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Solutions to Nehari and Hankel Approximation Problems Using Orthonormal Rational Functions

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***Abstract*— Compressed Hankel matrix is given by using orthonormal rational functions constructed from the Jury table. The solutions to the optimal and suboptimal Nehari problems, the solutions to the optimal and suboptimal Hankel approximation problems via the compressed Hankel matrix are given.**

I. Introduction

Various orthogonal functions play important roles in science and engineering. Examples include orthogonal polynomials, the standard basis functions in Fourier series or power series, wavelet functions. In this paper, we deal with orthogonal rational functions. The study of orthogonal rational functions has a long history. The idea of decomposing a linear system in term of orthogonal components, such as Laguerre functions, other than the functions in the standard Fourier series dates back to the work of Lee [15] and Wiener [19]. Kautz [13] formulated a more general class of orthogonal rational functions with two parameters. Heuberger et al. [10] developed a theory on construction of orthogonal rational functions using balanced realizations of inner transfer functions. The standard basis functions in power series, Laguerre functions and Kautz functions are special cases in this theory. A further generalization was presented by Ninness and Gustasson [17]. The studies in [10] and [17] are motivated by applications in system identification.

These recently developed orthogonal functions are generated through the balanced realization of inner transfer functions and hence rely on modern state space system theory. Some new investigation of the connection between advanced optimal and robust control problems and the classical tools for continuous time systems is recently carried out by Qiu [18]. The motivation is to develop elementary solutions to advanced optimal control problems so to make the advanced optimal control accessible to a wider audience. It is shown that the Routh table can be used to form orthonormal rational functions, to compute the $\mathcal{H}_2$ norm of a stable transfer function and can also be used to find the Hankel singular values and vectors, hence yielding the solution to the Hankel approximation and the Nehari problems. Since these problems play fundamental roles in $\mathcal{H}_\infty$ optimal control theory, their elementary solutions open the door for a simple, polynomial approach to $\mathcal{H}_\infty$ optimal control theory.

The Jury table and the Jury stability criterion are the counterparts of the Routh table and the Routh stability criterion in the discrete time case. The Jury table can also be used to construct orthonormal rational functions [4]. In this paper, we will study Hankel operator by using these orthonormal functions and give a compressed Hankel matrix representation and find the Hankel singular values and the corresponding Schmidt pairs. They will further be used to solve the optimal and suboptimal Nehari problem, the optimal and suboptimal Hankel approximation problem in the discrete time signal and system context.

II. Jury Table and Orthonormal Functions

Consider a stable polynomial

$$a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n,$$

where $a_i \in \mathbb{R}$ and $a_0 > 0$. It is said to be stable if all of its roots are inside the unit disk.

Construct the Jury table [12]

r_0	r_{00}	r_{01}	$\cdots$	$r_{0(n-1)}$	r_{0n}
r_0^*	r_{0n}	$r_{0(n-1)}$	$\cdots$	r_{01}	r_{00}
r_1	r_{10}	r_{11}	$\cdots$	$r_{1(n-1)}$	
r_1^*	$r_{1(n-1)}$	$r_{1(n-2)}$	$\cdots$	r_{10}	
$\vdots$	$\vdots$				
r_{n-1}	$r_{(n-1)0}$	$r_{(n-1)1}$			
r_{n-1}^*	$r_{(n-1)1}$	$r_{(n-1)0}$			
r_n	r_{n0}				

In the Jury table, the first row is copied from the coefficients of the polynomial,

$$r_{00} = a_0,\ r_{01} = a_1,\ \ldots,\ r_{0(n-1)} = a_{n-1},\ r_{0n} = a_n.$$

The row r_i^*, $i = 0, \cdots, n-1$, is obtained by writing the elements of the preceding row in the reverse order. The row r_{i+1}, $i = 0, \cdots, n-1$, is computed from its two preceding rows r_{i-1} and r_{i-1}^* as

$$r_{(i+1)j} = \frac{1}{r_{i0}} \begin{vmatrix} r_{ij} & r_{i(n-i)} \\ r_{i(n-i-j)} & r_{i0} \end{vmatrix}, \qquad (1)$$

for $i = 0, \ldots, n-1$, $j = 0, \ldots, n-i-1$.

