 0 \\ 0 & I - \Pi_{\mathcal{Y}_r} & 0 \\ 0 & 0 & I \end{bmatrix} H_F|_{(\mathcal{Y}_r \oplus \mathcal{R}^n)} \right\|^2,$$

where

$$H_F = \begin{bmatrix} -D'S'^{-1} & -(B' + D'L')(X^{-1} + Y)^{-\frac{1}{2}} \\ S'^{-1} & L'(X^{-1} + Y)^{-\frac{1}{2}} \\ Y^{\frac{1}{2}}C'S'^{-1} & Y^{\frac{1}{2}}(A + LC)'(X^{-1} + Y)^{-\frac{1}{2}} \end{bmatrix}. \tag{23}$$

This theorem is straitforward after the Lemma 4, which is obtained by slightly modifying the result in (Georgiou and McFarlane, 1989), is introduced.

For a nominal plant P with $\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$, there are normalized left coprime factorizations $P = \tilde{M}^{-1}\tilde{N}$ with $\hat{\tilde{N}}(0) \in \mathcal{N}(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$ and $\hat{\tilde{M}}(0) \in \mathcal{N}(\{\mathcal{Y}_r\})$. One particular realization of such factorization is

$$\begin{bmatrix} \hat{\tilde{N}} & \hat{\tilde{M}} \end{bmatrix} = \left[\begin{array}{c|cc} (A+LC) & B+LD & L \\ \hline S^{-1}C & S^{-1}D & S^{-1} \end{array} \right].$$

Lemma 4. Let $P = \tilde{M}^{-1}\tilde{N}$ be a normalized left coprime factorization with $\hat{\tilde{M}}(0) \in \mathcal{N}(\{\mathcal{Y}_r\})$ and $\hat{\tilde{N}}(0) \in \mathcal{N}(\{\mathcal{U}_r\},\{\mathcal{Y}_r\})$. The optimal robust stabilization margin is

$$\sup_C b_{P,C} =$$

$$\left\{ 1 - \inf_{\substack{\hat{U},\hat{V} \in \mathcal{RH}_\infty \\ \hat{U}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\}) \\ \hat{V}(0) \in \mathcal{N}(\{\mathcal{Y}_r\})}} \left\| \begin{bmatrix} -\hat{\tilde{N}}^{\sim} \\ \hat{\tilde{M}}^{\sim} \end{bmatrix} + \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right\|^2 \right\}^{\frac{1}{2}} . \quad (24)$$

Let $\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix}$ be a solution of the minimization problem in (24), then an optimal robust stabilizing controller is given by $C = UV^{-1}$ with $\hat{C}(0) \in \mathcal{N}(\{\mathcal{Y}_r\},\{\mathcal{U}_r\})$.

The optimization problem in Lemma 4 can be solved in the same way as that in Lemma 3. Note that

$$\begin{bmatrix} -\hat{\tilde{N}}^{\sim} \\ \hat{\tilde{M}}^{\sim} \end{bmatrix}^{\sim} = \left[\begin{array}{c|cc} A+LC & -B-LD & L \\ \hline S^{-1} & -S^{-1}D & S^{-1} \end{array} \right].$$

Parallel to the continuous-time case, this system's controllability Grammian P_F and the observability Grammian Q_F are

$$P_F = Y, \;\; Q_F = X(I+YX)^{-1},$$

where X, Y are the solutions of the Riccati equations (6-7). Now Theorem 2 is clear.

We can also get another formula for $\sup_C b_{P,C}$ symmetric to (23) by considering the normalized right coprime factorization of P. The optimization problem in Theorem 2 has a bigger size than that in Theorem 1. These two theorems actually give the same optimal robust stabilization margin.

In summary, the robust stabilization problem for a general multirate system can be solved as follows
1) Obtain the solutions X, Y of the two Riccati equations (6-7) to form the matrices in (12) or (23).
2) Compute the optimal robust stabilization margin by Theorems 1 or 2.
3) Use the algorithm in (Georgiou and Khargonekar, 1987) to get a solution of the minimization problem of (24), then the optimal robust stabilizing controller is $C = UV^{-1}$.

4. CONCLUSION

In this paper, we present a state space solution to the robust stabilization problem of discrete-time periodic and multirate systems. First, we show how the robust stabilization problem of multirate systems with gap and ν-gap metric uncertainty can be converted to a constrained $\mathcal{H}_\infty$ optimal control problem. The optimal robust stabilization margin is explicitly computed and a method is given to design an optimal controll. The computational burden is to solve two Riccati equations and an extended Parrot problem.

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