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Stabilization of LTI Systems with Planar Anti-stable Dynamics Using Saturated Linear Feedback

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Abstract

In this paper, we first study the stabilization of an LTI anti-stable planar system with a saturated linear state feedback. We show that the domain of attraction of such a system under any saturated linear stabilizing feedback can be obtained easily by simulating the time-reversed closed-loop system. We then show that a saturated linear state feedback can be designed for such a system so that the equilibrium of the closed-loop system has a domain of attraction that is arbitrarily close to the null controllable region. Finally we present an extension of this result to general LTI systems with planar anti-stable dynamics.

1 Introduction

Two fundamental issues relating to the control of a system are its controllability and stabilizability. These issues are well-known to be difficult in the presence of input saturation even when the system itself is linear. They have been focuses of study of linear systems that have no poles in the open right half of the complex plane (we will call such systems semi-stable systems) and are now well addressed. For example, it is well-known [8, 10, 11] that such systems are globally null controllable with bounded controls as long as they are controllable in the usual linear system sense. Based on this fact, extensive literature has been devoted to the control of semi-stable systems using bounded control. In [12] and [14], nonlinear globally asymptotically stabilizing feedback laws were designed. Later, saturated linear state feedback laws were constructed so that the closed-loop system has a domain of attraction containing an arbitrarily prescribed bounded region, see, e.g., [5, 6, 7, 9]. In these papers, the feedback gains are kept small so that within a prescribed region of states, the control signal will not exceed the saturation level. It was also recognized that if the feedback is designed by the LQ method, then the feedback can be amplified by any positive gain while keeping the domain of attraction no smaller that the estimated one (a level set) achieved by the original feedback. This positive gain is then utilized to improve other performance of the closed loop system, see [4, 9].

The stabilization of an exponentially unstable system with saturated input is a much harder problem. Even the analysis problem of describing the domain of attraction of a closed loop system under a fixed control is not sufficiently addressed. Although a subset of the domain of attraction can be estimated, even some performance can be guaranteed within this subset, it is not clear how conservative this estimation is, nor is it clear how to enlarge this subset to meet the performance requirements.

We begin with the study of anti-stable planar systems (an LTI system is said to be anti-stable if it has all the poles in the open right half of the complex plane). We show that for such a system its domain of attraction under a saturated stabilizing linear state feedback can be easily obtained from the unique stable limit cycle of the time-reversed closed-loop system. Then a saturated linear state feedback is designed so that the domain of attraction is arbitrarily close to its null controllable region. Finally we show that for a higher order system with only two anti-stable modes, a switched saturated linear state feedback (with only one switch) can be designed so that the domain of attraction contains any prescribed compact subset of the null controllable region. The stabilization of general unstable systems with more than two anti-stable modes are left for future study.

2 Preliminaries and Notation

Consider a single input system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state and $u(t) \in \mathbf{R}$ is the control. A control signal u is said to be *admissible* if $|u(t)| \leq 1$ for all $t \geq 0$. In this paper, we are interested in stabilizing (1) with some simple control strategies such that the closed-loop system has a desired domain of attraction. Since the domain of attraction must lie within the null controllable region of (1), our objective

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in this paper is to make the domain of attraction close to the null controllable region. Recall that the null controllable region is defined as follows.

Definition 1

(a) *A state x_0 is said to be null controllable if there exists $T \in [0, \infty)$ and an admissible control u such that the state trajectory x of the system satisfies $x(0) = x_0$ and $x(T) = 0$.*

(b) *The set of all null controllable states is called the null controllable region of the system and is denoted by $\mathcal{C}$.*

For general systems, $\mathcal{C}$ has the following properties.