In general, the Jury table cannot be completely constructed when $r_{i0} = 0$ for some $1 \leq i < n$. In this case,

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there is no need to complete the rest of the table since the polynomial is unstable.

Consider the set of strictly proper rational functions with denominator $a(z)$

$$\mathcal{X}_a = \left\{ \frac{b(z)}{a(z)},\ \deg b(z) < \deg a(z) \right\}. \quad (2)$$

Clearly, $\mathcal{X}_a$ is an n-dimensional subspace of $\mathcal{RH}_2$. In applications, as evidenced later in this paper, it is desirable to find a basis, or better an orthonormal basis of $\mathcal{X}_a$.

The Jury table can be used to construct the orthonormal basis, see [2], [4] and [22]. Recall the Jury table of $a(z)$ and for the rows $r_i,\ i = 1, 2, \ldots, n$, define polynomials

$$\begin{aligned} r_1(z) &= r_{10}z^{n-1} + r_{11}z^{n-2} + \cdots + r_{1(n-1)} \quad (3) \\ &\vdots \\ r_{n-1}(z) &= r_{(n-1)0}z + r_{(n-1)1} \\ r_n(z) &= r_{n0}. \end{aligned}$$

Since $a(z)$ is stable, $r_{i0} > 0$, $|r_{i0}| > |r_{i(n-i)}|$, for $i = 1, 2, \ldots, n$. We can define

$$\alpha_i = \sqrt{\frac{r_{00}}{r_{i0}}},\ k_i = \frac{r_{i(n-i)}}{r_{i0}},\ i = 0, 1, 2, \ldots, n.$$

Theorem 1 *The functions* $E_i(z) = \alpha_i \frac{r_i(z)}{a(z)},\ i = 1, 2, \ldots, n.$ *form orthonormal basis of* $\mathcal{X}_a$.

III. Hankel Operator and Compressed Hankel Matrix

Hankel operators find various applications in engineering problems such as in model reduction and optimal control. Analysis and description of the Hankel matrix, the Hankel singular values and Schmidt pairs are the key for these applications and are studied in [1], [8] and [6].

Since the Hankel matrix is an infinite dimension matrix, it is not convenient for practical computation. We will define a compressed Hankel Matrix which has only finite dimension. It will be shown later in this paper that this compressed Hankel matrix is very useful in solving the Nehari and Hankel approximation problems.

Let $P_+ : \mathcal{L}_2 \to \mathcal{H}_2$ and $P_- : \mathcal{L}_2 \to \mathcal{H}_2^\perp$ denote the orthogonal projections such that

$$\begin{aligned} P_+\left(\sum_{k=-\infty}^{\infty} f(k)z^{-k}\right) &= \sum_{k=0}^{\infty} f(k)z^{-k}, \\ P_-\left(\sum_{k=-\infty}^{\infty} f(k)z^{-k}\right) &= \sum_{k=-\infty}^{-1} f(k)z^{-k}. \end{aligned}$$

Let $J : \mathcal{L}_2 \to \mathcal{L}_2$ denote the reversal operator and $S : \mathcal{L}_2 \to \mathcal{L}_2$ denote the backward shift operator such that

$$\begin{aligned} JF(z) &= F(z^{-1}) \\ SF(z) &= zF(z). \end{aligned}$$

Clearly J and S are both unitary operators. For any $F(z) = \frac{x(z)}{a(z)} \in \mathcal{X}_a$, we have

$$JF(z) = F(z^{-1}) = \frac{x^\sim(z)}{a^\sim(z)},$$

where $a^\sim(z) = z^n a(z^{-1})$ and $x^\sim(z) = z^n x(z^{-1})$.

Definition *Given a stable system with strictly proper transfer function* $G(z)$, *the associated Hankel operator* $\Gamma_G : \mathcal{H}_2^\perp \to \mathcal{H}_2$ *is defined by*

$$\Gamma_G U(z) = P_+(G(z)U(z)),\ U(z) \in \mathcal{H}_2^\perp.$$

It is well-known that Γ_G is a finite rank operator when $G(z)$ is rational.

