

Stabilization of Exponentially Unstable Linear Systems with Saturating Actuators

Tingshu Hu, Zongli Lin, and Li Qiu

Abstract—We study the problem of stabilizing exponentially unstable linear systems with saturating actuators. The study begins with planar systems with both poles exponentially unstable. For such a system, we show that the boundary of the domain of attraction under a saturated stabilizing linear state feedback is the unique stable limit cycle of its time-reversed system. A saturated linear state feedback is designed that results in a closed-loop system having a domain of attraction that is arbitrarily close to the null controllable region. This design is then utilized to construct state feedback laws for higher order systems with two exponentially unstable poles.

Index Terms—Actuator saturation, domain of attraction, null controllable region, semiglobal stabilization.

I. INTRODUCTION

We consider the problem of stabilizing exponentially unstable linear systems subject to actuator saturation. For systems that are not exponentially unstable, this stabilization problem has been focus of study and is now well addressed. For example, it was shown in [13] that a linear system subject to actuator saturation can be globally asymptotically stabilized by nonlinear feedback if and only if the system is asymptotically null controllable with bounded controls (ANCBC), which, as shown in [11], is equivalent to the system being stabilizable in the usual linear sense and having open-loop poles in the closed left-half plane. A nested feedback design technique for designing nonlinear globally asymptotically stabilizing feedback laws was proposed in [16] for a chain of integrators and was fully generalized in [14]. Alternative solutions to the global stabilization problem consisting of scheduling a parameter in an algebraic Riccati equation according to the size of the state vector was later proposed in [12], [17]. The question of whether or not a general linear ANCBC system subject to actuator saturation can be globally asymptotically stabilized by linear feedback was answered in [3], [15], where it was shown that a chain of integrators of length greater than two cannot be globally asymptotically stabilized by saturated linear feedback.

The notion of semiglobal asymptotic stabilization on the null controllable region for linear systems subject to actuator saturation was introduced in [7], [8]. The semi-global framework for stabilization requires feedback laws that yield a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes an *a priori* given (arbitrarily large) bounded subset of the null controllable region. In [7], [8], it was shown that, for linear ANCBC systems subject to actuator saturation, one can achieve semi-global asymptotic stabilization on the asymptotically null controllable region (the whole state space in this case) using linear feedback laws.

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Despite the existing results (see [2] for an extensive chronological bibliography on the subject), the general picture of stabilizing exponentially unstable linear systems with saturating actuators remains not as clear as that of ANCBC systems. It is clear that this kind of systems cannot be globally stabilized in any way since they are not globally null controllable. The largest possible region on which a system can be stabilized is the null controllable region. In [4], we gave an explicit description of the null controllable region for a general linear system in terms of a set of extremal trajectories of the (time reversed) antistable subsystem. We recall that a linear system is said to be antistable if all its poles are in the open right-half plane and semistable if all its poles are in the closed left-half plane. For example, for a second order antistable system, the boundary of its null controllable region is covered by at most two extremal trajectories; and for a third order antistable system, the set of extremal trajectories can be described in terms of parameters in a real interval.

Based on the description of the null controllable region in [4], we begin our study of stabilization with planar antistable systems. We show that for such a system the boundary of the domain of attraction under any stabilizing saturated linear state feedback is the unique stable limit cycle of its time-reversed system. Moreover, the domain of attraction is convex. We next show that any second order antistable linear system can be semiglobally asymptotically stabilized on its null controllable region by saturated linear feedback. That is, for any *a priori* given set in the interior of the null controllable region, there exists a saturated linear feedback law that yields a closed-loop system which has an asymptotically stable equilibrium whose domain of attraction includes this given set. This design is then utilized to construct state feedback laws for higher order systems with two exponentially unstable poles.

The remainder of this note is organized as follows. Section II contains a brief summary of the description of the null controllable region which will be used in this note. Section III determines the domain of attraction for a second order antistable linear system under any saturated stabilizing linear feedback law. Section IV constructs saturated feedback laws that achieve semiglobal asymptotic stability on the null controllable region for any linear systems having two exponentially unstable poles. Finally, Section V draws some brief conclusions.

II. RESULTS ON THE NULL CONTROLLABLE REGION

Consider a linear system

$$\dot{x}(t) = Ax(t) + bu(t), \quad |u| \leq 1 \quad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state and $u(t) \in \mathbf{R}$ is the control. Assume that (A, b) is stabilizable. The null controllable region of the system, denoted as \mathcal{C} , is defined to be the set of states that can be steered to the origin in a finite time by using a control u that is measurable and $|u(t)| \leq 1$ for all t . If A is antistable, then \mathcal{C} is a bounded convex open set. For a general unstable system, \mathcal{C} is the Cartesian product of the null controllable region of its semistable subsystem, which is the whole subspace, and that of its antistable subsystem. It was shown in [4] that $\partial\mathcal{C}$ (the boundary of \mathcal{C}) of an anti-stable system is composed of a set of extremal trajectories of its time reversed system. The time reversed system of (1) is

$$\dot{z}(t) = -Az(t) - bv(t), \quad |v| \leq 1. \quad (2)$$

Suppose that A is anti-stable, denote

$$\mathcal{E} := \{v(t) = \text{sgn}(c' e^{At} b), t \in \mathbf{R}: c \neq 0\} \quad (3)$$

and for a control v , $|v(t)| < 1$ for all $t \in \mathbf{R}$, denote the trajectory of (2) under the control of v as

$$\Phi(t, v) := \int_{-\infty}^t e^{-A(t-\tau)} b v(\tau) d\tau. \quad (4)$$

Since A is antistable, the integral in (4) exists for all $t \in \mathbf{R}$, so $\Phi(t, v)$ is well defined. It is shown in [4] that

$$\partial\mathcal{C} = \{\Phi(t, v): t \in \mathbf{R}, v \in \mathcal{E}\}. \quad (5)$$

In particular, for a second-order antistable system, if A has two real eigenvalues, then

$$\begin{aligned} \partial\mathcal{C} &= \left\{ \pm \left[e^{-At} z_e^- - \int_0^t e^{-A(t-\tau)} b d\tau \right] : t \in [0, \infty] \right\} \\ &= \{\pm(-2e^{-At} + I)A^{-1}b : t \in [0, \infty]\} \end{aligned} \quad (6)$$

where $z_e^- = A^{-1}b$ is the equilibrium point under the constant control $u = -1$; if A has a pair of complex eigenvalues $\alpha \pm j\beta$, $\alpha, \beta > 0$, then

$$\begin{aligned} \partial\mathcal{C} &= \left\{ \pm \left[e^{-At} z_s^- - \int_0^t e^{-A(t-\tau)} b d\tau \right] : t \in [0, T_p] \right\} \\ &= \{\pm[e^{-At} z_s^- - (I - e^{-At})A^{-1}b] : t \in [0, T_p]\} \end{aligned} \quad (7)$$

where $T_p = \pi/\beta$ and $z_s^- = (I + e^{-AT_p})^{-1}(I - e^{-AT_p})A^{-1}b$.