Proposition 1 *Assume that (A, b) is controllable.*

(a) *If A is semi-stable, then $\mathcal{C} = \mathbf{R}^n$.*

(b) *If A is anti-stable, then $\mathcal{C}$ is a bounded convex open set containing the origin.*

(c) *If $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ with $A_1 \in \mathbf{R}^{n_1 \times n_1}$ being anti-stable and $A_2 \in \mathbf{R}^{n_2 \times n_2}$ being semi-stable, and b is partitioned as $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ accordingly, then $\mathcal{C} = \mathcal{C}_1 \times \mathbf{R}^{n_2}$ where $\mathcal{C}_1$ is the null controllable region of the anti-stable system $\dot{x}_1(t) = A_1 x_1 + b_1 u(t)$.*

Statement (a) is well-known [8, 10, 11]. Statements (b) and (c) are proved in [2]. Because of this proposition, we only need to study the null controllable regions of anti-stable systems.

In the study of the null controllable region, the time-reversed system

$$\dot{z}(t) = -Az(t) - bv(t) \tag{2}$$

plays an important role. Denote the boundary of $\mathcal{C}$ as $\partial\mathcal{C}$. Now let us restrict our study to planar anti-stable systems. The following results, developed in [3], will be used.

Proposition 2 *Suppose that $A \in \mathbf{R}^{2\times 2}$ has two real positive eigenvalues. Then*

$$\partial\mathcal{C} = \left\{\pm(-2e^{-At} + I)A^{-1}b : \; t \in [0, \infty]\right\}; \tag{3}$$

suppose that $A \in \mathbf{R}^{2\times 2}$ has a pair of conjugate complex eigenvalues, $\alpha \pm j\beta, \alpha, \beta > 0$. Let $T_p = \frac{\pi}{\beta}$, $z_s^- = (I + e^{-AT_p})^{-1}(I - e^{-AT_p})A^{-1}b$. Then

$$\partial\mathcal{C} = \left\{\pm[e^{-At}z_s^- - (I - e^{-At})A^{-1}b] : \; t \in [0, T_p]\right\}. \tag{4}$$

The description of the null controllable regions paves the way for the study of stabilization of systems with input saturation. Consider the open loop system

$$\dot{x}(t) = Ax(t) + bu(t) \tag{5}$$

with admissible control $|u(t)| \leq 1$. A saturated linear state feedback is given by $u = \sigma(fx)$ where $f \in \mathbf{R}^{1\times n}$ is the feedback gain and $\sigma(\cdot)$ is the saturation function

$$\sigma(s) = \begin{cases} 1 & , \quad s \geq 1 \\ s & , \quad |s| < 1 \\ -1 & , \quad s \leq -1 \end{cases}.$$

Such a feedback is said to be stabilizing if $A + bf$ is stable. With a saturated linear state feedback applied, the closed-loop system is

$$\dot{x}(t) = Ax(t) + b\sigma[fx(t)]. \tag{6}$$

Denote the state transition map of (6) by $\phi : (t, x_0) \mapsto x(t)$. Then the domain of attraction $\mathcal{S}$ of the equilibrium $x = 0$ of (6) is defined by

$$\mathcal{S} = \left\{x_0 \in \mathbf{R}^n : \lim_{t\to\infty} \phi(t, x_0) = 0\right\}.$$

Obviously, $\mathcal{S}$ must lie within the null controllable region $\mathcal{C}$ of system (5). Therefore, a design problem is to choose the state feedback gain so that $\mathcal{S}$ is close to $\mathcal{C}$.

This seemingly simple task is actually quite nontrivial, even for semi-stable systems. In the past few years, extensive research has been reported on the stabilization of semi-stable plant, see, for example, [5, 6, 9, 12, 13, 14]. The problem for exponentially unstable systems is much harder. In this paper, we will first deal with planar anti-stable systems, then extend the results to higher order systems with only two anti-stable modes.

3 Domain of attraction

Consider system (6). Assume that $A \in \mathbf{R}^{2\times 2}$ and A is anti-stable. In this section, we analyze the domain of attraction of the equilibrium $x = 0$ of (6). In [1], it was shown that the boundary of $\mathcal{S}$, denoted by $\partial\mathcal{S}$, is a closed orbit, but no method to find this closed orbit is provided. Generally, only a subset of $\mathcal{S}$ lying between $fx = 1$ and $fx = -1$ is detected as a level set for some Lyapunov function, see, for example, [4]. Let P be a positive definite matrix such that $(A+bf)'P+P(A+bf)$ is negative definite and since $\{z \in \mathbf{R}^2 : -1 < fz < 1\}$ is an open neighborhood of the origin, it must contain

$$\mathcal{Q}_0 := \{z \in \mathbf{R}^2 : z'Pz \leq r_0\} \tag{7}$$

for some $r_0 > 0$. Clearly $\mathcal{Q}_0 \subset \mathcal{S}$. We will see latter that this estimate of the stability region can be very conservative (see, for example, Fig. 1).