Lemma 1 [6] *Let* $G(z) = \frac{b(z)}{a(z)}$ *be a strictly proper stable transfer function. Then*

$$\begin{aligned} \text{Im}\,\Gamma_G &= S\mathcal{X}_a, \\ (\text{Ker}\,\Gamma_G)^\perp &= J\mathcal{X}_a. \end{aligned}$$

The Hankel operator Γ_G is the orthogonal direct sum of a zero operator and a compression of Γ_G mapping $J\mathcal{X}_a$ into $S\mathcal{X}_a$. Everything interesting about it is contained in the compression.

This compressed Hankel operator can be represented by a matrix if we choose a basis in $(\text{Ker}\,\Gamma_G)^\perp$ and a basis in Im Γ_G. Note that both $(\text{Ker}\,\Gamma_G)^\perp$ and Im Γ_G are isomorphic to $\mathcal{X}_a$. Hence we can use the orthonormal basis of $\mathcal{X}_a$

$$E(z) := [\ E_1(z) \quad E_2(z) \quad \cdots \quad E_n(z)\ ]$$

defined in Theorem 1 to form an orthonormal basis in $(\text{Ker}H_G)^\perp$

$$E(z^{-1}) = [\ E_1(z^{-1}) \quad E_2(z^{-1}) \quad \ldots \quad E_n(z^{-1})\ ]$$

and one in $\text{Im}H_G$

$$zE(z) = [\ zE_1(z) \quad zE_2(z) \quad \ldots \quad zE_n(z)\ ].$$

We call the matrix representation under this basis *Compressed Hankel Matrix* and denote it by H_G. The singular values of H_G are the Hankel singular values of $G(z)$ and are denoted by $\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n$. We assume that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n.$$

The largest singular value is called the Hankel norm of $G(z)$ and is denoted by $\|G(z)\|_H$. Let (u_i, v_i) be a left and right singular vectors of H_G corresponding to σ_i and let

$$\begin{aligned} U_i(z) &= E(z^{-1})u_i \\ V_i(z) &= zE(z)v_i. \end{aligned}$$

Then $(U_i(z), V_i(z))$ is a Schmidt pair of Γ_G corresponding to σ_i.

We are interested in computing the Hankel singular values and Schmidt pairs of Γ_G, the key is to find H_G from $G(z) = \frac{b(z)}{a(z)}$.

For any $U(z) = \frac{x^{\sim}(z)}{a^{\sim}(z)} \in J\mathcal{X}_a$,

$$\Gamma_G U(z) = P_+ \left[\frac{b(z)}{a(z)} \frac{x^{\sim}(z)}{a^{\sim}(z)}\right] = P_+ \left[\frac{b(z)}{a^{\sim}(z)} \frac{x^{\sim}(z)}{a(z)}\right].$$

Define a new operator $T : S\mathcal{X}_a \longrightarrow S\mathcal{X}_a$ by

$$T\frac{x^{\sim}(z)}{a(z)} = P_+ \left[z\frac{x^{\sim}(z)}{a(z)}\right]. \qquad (4)$$

Note that

$$P_+ \left[z\frac{x^{\sim}(z)}{a(z)}\right] = P_+ \left[\frac{z\beta(z)}{a(z)} + z\gamma\right] = \frac{z\beta(z)}{a(z)} \in S\mathcal{X}_a,$$

where γ is some constant and $\beta(z)$ is a polynomial with $\deg \beta(z) < n$. Hence $T\frac{x^{\sim}(z)}{a(z)} \in S\mathcal{X}_a$ and T is well defined. Then

$$T^i \frac{x^{\sim}(z)}{a(z)} = P_+ \left[z^i \frac{x^{\sim}(z)}{a(z)}\right], i = 1, 2, \ldots.$$