III. DOMAIN OF ATTRACTION UNDER SATURATED LINEAR STATE FEEDBACK

Also consider the open-loop system (1). A saturated linear state feedback is given by $u = \sigma(fx)$, where $f \in \mathbf{R}^{1 \times n}$ is the feedback gain and $\sigma(\cdot)$ is the saturation function $\sigma(s) = \text{sgn}(s) \min\{1, |s|\}$. Such a feedback is said to be stabilizing if $A + bf$ is asymptotically stable. With a saturated linear state feedback applied, the closed-loop system is

$$\dot{x}(t) = Ax(t) + b\sigma(fx(t)). \quad (8)$$

Denote the state transition map of (8) by $\phi: (t, x_0) \mapsto x(t)$. The domain of attraction S of the equilibrium $x = 0$ of (8) is defined by

$$S := \left\{ x_0 \in \mathbf{R}^n : \lim_{t \rightarrow \infty} \phi(t, x_0) = 0 \right\}.$$

Obviously, S must lie within the null controllable region \mathcal{C} of the system (1). Therefore, a design problem is to choose a state feedback gain so that S is arbitrarily close to \mathcal{C} . We refer to this problem as semiglobal stabilization on the null controllable region. We will first deal with antistable planar systems, then extend the results to higher order systems with only two antistable modes.

For the system (8), assume that $A \in \mathbf{R}^{2 \times 2}$ is anti-stable. In [1], it was shown that the boundary of S , denoted by ∂S , is a closed orbit, but no method to find this closed orbit is provided. Generally, only a subset of S lying between $fx = 1$ and $fx = -1$ is detected as a level set of some Lyapunov function (see, e.g., [5]). Let P be a positive-definite matrix such that $(A + bf)'P + P(A + bf)$ is negative-definite. Since $\{z \in \mathbf{R}^2: -1 < fz < 1\}$ is an open neighborhood of the origin, it must contain

$$\mathcal{Q}_0 := \{z \in \mathbf{R}^2: z'Pz \leq r_0\} \quad (9)$$

for some $r_0 > 0$. Clearly, \mathcal{Q}_0 is an invariant set inside S . However, \mathcal{Q}_0 as an estimation of the domain of attraction can be very conservative (see, e.g., Fig. 1).

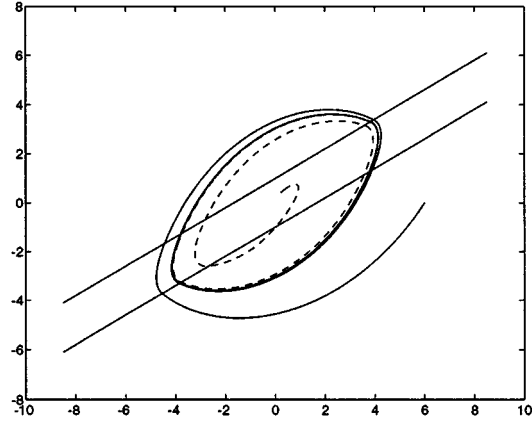


Fig. 1. Determination of ∂S from the limit cycle.

Lemma 3.1 [1]: The origin is the unique equilibrium point of system (8).

Let us introduce the time-reversed system of (8)

$$\dot{z}(t) = -Az(t) - b\sigma(fz(t)). \quad (10)$$

Clearly (10) also has only one equilibrium point, an unstable one, at the origin. Denote the state transition map of (10) by $\psi: (t, z_0) \mapsto z(t)$.

Theorem 3.1: ∂S is the unique limit cycle of planar systems (8) and (10). Furthermore, ∂S is the positive limit set of $\psi(\cdot, z_0)$ for all $z_0 \neq 0$.

This theorem says that ∂S is the unique limit cycle of (8) and (10). This limit cycle is a stable one for (10) (in a global sense), but an unstable one for (8). Therefore, it is easy to determine ∂S by simulating the time-reversed system (10). Shown in Fig. 1 is a typical result, where two trajectories, one starting from outside, the solid curve, and the other starting from inside, the dashed curve, both converge to the unique limit cycle. The straight lines in Fig. 1 are $fz = 1$ and $fz = -1$.

To prove Theorem 3.1, we need the following two lemmas, proofs of which can be found in [4].

Lemma 3.2: Suppose that $A \in \mathbf{R}^{2 \times 2}$ is anti-stable and (f, A) is observable. Given a $c > 0$, let x_1, x_2, y_1 and y_2 ($x_1 \neq x_2$) be four points on the line $fx = c$, satisfying

$$y_1 = e^{AT_1}x_1, \quad y_2 = e^{AT_2}x_2,$$

for some $T_1, T_2 > 0$ and

$$fe^{At_1}x_1 > c \quad fe^{At_2}x_2 > c, \quad \forall t_1 \in (0, T_1), \quad t_2 \in (0, T_2)$$

then, $\|y_1 - y_2\| > \|x_1 - x_2\|$.

Shown in Fig. 2 is an illustration of Lemma 3.2. The curve from x_i to y_i is $x(t) = e^{At}x_i$, $t \in [0, T_i]$, a segment of a trajectory of the autonomous system $\dot{x} = Ax$. Lemma 3.2 indicates that if any two different trajectories leave a straight line on the same side, they will be further apart when they return to it.