Lemma 1 *[1] The origin is the unique equilibrium point of system (6).*

Let us introduce the time-reversed system of (6):

$$\dot{z}(t) = -Az(t) - b\sigma[fz(t)]. \tag{8}$$

Clearly (8) also has only one equilibrium point, an unstable one, at the origin. By applying the describing function method, one can predict that (6) and (8) have a unique limit cycle. Due to the approximate nature of the describing function analysis, we cannot expect it to give a definite answer. Further investigation shows that the prediction is indeed correct. Denote the state transition map of (8) by $\psi : (t, z_0) \mapsto z(t)$.

Theorem 1 *∂S is the unique limit cycle of systems (6) and (8). Furthermore, ∂S is the positive limit set of $\psi(\cdot, z_0)$ for all $z_0 \neq 0$.*

This theorem says that ∂S is the unique limit cycle of (6) and (8). This limit cycle is stable for (8) (in a global sense) but unstable for (6). Therefore, it is easy to determine ∂S by simulating the time-reversed system (8). See Fig. 1 for a typical result, where two trajectories, one starting from outside, the solid curve, and the other starting from inside, the dashed curve, both converge to the unique limit cycle. The straight lines in Fig. 1 are $fz = 1$ and $fz = -1$.

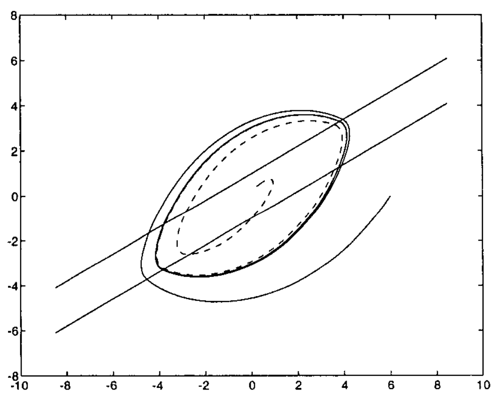

Figure 1: Determination of ∂S from the limit cycle

To prove Theorem 1, we need the following two lemmas, whose proofs are omitted due to space limitation.

Lemma 2 *Suppose that $A \in \mathbf{R}^{2\times 2}$ is anti-stable and (f, A) is observable. Given $c > 0$, let x_1, x_2, y_1 and y_2 ($x_1 \neq x_2$) be four points on the line $fx = c$, satisfying*

$$y_1 = e^{AT_1}x_1, \quad y_2 = e^{AT_2}x_2$$

for some $T_1, T_2 > 0$ and

$$fe^{At_1}x_1 \geq c,\ fe^{At_2}x_2 \geq c, \quad \forall t_1 \in [0, T_1],\ t_2 \in [0, T_2],$$

then $\|y_1 - y_2\| > \|x_1 - x_2\|$.

Lemma 2 indicates that if any two different trajectories leave a straight line on the same side, they will be further apart when they return to it.

Lemma 3 *: Suppose that $A \in \mathbf{R}^{2\times 2}$ is asymptotically stable and (f, A) is observable. Given $c > 0$, let x_1, x_2 be two points on the line $fx = c$ and y_1, y_2 be two points on $fx = -c$ such that*

$$y_1 = e^{AT_1}x_1, \quad y_2 = e^{AT_2}x_2$$

for some $T_1, T_2 > 0$, and

$$|fe^{At_1}x_1| \leq c,\ |fe^{At_2}x_2| \leq c, \ \ \forall t_1 \in [0, T_1],\ t_2 \in [0, T_2],$$

then $\|y_1 - y_2\| > \|x_1 - x_2\|$.