Let

$$F(z) = \frac{b(z)}{a^{\sim}(z)} = \sum_{k=1}^{\infty} f(k) z^k,$$

then $F(T)$ is well defined by

$$\begin{aligned} F(T)\frac{x^{\sim}(z)}{a(z)} &= \sum_{k=1}^{\infty} f(k) T^k \frac{x^{\sim}(z)}{a(z)} \\ &= \sum_{k=1}^{\infty} f(k) P_+ \left[\frac{z^k x^{\sim}(z)}{a(z)}\right] \\ &= P_+ \left[\sum_{k=1}^{\infty} f(k) \frac{z^k x^{\sim}(z)}{a(z)}\right] \\ &= P_+ \left[\frac{b(z)}{a^{\sim}(z)} \frac{x^{\sim}(z)}{a(z)}\right]. \end{aligned}$$

Let us also define a unitary mapping $K : \mathcal{X}_a \rightarrow \mathcal{X}_a$ by

$$K\frac{x(z)}{a(z)} = \frac{x^{\sim}(z)}{za(z)},$$

then we have

$$\Gamma_G \frac{x^{\sim}(z)}{a^{\sim}(z)} = F(T) SKJ \frac{x^{\sim}(z)}{a^{\sim}(z)}.$$

We denote the matrix representation of T, K under the above basis by T_E, K_E. Then we get the following theorem. Similar result can be found in [22].

Theorem 2 *Construct the Jury table of $a(z)$. Define matrix A as in (10) and M as:*

$$M = \begin{bmatrix} \alpha_1 r_{10} & 0 & \cdots & 0 \\ \alpha_1 r_{11} & \alpha_2 r_{20} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \alpha_1 r_{1(n-1)} & \alpha_2 r_{2(n-2)} & \cdots & \alpha_n r_{n0} \end{bmatrix}.$$

Then

(1)

$$T_E = A, \; K_E = M^{-1} \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & \cdots & 0 \end{bmatrix} M; \qquad (5)$$

(2)

$$\begin{aligned} H_G &= a^{\sim}(A)^{-1} b(A) M^{-1} \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & \cdots & 0 \end{bmatrix} M \qquad (6) \\ &= r_1^*(A)^{-1} b(A) M^{-1} \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ 1 & \cdots & 0 \end{bmatrix} M. \qquad (7) \end{aligned}$$

where

$$r_1^*(z) = z^{n-1} r_1(z^{-1}).$$

The adjoint Hankel operator $\Gamma_G^* : \mathcal{H}_2 \rightarrow \mathcal{H}_2^{\perp}$ is given by

$$H_G^* U(z) = P_-(G(z^{-1})U(z)), \; U(z) \in \mathcal{H}_2$$

and

$$\begin{aligned} \text{Im } \Gamma_G^* &= J\mathcal{X}_a, \\ (\text{Ker } \Gamma_G^*)^{\perp} &= S\mathcal{X}_a. \end{aligned}$$

Corollary 1 *The adjoint Hankel operator Γ_G^* satisfies*

$$\Gamma_G^* = SJ\Gamma_G SJ. \qquad (8)$$

Remark 1 : Corollary 1 implies that the compressed matrix representation of Γ_G^* is also H_G. By definition, the matrix representation of Γ_G^* is the transpose of that of Γ_G. Hence H_G must be symmetric.

Since H_G is symmetric, it is easy to show that

$$U_i(z) = \epsilon z V_i(z^{-1}) = \epsilon SJV_i(z) \qquad (9)$$

where $\epsilon = 1$ or $\epsilon = -1$. This fact may offer some simplification in the computation.

$$A = \begin{bmatrix} -k_0 k_1 & \alpha_1/\alpha_2 & \cdots & 0 & 0 \\ -k_0 k_2 \alpha_1/\alpha_2 & -k_1 k_2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -k_0 k_{n-1} \alpha_1/\alpha_{n-1} & -k_1 k_{n-1} \alpha_2/\alpha_{n-1} & \cdots & -k_{n-2} k_{n-1} & \alpha_{n-1}/\alpha_n \\ -k_0 k_n \alpha_1/\alpha_n & -k_1 k_n \alpha_2/\alpha_n & \cdots & -k_{n-2} k_n \alpha_{n-1}/\alpha_n & -k_{n-1} k_n \end{bmatrix}. \qquad (10)$$