Lemma 3.3: Suppose that $A \in \mathbf{R}^{2 \times 2}$ is asymptotically stable and (f, A) is observable. Given a $c > 0$, let x_1, x_2 be two points on the line $fx = c$ and y_1, y_2 be two points on $fx = -c$ such that

$$y_1 = e^{AT_1}x_1, \quad y_2 = e^{AT_2}x_2$$

for some $T_1, T_2 > 0$, and

$$\begin{aligned} |fe^{At_1}x_1| < c, \quad |fe^{At_2}x_2| < c, \\ \forall t_1 \in (0, T_1), \quad t_2 \in (0, T_2) \end{aligned}$$

then, $\|y_1 - y_2\| > \|x_1 - x_2\|$.

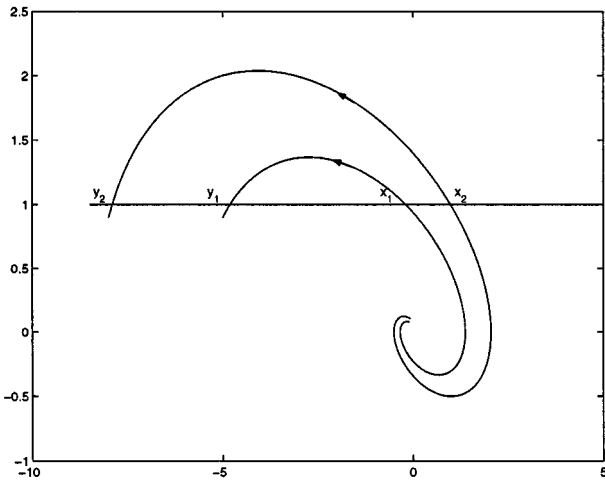


Fig. 2. Illustration of Lemma 3.2.

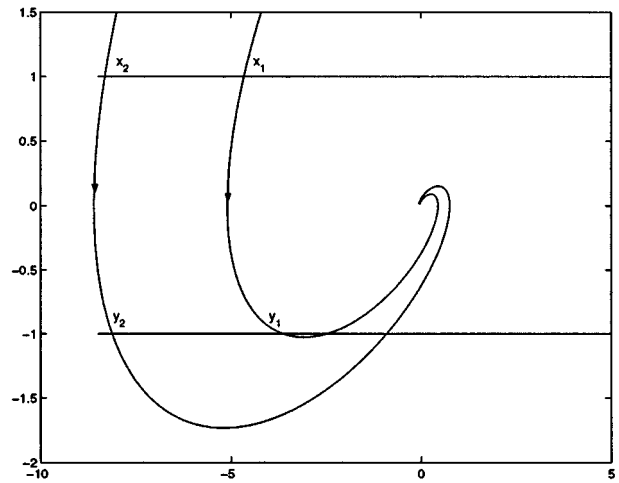


Fig. 3. Illustration of Lemma 3.3.

Shown in Fig. 3 is an illustration of Lemma 3.3. It says that if two different trajectories of the autonomous system $\dot{x} = Ax$ enter the region between $fx = c$ and $fx = -c$, they will be further apart when they leave the region. Notice that in Lemma 3.2, A is antistable, and in Lemma 3.3, A is asymptotically stable.

Proof of Theorem 3.1: We first prove that for the system (10), every trajectory $\psi(t, z_0)$, $z_0 \neq 0$, converges to a periodic orbit as $t \rightarrow \infty$. Recall that Q_0 [defined in (9)] lies within the domain of attraction of the equilibrium $x = 0$ of (8) and is an invariant set. It follows that, for every state $z_0 \neq 0$ of (10), there is some $t_0 \geq 0$ such that $\psi(t, z_0)$ lies outside Q_0 for all $t \geq t_0$. The state transition map of the system (10) is

$$\psi(t, z_0) = e^{-At} z_0 - \int_0^t e^{-A(t-\tau)} b \sigma(fz(\tau)) d\tau. \quad (11)$$

Since $-A$ is stable, the first term converges to the origin. Since $|\sigma(fz(\tau))| \leq 1$, the second term belongs to \mathcal{C} , the null controllable region of (1), for all t . It follows that there exists an $r_1 > r_0$ such that $\psi^1(t, z_0) P \psi(t, z_0) \leq r_1 < \infty$ for all $t \geq t_0$. Let $Q = \{z \in \mathbf{R}^2: r_0 \leq z^1 P z \leq r_1\}$. Then $\psi(t, z_0)$, $t \geq t_0$, lies entirely in Q . It follows from the Poincaré-Bendixon theorem that $\psi(t, z_0)$ converges to a periodic orbit.

The preceding paragraph shows that (8) and (10) have periodic orbits. We claim that the system (8) and (10) each has only one periodic orbit. For direct use of Lemma 3.2 and Lemma 3.3, we prove this claim through the original system (8).

First notice that a periodic orbit must enclose the unique equilibrium point $x = 0$ by the index theory, see e.g., [6], and must be symmetric to the origin ($-\Gamma$ is a periodic orbit if Γ is, hence if the periodic orbit is not symmetric, there will be two intersecting trajectories). Also, it cannot be completely contained in the linear region between $fx = 1$ and $fx = -1$. (Otherwise the asymptotically stable linear system $\dot{x} = (A + bf)x$ would have a closed trajectory in this region. This is impossible). Hence, it has to intersect each of the lines $fx = \pm 1$ at least twice. Assume without loss of generality that (f, A, b) is in the observer canonical form, i.e., $f = [0 \ 1]$, $A = \begin{bmatrix} 0 & -a_1 \\ 1 & -a_2 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, with $a_1, a_2 > 0$, and denote $x = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$. In this case, $fx = \pm 1$ are horizontal lines. The stability of $A + bf$ requires that $-a_1 + b_1 < 0$ and $a_2 + b_2 < 0$. Observe that on the line $fx = 1$, we have $\xi_2 = 1$ and $\xi_2 = \xi_1 + a_2 + b_2$. Hence, if $\xi_1 > -a_2 - b_2$, then $\xi_2 > 0$, i.e., the trajectories go upwards; if $\xi_1 < -a_2 - b_2$, then $\xi_2 < 0$, i.e., the trajectories go downwards. This implies that any periodic orbit crosses