This lemma says that if two different trajectories of the autonomous system $\dot{x} = Ax$ enter the region between $fx = c$ and $fx = -c$, they will be further apart when they leave the region.

Proof of Theorem 1: We first prove that for system (8), every trajectory $\psi(t, z_0)$, $z_0 \neq 0$, converges to a limit cycle as $t \to \infty$. Recall that $\mathcal{Q}_0$(defined in (7)) lies within the domain of attraction of (6) and is an invariant set. It follows that, for every state $z_0 \neq 0$ of (8), there is some $t_0 \geq 0$ such that $\psi(t, z_0)$ lies outside $\mathcal{Q}_0$ for all $t \geq t_0$. The state transition map of system (8) is,

$$\psi(t, z_0) = e^{-At}z_0 - \int_0^t e^{-A(t-\tau)} b\sigma(fz(\tau))\mathrm{d}\tau \qquad (9)$$

Since $-A$ is stable, the first term converges to the origin. Since $|\sigma(fz(\tau))| \leq 1$, the second term belongs to $\mathcal{C}$, the null controllable region of (5), for all t. It follows that there exists an $r_1 > r_0$ such that $\psi'(t, z_0)P\psi(t, z_0) \leq r_1 < \infty$ for all $t \geq t_0$. Let $\mathcal{Q} = \{z \in \mathbf{R}^2 : r_0 \leq z'Pz \leq r_1\}$. Then $\psi(t, z_0)$, $t \geq t_0$, lies entirely in $\mathcal{Q}$. It follows from the Poincaré-Bendixon theorem that $\psi(t, z_0)$ converges to a limit cycle.

The preceding paragraph shows that (6) and (8) have limit cycles. We claim that system (6) and (8) each has only one limit cycle. For direct use of Lemma 2 and Lemma 3, we prove this claim through the original system (6).

First notice that a limit cycle must be symmetric to the origin Also, it cannot be completely contained in the linear region between $fx = 1$ and $fx = -1$. Hence it has to intersect each of the lines $fx = \pm 1$ at least twice. Assume without loss of generality that (f, A, b) is in the observer canonical form, i.e., $f = \begin{bmatrix} 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & -a_1 \\ 1 & a_2 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, with $a_1, a_2 > 0$, and denote $x = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$. In this case, $fx = \pm 1$ are horizontal lines. The stability of $A + bf$ requires that $-a_1 + b_1 < 0$ and $a_2 + b_2 < 0$. Observe that on the line $fx = 1$, we have $\xi_2 = 1$ and $\dot{\xi}_2 = \xi_1 + a_2 + b_2$. Hence, if $\xi_1 > -a_2 - b_2$, then $\dot{\xi}_2 > 0$, i.e., the trajectories go upwards; if $\xi_1 < -a_2 - b_2$, then $\dot{\xi}_2 < 0$, i.e., the trajectories go downwards. This implies that any limit cycle crosses $fx = 1$ exactly twice and similarly for $fx = -1$. It also implies that a limit cycle goes anti-clockwisely.

Now suppose on the contrary that (6) has two different limit cycles Γ_1 and Γ_2, with Γ_1 enclosed by Γ_2, as illustrated in Fig. 2. Note that (6) has only one equilibrium point. Hence all its limit cycles must be ordered by enclosement. Let x_1 and y_1 be the two intersections of Γ_1 with $fx = 1$, and x_2, y_2 be the two intersections of Γ_2 with $fx = 1$. Then along Γ_1, the trajectory goes from x_1 to $y_1, -x_1, -y_1$ and returns to x_1; and along Γ_2, the trajectory goes from x_2 to $y_2, -x_2, -y_2$ and returns to x_2.