IV. Optimal and Suboptimal Nehari Problem

In this section, we apply the materials in the last section to the solutions of the optimal and suboptimal Nehari problem. The Nehari problem [16] plays an important role in robust and optimal control, it is an approximation problem with respect to the $\mathcal{L}_\infty$ norm: Given a stable strictly proper system $G(z) = \frac{b(z)}{a(z)}$, find $Q(z) \in \mathcal{H}_\infty$ to minimize

$$\|G(z^{-1}) - Q(z)\|_\infty.$$

The following theorem is well-known [1], see also [6], [21].

Theorem 3 *Let $(U_1(z), V_1(z))$ be the Schmidt pair of H_G corresponding to the largest Hankel singular value σ_1. Then*

$$\min_{Q(z)\in\mathcal{H}_\infty} \|G(z^{-1}) - Q(z)\|_\infty = \sigma_1,$$

and the unique minimizing $Q(z)$ is given by

$$Q(z) = G(z^{-1}) - \sigma_1 \frac{V_1(z^{-1})}{U_1(z^{-1})}.$$

Since the Hankel singular values and Schmidt pairs can be obtained using the orthonormal basis constructed from the Jury table, a computational method for solving the Nehari problem is thus obtained.

The key point of Nehari's theorem is that the lower bound of $\|G(z^{-1}) - Q(z)\|_\infty$ is achievable, i.e., there exists a $Q(z) \in \mathcal{H}_\infty$ such that $\|G(z)\|_H = \|G(z^{-1}) - Q(z)\|_\infty$. If however, we look for a $Q(z) \in \mathcal{H}_\infty$ such that $\|G(z^{-1}) - Q(z)\|_\infty \le \gamma$ with $\|G(z)\|_H < \gamma$, then $Q(z)$ is called a suboptimal Nehari complement of $G(z^{-1})$.

The suboptimal Nehari problem is to characterize all suboptimal Nehari complements of a given $G(z^{-1})$ and is studied in [5], [3] and [7], the methods in these papers are all related to the state space system theory. Our approach to the solution will be based on the orthonormal basis and the compressed Hankel matrix Γ_G in Theorem 2.

We also define the entropy of $F(z)$ as

$$\mathcal{I}[F(z)] = -\frac{\gamma^2}{2\pi}\int_{-\pi}^{\pi} \ln[1 - \gamma^{-2}F(e^{-j\omega})F(e^{j\omega})]d\omega.$$

Theorem 4 *Let $G(z) = \frac{b(z)}{a(z)} \in \mathcal{H}_\infty$ be rational, strictly proper and $\|G(z)\|_H < \gamma$. Expand $G(z)$ as*

$$G(z) = E(z)\beta$$

and let

$$\alpha = \sqrt{1 + \beta'(\gamma^2 I - H_G^2)^{-1}\beta} \quad (11)$$

$$X(z) = \gamma E(z)(\gamma^2 I - H_G^2)^{-1}\beta/\alpha \quad (12)$$

$$Y(z) = [1 + zE(z)H_G(\gamma^2 I - H_G^2)^{-1}\beta]/\alpha \quad (13)$$

(1) Define

$$V(z) = \begin{bmatrix} V_{11}(z) & V_{12}(z) \\ V_{21}(z) & V_{22}(z) \end{bmatrix}, \quad (14)$$

where

$$\begin{aligned} V_{11}(z) &= Y(z^{-1}) - \gamma^{-1}G(z^{-1})X(z) \\ V_{12}(z) &= X(z^{-1}) - \gamma^{-1}G(z^{-1})Y(z) \\ V_{21}(z) &= X(z) \\ V_{22}(z) &= Y(z) \end{aligned} \quad (15)$$

Then the set of all $Q(z)$ such that $\|G(z^{-1}) - Q(z)\|_\infty \le \gamma$ is given by

$$\{Q(z) = -\gamma\mathcal{L}[V(z), R(z)] : R(z) \in \mathcal{H}_\infty, \|R(z)\|_\infty \le 1\},$$

where

$$\mathcal{L}[V(z), R(z)] = \frac{V_{11}(z)R(z) + V_{12}(z)}{V_{21}(z)R(z) + V_{22}(z)}.$$