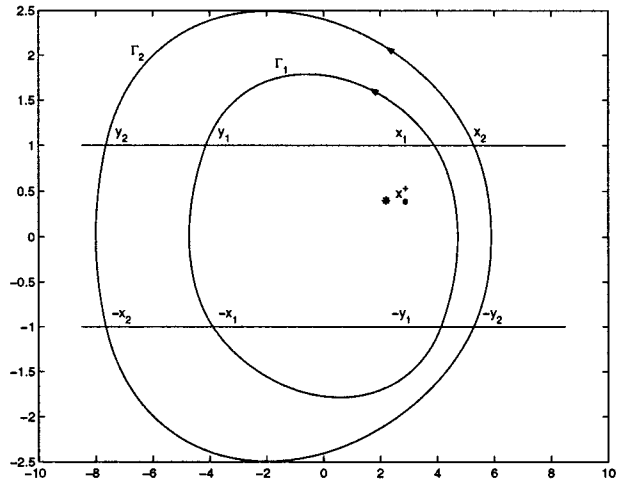


Fig. 4. Illustration for the proof of Theorem 3.1.

$fx = 1$ exactly twice and similarly for $fx = -1$. It also implies that a periodic orbit goes counter-clockwise.

Now, suppose on the contrary that (8) has two different periodic orbits Γ_1 and Γ_2 , with Γ_1 enclosed by Γ_2 , as illustrated in Fig. 4. Note that any periodic orbit must enclose the origin and any two trajectories cannot intersect. Hence, all the periodic orbits must be ordered by enclosure. Let x_1 and y_1 be the two intersections of Γ_1 with $fx = 1$, and x_2, y_2 be the two intersections of Γ_2 with $fx = 1$. Then, along Γ_1 , the trajectory goes from x_1 to $y_1, -x_1, -y_1$ and returns to x_1 ; along Γ_2 , the trajectory goes from x_2 to $y_2, -x_2, -y_2$ and returns to x_2 .

Let $x_e^+ = -A^{-1}b$. Since $x_1 \rightarrow y_1$ along Γ_1 and $x_2 \rightarrow y_2$ along Γ_2 are on trajectories of $\dot{x} = Ax + b$ (or $d(x - x_e^+)/dt = A(x - x_e^+)$), we have

$$y_1 - x_e^+ = e^{AT_1}(x_1 - x_e^+) \quad y_2 - x_e^+ = e^{AT_2}(x_2 - x_e^+)$$

for some $T_1, T_2 > 0$. Furthermore, $f(x_1 - x_e^+) = f(x_2 - x_e^+) = f(y_1 - x_e^+) = f(y_2 - x_e^+) = 1 - fx_e^+ > 0$ (since $fx_e^+ = b_1/a_1 < 1$) and for all x on the two pieces of trajectories, $f(x - x_e^+) \geq 1 - fx_e^+$. It follows from Lemma 3.2 that

$$\|y_2 - y_1\| > \|x_2 - x_1\|.$$

On the other hand, $y_1 \rightarrow -x_1$ along Γ_1 and $y_2 \rightarrow -x_2$ along Γ_2 are on trajectories of $\dot{x} = (A + bf)x$ satisfying $-x_1 = e^{(A+bf)T_3} y_1$ and

$-x_2 = e^{(A+bf)T_4}y_2$ for some $T_3, T_4 > 0$. It follows from Lemma 3.3 that

$$\|x_2 - x_1\| > \|y_2 - y_1\|$$

which is a contradiction. Therefore, Γ_1 and Γ_2 must be the same periodic orbit. This shows that the systems have only one periodic orbit and, hence, it is a limit cycle.

We have so far proven that both (8) and (10) have a unique limit cycle and every trajectory $\psi(t, z_0)$, $z_0 \neq 0$, of (10) converges to this limit cycle. This implies that a trajectory $\phi(t, x_0)$ of (8) converges to the origin if and only if x_0 is inside the limit cycle. This shows that the limit cycle is ∂S . \square

In the above proof, we also showed that ∂S is symmetric and has two intersections with $fx = 1$ and two with $fx = -1$. Another nice feature of S , as shown in [4], is that it is convex.

IV. SEMIGLOBAL STABILIZATION ON THE NULL CONTROLLABLE REGION

A. Second Order Antistable Systems

In this subsection, we continue to assume that $A \in \mathbf{R}^{2 \times 2}$ is antistable and (A, b) is controllable. We will show that the domain of attraction S of the equilibrium $x = 0$ of the closed-loop system (8) can be made arbitrarily close to the null controllable region \mathcal{C} by judiciously choosing the feedback gain f . To state the main result of this section, we need to introduce the Hausdorff distance. Let $\mathcal{X}_1, \mathcal{X}_2$ be two bounded subsets of \mathbf{R}^n . Then, their Hausdorff distance is defined as

$$d(\mathcal{X}_1, \mathcal{X}_2) := \max \left\{ \bar{d}(\mathcal{X}_1, \mathcal{X}_2), \bar{d}(\mathcal{X}_2, \mathcal{X}_1) \right\}$$

where

$$\bar{d}(\mathcal{X}_1, \mathcal{X}_2) = \sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} \|x_1 - x_2\|.$$

Here, the vector norm used is arbitrary.

Let P be the unique positive-definite solution of the following Riccati equation:

$$A'P + PA - Pbb'P = 0. \quad (12)$$

Note that this equation is associated with the minimum energy regulation, i.e., an LQR problem with cost

$$J = \int_0^\infty u'(t)u(t) dt.$$

The corresponding minimum energy state feedback gain is given by $f_0 = -b'P$. By the infinite gain margin and 50% gain reduction margin property of LQR regulators, the origin is a stable equilibrium of system

$$\dot{x}(t) = Ax(t) + b\sigma(kf_0x(t)) \quad (13)$$

for all $k > 0.5$. Let $\mathcal{S}(k)$ be the domain of attraction of the equilibrium $x = 0$ of (13).

Theorem 4.1: $\lim_{k \rightarrow \infty} d(\mathcal{S}(k), \mathcal{C}) = 0$.