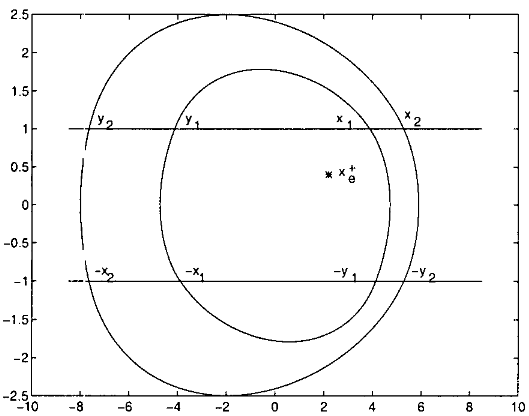

Figure 2: Illustration for the proof of Theorem 1

Let $x_e^+ = -A^{-1}b$. Since $x_1 \to y_1$ along Γ_1 and $x_2 \to y_2$ along Γ_2 are on trajectories of $\dot{x} = Ax + b$ (or $\mathrm{d}(x - x_e^+)/\mathrm{d}t = A(x - x_e^+)$), we have

$$y_1 - x_e^+ = e^{AT_1}(x_1 - x_e^+), \quad y_2 - x_e^+ = e^{AT_2}(x_2 - x_e^+)$$

for some $T_1, T_2 > 0$. Furthermore, $f(x_1 - x_e^+) = f(x_2 - x_e^+) = f(y_1 - x_e^+) = f(y_2 - x_e^+) = 1 - fx_e^+ > 0$ (since $fx_e^+ = \frac{b_1}{a_1} < 1$) and for all x on the two pieces of trajectories, $f(x - x_e^+) \geq 1 - fx_e^+$. It follows from Lemma 2 that

$$\|y_2 - y_1\| > \|x_2 - x_1\|.$$

On the other hand, $y_1 \to -x_1$ along Γ_1 and $y_2 \to -x_2$ along Γ_2 are on trajectories of $\dot{x} = (A + bf)x$ satisfying $-x_1 = e^{(A+bf)T_3}y_1$ and $-x_2 = e^{(A+bf)T_4}y_2$ for some $T_3, T_4 > 0$. It follows from Lemma 3 that

$$\|x_2 - x_1\| > \|y_2 - y_1\|,$$

which is a contradiction. Therefore, Γ_1 and Γ_2 must be the same limit cycle.

We have so far proven that every trajectory $\psi(t, z_0), z_0 \neq 0$ of (8) converges to a unique limit cycle. This implies that a trajectory $\phi(t, x_0)$ of (6) converges to the origin if and only if x_0 is inside the limit cycle. This shows that the limit cycle is $\partial\mathcal{S}$. □

Moreover, it can be shown that $\mathcal{S}$ has the following nice feature.

Proposition 3 *$\mathcal{S}$ is convex.*

4 Semi-Global Stabilization on the Null Controllable Region

In this section, we examine issues related to semi-global asymptotic stabilization on the null controllable region of linear systems with saturating actuators.

4.1 Second order anti-stable systems

In this subsection, we continue to assume that $A \in \mathbf{R}^{2\times 2}$ is anti-stable and (A, b) is controllable. To state the main result of this section, we need to introduce the Hausdorff distance. Let $\mathcal{X}_1, \mathcal{X}_2$ be two bounded subsets of $\mathbf{R}^n$. Then their Hausdorff distance is defined as:

$$d(\mathcal{X}_1, \mathcal{X}_2) := \max\{\vec{d}(\mathcal{X}_1, \mathcal{X}_2), \vec{d}(\mathcal{X}_2, \mathcal{X}_1)\},$$

where

$$\vec{d}(\mathcal{X}_1, \mathcal{X}_2) = \sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} \|x_1 - x_2\|.$$

Here the vector norm used is arbitrary.

Let P be the unique positive definite solution of the following Riccati equation,

$$A'P + PA - Pbb'P = 0. \tag{10}$$

Note that this equation is associated with the minimum energy regulation, i.e., an LQR problem with cost

$$J = \int_0^\infty u'(t)u(t)dt.$$

The corresponding minimum energy state feedback gain is given by $f_0 = -b'P$. The origin is a stable equilibrium of system

$$\dot{x}(t) = Ax(t) + b\sigma(kf_0x(t)) \tag{11}$$

for all $k > 0.5$. Let $\mathcal{S}(k)$ be the domain of attraction of the equilibrium $x = 0$ of (11).