(2) Define

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \frac{1}{Y(z)}\begin{bmatrix} U(z) & 1 \\ 1 & -X(z) \end{bmatrix}, \quad (16)$$

with

$$U(z) = X(z^{-1}) - \gamma^{-1}G(z^{-1})Y(z) \quad (17)$$

Then the set of all $Q(z)$ such that $\|G(z^{-1}) - Q(z)\|_\infty \le \gamma$ is given by

$$\{Q(z) = -\gamma\mathcal{F}[P(z), R(z)], R(z) \in \mathcal{H}_\infty, \|R(z)\|_\infty \le 1\},$$

where

$$\mathcal{F}[P(z), R(z)] = P_{11}(z) + P_{12}(z)R(z)(I - P_{22}(z)R(z))^{-1}P_{21}(z).$$

(3) By setting $R(z) = 0$, the unique $Q(z)$ satisfying $\|G(z^{-1}) - Q(z)\|_\infty \le \gamma$ which minimizes $\mathcal{I}[G(z^{-1}) - Q(z)]$ is given by

$$Q(z) = -\gamma V_{12}(z)V_{22}^{-1}(z) = -\gamma P_{11}(z).$$

and

$$G(z^{-1}) - Q(z) = \gamma\frac{X(z^{-1})}{Y(z)}.$$

Example 1

For

$$G(z) = \frac{b(z)}{a(z)} = \frac{\sqrt{2}z + 0.5}{z^2 + \sqrt{2}z + 0.5},$$

We wish to find all $Q(z) \in \mathcal{H}_\infty$ such that $\|G(z^{-1}) - Q(z)\|_\infty \le \gamma$ with $\gamma = 8$.

Construct the Jury table, we can get

$$\alpha_0 = 1, \alpha_1 = \frac{2\sqrt{3}}{3}, \alpha_2 = 2\sqrt{3}$$

$$k_0 = 0.5, k_1 = \frac{2\sqrt{2}}{3}, k_2 = 1$$

$$E_1(z) = \frac{\sqrt{3}/2z + \sqrt{6}/3z}{z^2 + \sqrt{2} + 0.5},\ E_2(z) = \frac{\sqrt{3}/6}{z^2 + \sqrt{2} + 0.5},$$

and

$$\beta = \begin{bmatrix} \frac{2\sqrt{6}}{3} & \frac{-5\sqrt{3}}{3} \end{bmatrix}'.$$

Hence,

$$A = \begin{bmatrix} -\frac{\sqrt{2}}{3} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{2\sqrt{2}}{3} \end{bmatrix}, \; M = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{6}}{3} & \frac{1}{\sqrt{12}} \end{bmatrix}.$$

and

$$H_G = \begin{bmatrix} 1.8856 & -3.3333 \\ -3.3333 & 3.7712 \end{bmatrix}, \sigma_1 = 6.2925.$$

Now let

$$\begin{aligned} X(z) &= \frac{0.43z + 0.2}{z^2 + \sqrt{2}z + 0.5} \\ Y(z) &= \frac{1.2z^2 + 1.37z + 0.42}{z^2 + \sqrt{2}z + 0.5}. \end{aligned}$$

So, $V(z) =$

$$\begin{pmatrix} \frac{0.83z^2 + 1.50z + 0.60}{z^2 + \sqrt{2}z + 0.5} & \frac{0.24z^2 + 0.14z}{z^2 + \sqrt{2}z + 0.5} \\ \frac{0.43z + 0.20}{z^2 + \sqrt{2}z + 0.5} & \frac{1.20z^2 + 1.37z + 0.42}{z^2 + \sqrt{2}z + 0.5} \end{pmatrix},$$

and $P(z) =$

$$\begin{pmatrix} \frac{0.24z^2 + 0.14z}{1.20z^2 + 1.37z + 0.42} & \frac{z^2 + \sqrt{2}z + 0.5}{1.20z^2 + 1.37z + 0.42} \\ \frac{z^2 + \sqrt{2}z + 0.5}{1.20z^2 + 1.37z + 0.42} & -\frac{0.43z + 0.20}{1.20z^2 + 1.37z + 0.42} \end{pmatrix}.$$