Proof: See the Appendix. \square

Note that the use of high gain feedback is crucial here. The minimum energy feedback f_0 itself does not give a domain of attraction close to \mathcal{C} . This is quite different from the related result in [8] and [9] for semistable open-loop systems. In these two papers, it was shown that if (A, B) is ANCBC, then low-gain feedback gives arbitrarily large domain of attraction.

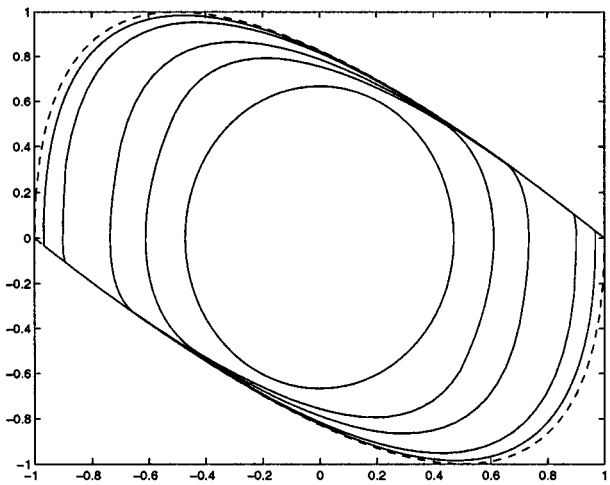


Fig. 5. Domains of attraction under different feedbacks.

Example 4.1: Let $A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Then $f_0 = [0 \ 3]$. In Fig. 5, the boundaries of the domains of attraction corresponding to different $f = kf_0$, $k = 0.50005, 0.65, 1, 3$, are plotted. The regions do become bigger for greater k . The outermost boundary is ∂C . When $k = 3$, it can be seen that ∂S is already very close to ∂C .

B. Higher Order Systems with Two Exponentially Unstable Poles

Consider the following open-loop system:

$$\dot{x}(t) = Ax(t) + bu(t) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \quad (14)$$

where $x = [x_1' \ x_2']'$, $x_1 \in \mathbf{R}^2$, $x_2 \in \mathbf{R}^n$, $A_1 \in \mathbf{R}^{2 \times 2}$ is antistable and $A_2 \in \mathbf{R}^n$ is semi-stable. Assume that (A, b) is controllable. Denote the null controllable region of the subsystem

$$\dot{x}_1(t) = A_1x_1(t) + b_1u(t)$$

as \mathcal{C}_1 , then the null controllable region of (14) is $\mathcal{C}_1 \times \mathbf{R}^n$ [4]. Given $\gamma_1, \gamma_2 > 0$, denote

$$\begin{aligned} \Omega_1(\gamma_1) &:= \{\gamma_1 x_1 \in \mathbf{R}^2: x_1 \in \bar{\mathcal{C}}_1\} \\ \Omega_2(\gamma_2) &:= \{x_2 \in \mathbf{R}^n: \|x_2\| \leq \gamma_2\}. \end{aligned} \quad (15)$$

When $\gamma_1 = 1$, $\Omega_1(\gamma_1) = \bar{\mathcal{C}}_1$ and when $\gamma_1 < 1$, $\Omega_1(\gamma_1)$ lies in the interior of \mathcal{C}_1 . In this section, we will show that given any $\gamma_1 < 1$ and $\gamma_2 > 0$, a state feedback can be designed such that $\Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$ is contained in the domain of attraction of the equilibrium $x = 0$ of the closed-loop system.

For $\epsilon > 0$, let $P(\epsilon) = \begin{bmatrix} P_1(\epsilon) & P_2'(\epsilon) \\ P_2(\epsilon) & P_3(\epsilon) \end{bmatrix} \in \mathbf{R}^{(2+n) \times (2+n)}$ be the unique positive-definite solution to the ARE

$$A'P + PA - Pbb'P + \epsilon^2 I = 0. \quad (16)$$

Clearly, as $\epsilon \downarrow 0$, $P(\epsilon)$ decreases. Hence, $\lim_{\epsilon \rightarrow 0} P(\epsilon)$ exists.

Let P_1 be the unique positive definite solution to the ARE

$$A_1'P_1 + P_1A_1 - P_1b_1b_1'P_1 = 0.$$

Then by the continuity property of the solution of the Riccati equation [19]

$$\lim_{\epsilon \rightarrow 0} P(\epsilon) = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $f(\epsilon) := -b'P(\epsilon)$. First, consider the domain of attraction of the equilibrium $x = 0$ of the following closed-loop system:

$$\dot{x}(t) = Ax(t) + b\sigma(f(\epsilon)x(t)). \quad (17)$$

It is easy to see that

$$D(\epsilon) := \left\{ x \in \mathbf{R}^{2+n} : x'P(\epsilon)x \leq 1/\|b'P^{1/2}(\epsilon)\|^2 \right\}$$

is contained in the domain of attraction of the equilibrium $x = 0$ of (17) and is an invariant set. Note that if $x_0 \in D(\epsilon)$, then $x(t) \in D(\epsilon)$ and $|f(\epsilon)x(t)| \leq 1$ for all $t > 0$. That is, $x(t)$ will stay in the linear region of the closed-loop system, and in $D(\epsilon)$.

Lemma 4.1: Denote

$$r_1(\epsilon) := \frac{1}{2\|P_1^{1/2}(\epsilon)\|\|b'P^{1/2}(\epsilon)\|},$$

$$r_2(\epsilon) := \frac{-\|P_2(\epsilon)\| + \sqrt{\|P_2(\epsilon)\|^2 + 3\|P_1(\epsilon)\|\|P_3(\epsilon)\|}}{\|P_3(\epsilon)\|} r_1(\epsilon).$$

Then

$$D_1(\epsilon) := \{x \in \mathbf{R}^{2+n} : \|x_1\| \leq r_1(\epsilon), \|x_2\| \leq r_2(\epsilon)\} \subset D(\epsilon).$$

Moreover, $\lim_{\epsilon \rightarrow 0} r_2(\epsilon) = \infty$, and $r_1(\epsilon)$ increases with an upper bound as ϵ tends to zero.