Theorem 2 $\lim_{k\to\infty} d(\mathcal{S}(k), \mathcal{C}) = 0$.

This shows that a second order anti-stable linear system can be semi-globally asymptotically stabilized on its null controllable region by saturated linear feedback.

Note that the use of high gain feedback is crucial here. The minimum energy feedback f_0 itself does not give a domain of attraction that is close to $\mathcal{C}$. This is quite different from the related result in [5, 7] for semi-stable systems. In these two papers, it was shown that if A is semi-stable and (A, b) controllable (of arbitrary dimension), then a near minimum energy feedback gives an arbitrarily large domain of attraction.

Example 1 Let $A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Then $f_0 = \begin{bmatrix} 0 & 3 \end{bmatrix}$. In Fig. 3, the boundaries of the domains of attraction corresponding to different $f = kf_0$, $k = 0.50005, 0.65, 1, 3$, are plotted. The regions do become bigger for greater k. The outermost boundary is $\partial\mathcal{C}$. When $k = 3$, it can be seen that $\partial\mathcal{S}$ is already very close to $\partial\mathcal{C}$.

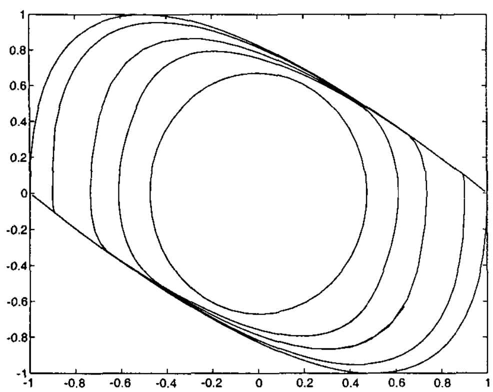

Figure 3: Domains of attraction under different feedbacks

4.2 Higher order systems with two exponentially unstable poles

Consider the following open-loop system

$$\dot{x}(t) = Ax(t) + bu(t) = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \quad (12)$$

where $x = \begin{bmatrix} x_1' & x_2' \end{bmatrix}'$, $x_1 \in \mathbf{R}^2, x_2 \in \mathbf{R}^n$, $A_1 \in \mathbf{R}^{2\times 2}$ is anti-stable and $A_2 \in \mathbf{R}^n$ is semi-stable. Assume that (A, b) is controllable. Denote the null controllable region of the subsystem

$$\dot{x}_1(t) = A_1 x_1(t) + b_1 u(t)$$

as $\mathcal{C}_1$, then the null controllable region of (12) is $\mathcal{C}_1 \times \mathbf{R}^n$. Given $\gamma_1, \gamma_2 > 0$, denote

$$\begin{aligned} \Omega_1(\gamma_1) &:= \{\gamma_1 x_1 \in \mathbf{R}^2 : \ x_1 \in \overline{\mathcal{C}}_1\} \\ \Omega_2(\gamma_2) &:= \{x_2 \in \mathbf{R}^n : \ \|x_2\| \le \gamma_2\}. \end{aligned}$$

When $\gamma_1 = 1$, $\Omega_1(\gamma_1) = \overline{\mathcal{C}}_1$ and when $\gamma_1 < 1$, $\Omega_1(\gamma_1)$ lies in the interior of $\mathcal{C}_1$. In this section, we will show that given any $\gamma_1 < 1$ and $\gamma_2 > 0$, a state feedback can be designed such that $\Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$ is contained in the domain of attraction of the equilibrium $x = 0$ of the closed-loop system.