By setting $R(z) = 0$, the unique $Q(z)$ satisfying $\|G(z^{-1}) - Q(z)\|_\infty \leq 8$ which minimizes $\mathcal{I}[G(z^{-1}) - Q(z)]$ is given by

$$Q(z) = -8\frac{0.24z^2 + 0.14z}{1.20z^2 + 1.37z + 0.42}.$$

Note that when $\gamma = \sigma_1$, $\gamma^2 I - H_G^2$ becomes singular and its inverse doesn't exist. Hence we couldn't get the optimal solution by just let $\gamma \to \sigma_1$ in the suboptimal solution. That's the reason why the solutions to optimal and suboptimal problems are so different in their formulas. The same gap exists for the state space solutions. We will give an alternative algorithm which gives the optimal and suboptimal solution in one unified formula.

Theorem 5 *Let $G(z) = \frac{b(z)}{a(z)} \in \mathcal{H}_\infty$ be rational, strictly proper and $\|G(z)\|_H \leq \gamma$. Expand $G(z)$ as*

$$G(z) = E(z)\beta$$

and let

$$\begin{aligned} \alpha &= \sqrt{1 - \beta'(\gamma^2 I - (AH_G)^2)^{-1}\beta} && (19) \\ X(z) &= \gamma E(z)(\gamma^2 I - (AH_G)^2)^{-1}\beta && (20) \\ Y(z) &= 1 + E(z)AH_G(\gamma^2 I - (AH_G)^2)^{-1}\beta && (21) \end{aligned}$$

Define

$$\begin{aligned} P(z) &= \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} \\ &= \frac{1}{Y(z)} \begin{bmatrix} U(z) & \alpha \\ \alpha & -X(z) \end{bmatrix}, \qquad (22) \end{aligned}$$

with

$$U(z) = X(z^{-1}) - \gamma^{-1}G(z^{-1})Y(z) \qquad (23)$$

Then the set of all $Q(z)$ such that $\|G(z^{-1}) - Q(z)\|_\infty \leq \gamma$ is given by

$$\{Q(z) = -\gamma\mathcal{F}[P(z), R(z)], R(z) \in \mathcal{H}_\infty, \|R(z)\|_\infty \leq 1\},$$

where

$$\begin{aligned} &\mathcal{F}[P(z), R(z)] \\ &= P_{11}(z) + P_{12}(z)R(z)(I - P_{22}(z)R(z))^{-1}P_{21}(z). \end{aligned}$$

Example 2

Consider the same system

$$G(z) = \frac{b(z)}{a(z)} = \frac{\sqrt{2}z + 0.5}{z^2 + \sqrt{2}z + 0.5},$$

We wish to find all $Q(z) \in \mathcal{H}_\infty$ such that $\|G(z^{-1}) - Q(z)\|_\infty \leq \gamma$ with $\gamma = 8$ and $\gamma = 6.2925 = \sigma_1$.

From Example 1 and Theorem 5, for $\gamma = 8$, we get $\alpha = 0.83$ and

$$\begin{aligned} X(z) &= 0.83\frac{0.43z + 0.2}{z^2 + \sqrt{2}z + 0.5} \\ Y(z) &= 0.83\frac{1.2z^2 + 1.37z + 0.42}{z^2 + \sqrt{2}z + 0.5} \end{aligned}$$

and $P(z) =$

$$\begin{pmatrix} \frac{0.24z^2 + 0.14z}{1.20z^2 + 1.37z + 0.42} & \frac{z^2 + \sqrt{2}z + 0.5}{1.20z^2 + 1.37z + 0.42} \\ \frac{z^2 + \sqrt{2}z + 0.5}{1.20z^2 + 1.37z + 0.42} & -\frac{0.43z + 0.20}{1.20z^2 + 1.37z + 0.42} \end{pmatrix}.$$

Note that $P(z)$ is exactly the same as in Example 1.