Proof: It can be verified that

$$\|P_1(\epsilon)\|r_1^2(\epsilon) + 2\|P_2(\epsilon)\|r_1(\epsilon)r_2(\epsilon) + \|P_3(\epsilon)\|r_2^2(\epsilon) = \frac{1}{\|b'P^{1/2}(\epsilon)\|^2}. \quad (18)$$

So for all $x \in D_1(\epsilon)$, $x'P(\epsilon)x \leq (1/\|b'P^{1/2}(\epsilon)\|^2)$, i.e., $D_1(\epsilon) \subset D(\epsilon)$. By the definition of $r_1(\epsilon)$ and $r_2(\epsilon)$, we have

$$r_2(\epsilon) = \frac{3\|P_1(\epsilon)\|}{\|P_2(\epsilon)\| + \sqrt{\|P_2(\epsilon)\|^2 + 3\|P_1(\epsilon)\|\|P_3(\epsilon)\|}} \cdot \frac{1}{2\|P_1^{1/2}(\epsilon)\|\|b'P^{1/2}(\epsilon)\|}.$$

Since as ϵ goes to zero, $P_2(\epsilon)$, $P_3(\epsilon) \rightarrow 0$, and $P_1(\epsilon) \rightarrow P_1$, so $r_1(\epsilon)$ is bounded whereas $r_2(\epsilon) \rightarrow \infty$. It follows from the monotonicity of $P(\epsilon)$ that r_1 is a monotonically decreasing function of ϵ . \square

Theorem 4.2: Let $f_0 = -b_1'P_1$. For any $\gamma_1 < 1$ and $\gamma_2 > 0$, there exist a $k > 0.5$ and an $\epsilon > 0$ such that $\Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$ is contained in the domain of attraction of the equilibrium $x = 0$ of the closed-loop system

$$\dot{x}(t) = Ax(t) + bu(t) \quad (19)$$

where

$$u(t) = \begin{cases} \sigma(kf_0x_1(t)), & x \notin D(\epsilon) \\ \sigma(f(\epsilon)x(t)), & x \in D(\epsilon). \end{cases} \quad (19)$$

Proof: Since $\gamma_1 < 1$, by Theorem 4.1, there exists a $k > 0.5$ such that $\Omega_1(\gamma_1)$ lies in the interior of the domain of attraction of the equilibrium $x_1 = 0$ of

$$\dot{x}_1(t) = A_1x_1(t) + b_1\sigma(kf_0x_1(t)). \quad (20)$$

Let $\epsilon_0 > 0$ be given. For an initial state $x_{10} \in \Omega_1(\gamma_1)$, denote the trajectory of (20) as $\psi(t, x_{10})$. Define

$$T(x_{10}) := \min\{t \geq 0 : \|\psi(t, x_{10})\| \leq r_1(\epsilon_0)\}$$

then $T(x_{10})$ is the time when $\psi(t, x_{10})$ first enters the ball $\{x_1 \in \mathbf{R}^2 : \|x_1\| \leq r_1(\epsilon_0)\}$. Let

$$T_M = \max\{T(x_{10}) : x_{10} \in \partial\Omega_1(\gamma_1)\} \quad (21)$$

and

$$\gamma = \max_{t \in [0, T_M]} \|e^{A_2t}\| \gamma_2 + \int_0^{T_M} \|e^{A_2(T_M-\tau)}b_2\| d\tau \quad (22)$$

then by Lemma 4.1, there exists an $\epsilon < \epsilon_0$ such that $r_1(\epsilon) \geq r_1(\epsilon_0)$, $r_2(\epsilon) \geq \gamma$ and

$$D_1(\epsilon) = \{x \in \mathbf{R}^{2+n} : \|x_1\| \leq r_1(\epsilon), \|x_2\| \leq r_2(\epsilon)\} \subset D(\epsilon)$$

lies in the domain of attraction of the equilibrium $x = 0$ of (17).

Now consider an initial state of (19), $x_0 \in \Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$. If $x_0 \in D(\epsilon)$, then $x(t)$ will go to the origin since $D(\epsilon)$ is an invariant set and is contained in the domain of attraction. If $x_0 \notin D(\epsilon)$, we conclude that $x(t)$ will enter $D(\epsilon)$ at some $T \leq T_M$ under the control $u = \sigma(kf_0x_1(t))$. Observe that under this control, $x_1(t)$ goes along a trajectory of (20). If there is no switch, $x_1(t)$ will hit the ball $\{x_1 \in \mathbf{R}^2 : \|x_1\| \leq r_1(\epsilon_0)\}$ at $T(x_{10})$. Clearly $T(x_{10}) \leq T_M$ and at this instant $\|x_2(T(x_{10}))\| \leq \gamma \leq r_2(\epsilon)$, so $x(T(x_{10})) \in D_1(\epsilon)$. Thus, we see that if there is no switch, $x(t)$ will be in $D_1(\epsilon)$ at $T(x_{10})$. Since $D_1(\epsilon) \subset D(\epsilon)$, $x(t)$ must have entered $D(\epsilon)$ at some earlier time $T \leq T(x_{10}) \leq T_M$. So we have that conclusion. With the switching control applied, once $x(t)$ enters the invariant set $D(\epsilon)$, it will remain in it and go to the origin asymptotically. \square

V. CONCLUSION

We provided a simple semiglobal stabilization strategy for exponentially unstable linear systems with saturating actuators. For a planar antistable system, the controllers are saturated linear state feedbacks and for higher order systems with two antistable modes, the controllers are piecewise linear state feedbacks with only one switch.

APPENDIX PROOF OF THEOREM 4.1

For simplicity and without loss of generality, we assume that

$$A = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix}, \quad a_1, a_2 > 0, \quad b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Since A is anti-stable and (A, b) is controllable, A, b can always be transformed into this form. Suppose that A has already taken this form and $b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Let $V = [-A^{-1}b \quad -b]$, then V is nonsingular and it can be verified that $V^{-1}AV = A$ and $V^{-1}b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

With this special form of A and b , we have

$$P = \begin{bmatrix} \frac{a_2}{a_1} & 0 \\ 0 & a_2 \end{bmatrix}$$

$f_0 = [0 \quad 2a_2]$, $A + kbf_0 = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2(1-2k) \end{bmatrix}$, $z_e^+ = -A^{-1}b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $z_e^- = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. We also have $f_0A^{-1}b = 0$.

For a given $k > 0.5$, (13) has a unique limit cycle which is the boundary of $S(k)$. To visualize the proof, ∂C and $\partial S(k)$ for some k are plotted in Fig. 6, where the inner closed curve is $\partial S(k)$, and the outer one is ∂C .

We recall that when the eigenvalues of A are real [see (6)]

$$\partial C = \left\{ \pm \left[e^{-At}z_e^- - \int_0^t e^{-A(t-\tau)}b d\tau \right] : t \in [0, \infty) \right\} \quad (23)$$

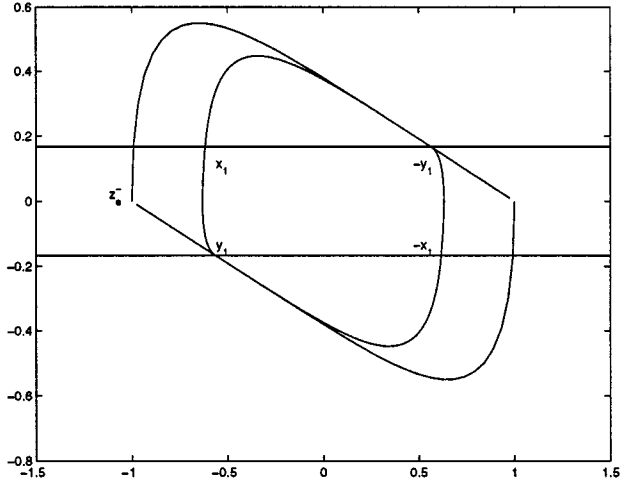


Fig. 6. The domain of attraction and the null controllable region.

and when the eigenvalues of A are complex [see (7)]

$$\partial C = \left\{ \pm \left[e^{-At} z_s^- - \int_0^t e^{-A(t-\tau)} b d\tau \right] : t \in [0, T_p] \right\}. \quad (24)$$

On the other hand, $\partial S(k)$ is the limit cycle of the time reversed system of (13),

$$\dot{z}(t) = -Az(t) - b\sigma(kf_0z(t)). \quad (25)$$

Here, the limit cycle as a trajectory goes clockwise. From the proof of Theorem 3.1, we know that the limit cycle is symmetric and has two intersections with $kf_0z = 1$ and two with $kf_0z = -1$, see Fig. 6. Let T be the time required for the limit cycle trajectory to go from y_1 to x_1 , and T_2 the time from x_1 to $-y_1$, then

$$\begin{aligned} \partial S(k) = & \left\{ \pm e^{-(A+kbf_0)t} y_1 : t \in [0, T] \right\} \\ & \cup \left\{ \pm \left[e^{-At} x_1 - \int_0^t e^{-A(t-\tau)} b d\tau \right] : t \in [0, T_2] \right\}. \end{aligned} \quad (26)$$

Here and in the sequel, the dependence of x_1 , y_1 , T and T_2 on k is omitted for simplicity.

As $k \rightarrow \infty$, the distance between the line $kf_0z = 1$ and $kf_0z = -1$ approaches zero. By comparing (23), (24) and (26), we see that to prove the theorem, it suffices to show

$$\begin{aligned} \lim_{k \rightarrow \infty} T &= 0, & \lim_{k \rightarrow \infty} x_1 &= \lim_{k \rightarrow \infty} y_1 = z_e^- \text{ (or } z_s^-) \\ \lim_{k \rightarrow \infty} T_2 &= \infty \text{ (or } T_p). \end{aligned}$$

In this case, the length of the part of the limit cycle between the lines $kf_0z = 1$ and $kf_0z = -1$ will tend to zero. We will first show that $\lim_{k \rightarrow \infty} T = 0$.

Let

$$x_1 = \begin{bmatrix} x_{11} \\ 1 \\ 2ka_2 \end{bmatrix}, \quad y_1 = \begin{bmatrix} y_{11} \\ -1 \\ -2ka_2 \end{bmatrix}$$

then $kf_0x_1 = 1$, $kf_0y_1 = -1$

$$\begin{bmatrix} x_{11} \\ 1 \\ 2ka_2 \end{bmatrix} = e^{-(A+kbf_0)T} \begin{bmatrix} y_{11} \\ -1 \\ -2ka_2 \end{bmatrix} \quad (27)$$

and

$$\left| kf_0 e^{-(A+kbf_0)t} \begin{bmatrix} y_{11} \\ -1 \\ -2ka_2 \end{bmatrix} \right| \leq 1, \quad \forall t \in [0, T].$$

We also note that the upward movement of the trajectory at x_1 and y_1 implies that $x_{11} < (2k-1)/2k$, $y_{11} < (1-2k)/2k$.

As $k \rightarrow \infty$, $A + kbf_0 = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2(1-2k) \end{bmatrix}$ has two distinct real eigenvalues $-\lambda_1$ and $-\lambda_2$. (Their dependence on k is also omitted.) Assume $\lambda_2 > \lambda_1$. Since $\lambda_1\lambda_2 = a_1$ and $\lambda_1 + \lambda_2 = a_2(2k-1)$, we have $\lim_{k \rightarrow \infty} \lambda_1 = 0$, $\lim_{k \rightarrow \infty} \lambda_2 = +\infty$.

With the special form of $A + kbf_0$, it can be verified that

$$e^{(A+kbf_0)T} = \begin{bmatrix} \lambda_2 & \lambda_1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 T} & 0 \\ 0 & e^{-\lambda_2 T} \end{bmatrix} \begin{bmatrix} \lambda_2 & \lambda_1 \\ 1 & 1 \end{bmatrix}^{-1}.$$

Hence, from (27), we obtain

$$\begin{aligned} x_{11} &= \frac{1}{2ka_2} \frac{\lambda_2 - \lambda_1 + \lambda_2 e^{-\lambda_2 T} - \lambda_1 e^{-\lambda_1 T}}{e^{-\lambda_2 T} - e^{-\lambda_1 T}} \\ y_{11} &= \frac{1}{2ka_2} \frac{\lambda_2 - \lambda_1 + \lambda_2 e^{\lambda_2 T} - \lambda_1 e^{\lambda_1 T}}{e^{\lambda_1 T} - e^{\lambda_2 T}}. \end{aligned}$$