For $\gamma = 6.2925 = \sigma_1$, we get $\alpha = 0$ and

$$\begin{aligned} X(z) &= \frac{z + 0.5}{z^2 + \sqrt{2}z + 0.5} = U_1(z^{-1}) \\ Y(z) &= \frac{z^2 + 0.5z}{z^2 + \sqrt{2}z + 0.5} = V_1(z), \end{aligned}$$

where $(U_1(z), V_1(z))$ is the Schmidt pair corresponding to σ_1. So,

$$P(z) = \begin{pmatrix} \frac{0.84z + 0.5}{z^2 + 0.5z} & 0 \\ 0 & -\frac{0.43z + 0.2}{z^2 + 0.5z} \end{pmatrix}$$

and

$$Q(z) = -\gamma P_{11}(z) = G(z^{-1}) - \gamma\frac{X(z^{-1})}{Y(z)}.$$

Hence,

$$\|G(z^{-1}) - Q(z)\|_\infty = \gamma \left\| \frac{X(z^{-1})}{Y(z)} \right\|_\infty = \gamma.$$

and Q is the optimal solution of Nehari problem.

V. Optimal and Suboptimal Hankel Norm Approximation Problems

In this section, we will study the optimal and suboptimal Hankel norm approximation problems. We first take a look at the optimal Hankel norm approximation problem. Given a stable system with strictly proper transfer function, we want to find a strictly stable lower order system to approximate the high order system so that the Hankel norm of the error is minimized. The solution is given by the following known theorem.

Theorem 6 *Let $(U_{k+1}(z), V_{k+1}(z))$ be the Schmidt pair of H_G corresponding to $(k+1)$-st Hankel singular value σ_{k+1}. Then*

$$\min_{\text{order } \tilde{G}(z) \le k} \|G(z) - \tilde{G}(z)\|_H = \sigma_{k+1},$$

and the stable minimizing $\tilde{G}(z)$ is given by

$$\tilde{G}(z) = G(z) - P_+ \left[\sigma_{k+1} \frac{V_{k+1}(z)}{U_{k+1}(z)} \right] + c,$$

where c is any constant.

The minimum to the Hankel norm approximation is σ_{k+1}, If however we look for a stable $\tilde{G}(z)$ with order $\tilde{G}(z) \le k$ such that $\|G(z) - \tilde{G}(z)\|_H \le \gamma$ with $\sigma_{k+1} \le \gamma < \sigma_k$, then $\tilde{G}(z)$ is not unique. The suboptimal Hankel norm approximation problem is to characterize all such $\tilde{G}(z)$ for a given $G(z)$. This problem is also studied in [9], [3], [8].

The solution to this problem is closely related to the so called Nehari-Takagi problem, see [9] and [3]. It is known that solution $Q(z)$ to the Nehari-Takagi problem is given by the same formula as in the suboptimal Nehari problem, but $Q(z)$ doesn't belong to $\mathcal{H}_\infty$ anymore, $Q(z)$ will have precisely k poles outside the unit disk.

Replacing z by z^{-1}, it's easy to get the solution to the suboptimal Hankel approximation problem.

Theorem 7 *Let $G(z) \in \mathcal{H}_\infty$ be rational, proper and with singular values $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n$, also let $\sigma_{k+1} \le \gamma < \sigma_k$. Then the set of all stable $\tilde{G}(z)$ with order $\tilde{G}(z) \le k$ such that*

$$\|G(z) - \tilde{G}(z)\|_H \le \gamma$$

is given by

$$\tilde{G}(z) = P_+[Q(z^{-1})] + c,$$

where $Q(z)$ is given by Theorem 5 and c is any constant.

VI. Conclusion

Compressed Hankel matrix is given by using orthonormal rational functions constructed from the Jury table. The solutions to the optimal and suboptimal Nehari problems, the solutions to the optimal and suboptimal Hankel approximation problems via the compressed Hankel matrix are given.

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