Since $y_{11} < (1-2k)/2k = -(\lambda_1 + \lambda_2)/(2ka_2)$ and $e^{\lambda_1 T} - e^{\lambda_2 T} < 0$, we have

$$\lambda_1 e^{\lambda_2 T} < \lambda_2 - \lambda_1 + \lambda_2 e^{\lambda_1 T} < 2\lambda_2 e^{\lambda_1 T}$$

and

$$T < \frac{\ln \frac{2\lambda_2}{\lambda_1}}{\lambda_2 - \lambda_1} = \frac{1}{\lambda_2 - \lambda_1} \ln \frac{2\lambda_2^2}{a_1}$$

noting that $\lambda_1 = a_1/\lambda_2$. Since $\lim_{k \rightarrow \infty} \lambda_2 = \infty$, we get $\lim_{k \rightarrow \infty} T = 0$. It follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{y_{11}}{x_{11}} &= \lim_{k \rightarrow \infty} \frac{\lambda_2 - \lambda_1 + \lambda_2 e^{\lambda_2 T} - \lambda_1 e^{\lambda_1 T}}{(\lambda_2 - \lambda_1)e^{(\lambda_1 + \lambda_2)T} + \lambda_2 e^{\lambda_1 T} - \lambda_1 e^{\lambda_2 T}} \\ &= \lim_{k \rightarrow \infty} \frac{\lambda_2 - \lambda_1 \frac{1 + e^{\lambda_1 T}}{1 + e^{\lambda_2 T}}}{\lambda_2 e^{\lambda_1 T} - \lambda_1 \frac{e^{\lambda_2 T}(1 + e^{\lambda_1 T})}{1 + e^{\lambda_2 T}}} = 1 \end{aligned}$$

where we have used the fact that $\lim_{k \rightarrow \infty} \lambda_1 = 0$. Since x_1 and y_1 are bounded by the null controllable region, we have

$$\lim_{k \rightarrow \infty} (y_{11} - x_{11}) = 0. \quad (28)$$

On the limit cycle of (25), we also have

$$-y_1 = e^{-AT_2} x_1 - \int_0^{T_2} e^{-A(T_2-\tau)} b d\tau$$

i.e.,

$$\begin{aligned} \begin{bmatrix} y_{11} \\ -1 \\ -2ka_2 \end{bmatrix} &= -e^{-AT_2} \begin{bmatrix} x_{11} \\ 1 \\ 2ka_2 \end{bmatrix} + (I - e^{-AT_2})A^{-1}b \\ (I + e^{-AT_2}) \begin{bmatrix} y_{11} \\ 0 \end{bmatrix} &= (I - e^{-AT_2})A^{-1}b \\ &+ e^{-AT_2} \begin{bmatrix} y_{11} - x_{11} \\ -1 \\ -2ka_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2ka_2 \end{bmatrix}. \end{aligned}$$

It follows from (28) that

$$\lim_{k \rightarrow \infty} \left\{ \begin{bmatrix} y_{11} \\ 0 \end{bmatrix} - (I + e^{-AT_2})^{-1} (I - e^{-AT_2}) A^{-1} b \right\} = 0.$$

Hence

$$\lim_{k \rightarrow \infty} [0 \quad 1] (I + e^{-AT_2})^{-1} (I - e^{-AT_2}) A^{-1} b = 0.$$

For different cases, it can be shown from the above equality that

1) if the eigenvalues of A are real, then

$$\lim_{k \rightarrow \infty} T_2 = \infty, \quad \lim_{k \rightarrow \infty} y_1 = \lim_{k \rightarrow \infty} x_1 = \lim_{k \rightarrow \infty} \begin{bmatrix} y_{11} \\ 0 \end{bmatrix} = z_e^-;$$

2) if the eigenvalues of A are complex, then

$$\lim_{k \rightarrow \infty} T_2 = T_p, \quad \lim_{k \rightarrow \infty} y_1 = \lim_{k \rightarrow \infty} x_1 = \lim_{k \rightarrow \infty} \begin{bmatrix} y_{11} \\ 0 \end{bmatrix} = z_s^-.$$

This completes the proof. \square

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On Stabilization and Spectrum Assignment in Periodically Time-Varying Continuous-Time Systems

Joseph J. Yamé and Raymond Hanus

Abstract—This note discusses the stabilization and spectrum assignment problems in linear periodically time-varying (LPTV) continuous-time systems with sampled state or output feedback. The hybrid nature of the overall feedback system in this case imposes some carefulness in handling classical concepts related to purely LPTV continuous-time systems. In particular, this note points out the fact that the stabilization of such systems by periodic feedback gains with sampled state or output does not imply the relocation of the original characteristic exponents of the LPTV systems as stated previously in the literature. It is also shown that the concept of monodromy matrix as extended to LPTV hybrid systems has not all the features of a true monodromy matrix.

Index Terms—Characteristic exponents, monodromy matrix, periodic systems, sampled-data feedback.

I. INTRODUCTION

The foundations of the theory of periodic systems can be traced back to the work of Floquet who first brought the initial time-varying system into a transformed equivalent one with a time-invariant evolution matrix [9]. An important result of Floquet theory states that the stability of a linear periodically time-varying (LPTV) continuous-time system can be inferred from the location of the eigenvalues of the time-invariant matrix of its transformed equivalent. These eigenvalues are called the characteristic exponents or Poincaré exponents of the periodic system. In control engineering, the interest for continuous-time periodic systems is mainly motivated by numerous application-oriented problems such as sampled-data control, multirate digital control, generalized hold design, control of mechanical systems in rotation, etc. A central issue in the control literature centers around the stabilization of LPTV continuous-time systems by periodic controllers and essentially two different approaches have been used for the design of such stabilizing controllers. In the first approach [3], the input to the periodic controller is a continuous-time signal, whereas in the second approach, the input is a sampled signal. In this latter case, an important feature of the overall system is its hybrid continuous/discrete nature. It has been shown that when a LPTV system is controllable, the whole "monodromy matrix" is assignable by periodic feedback gains with sampled state or sampled output feedback [6], [1]. In [4], a further step has been taken by arguing that the *characteristic exponents* are all relocated with this feedback scheme. This note motivated by this last statement discusses issues pertaining to the *relationship* between the stabilization

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