For $\epsilon > 0$, let $P(\epsilon) = \begin{bmatrix} P_1(\epsilon) & P_2(\epsilon) \\ P_2'(\epsilon) & P_3(\epsilon) \end{bmatrix} \in \mathbf{R}^{(2+n)\times(2+n)}$ be the unique positive definite solution to the ARE

$$A'P + PA - Pbb'P + \epsilon^2 I = 0. \quad (13)$$

Clearly, as $\epsilon \downarrow 0$, $P(\epsilon)$ decreases. Hence $\lim_{\epsilon\to 0} P(\epsilon)$ exists. Let P_1 be the unique positive definite solution to the ARE

$$A_1'P_1 + P_1A_1 - P_1b_1b_1'P_1 = 0.$$

Then by the continuity property of the solution of the Riccati equation [16],

$$\lim_{\epsilon\to 0} P(\epsilon) = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $f(\epsilon) := -b'P(\epsilon)$. First, consider the domain of attraction of the equilibrium $x = 0$ of the following closed-loop system

$$\dot{x}(t) = Ax(t) + b\sigma(f(\epsilon)x(t)). \quad (14)$$

It is easy to see that

$$D(\epsilon) := \left\{x \in \mathbf{R}^{2+n} : \ x'P(\epsilon)x \le 1/\|b'P^{\frac{1}{2}}(\epsilon)\|^2\right\}.$$

is contained in the domain of attraction of the equilibrium $x = 0$ of (14) and is an invariant set.

Theorem 3 *Let $f_0 = -b_1P_1$. For any $\gamma_1 < 1$ and $\gamma_2 > 0$, there exist $k > 0.5$ and $\epsilon > 0$ such that $\Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$ is contained in the domain of attraction of the equilibrium $x = 0$ of the following closed-loop system*

$$\dot{x}(t) = Ax(t) + bu(t), \ u(t) = \begin{cases} \sigma(kf_0x_1(t)), & x \notin D(\epsilon) \\ \sigma(f(\epsilon)x(t)), & x \in D(\epsilon) \end{cases} \quad (15)$$

Example 2 Consider an open-loop system described by (12) with

$$A = \begin{bmatrix} 0.6 & -0.8 & 0 & 0 & 0 \\ 0.8 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

The desired domain of attraction is $\Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$ with $\gamma_1 = 0.9, \gamma_2 = 10$.

The design result is,

$$\begin{aligned} kf_0 &= \begin{bmatrix} 0.1360 & -0.748 \end{bmatrix}, \\ f(\epsilon) &= \begin{bmatrix} 0.120097 & -0.660525 & 0 & 0.000949 & 0 \end{bmatrix}. \end{aligned}$$

Fig. 4 is the time response $x_3(t)$ of (15) with an initial state $x_0 = \begin{bmatrix} 4.7005 & 0.70001 & 10 & 0 & 0 \end{bmatrix}'$, which is on the boundary of $\Omega_1(\gamma_1) \times \Omega_2(\gamma_2)$. The figure shows that the convergence is very slow. This is because $A + bf(\epsilon)$ has a pair of eigenvalues very close to the imaginary axis, $-0.0005 \pm j1$.

The convergence can be accelerated after the state enters $D(\epsilon)$ by applying the piecewise-linear control(PLC) law of [15]. The idea is as follows: select

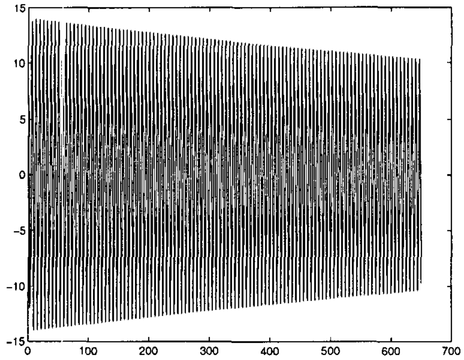

Figure 4: Time response of x_3

a chain of ϵ_i, $\epsilon_N > \epsilon_{N-1} > \cdots > \epsilon_1 > \epsilon$, compute $P(\epsilon_i)$, $f(\epsilon_i)$, then

$$D(\epsilon_i) = \left\{ x \in \mathbf{R}^{2+n} : x'P(\epsilon_i)x \leq \frac{1}{||b'P^{\frac{1}{2}}(\epsilon_i)||^2} \right\}$$

is a sequence of nested invariant sets corresponding to each feedback control $u(t) = \sigma(f(\epsilon_i)x(t))$, i.e.,

$$D(\epsilon_N) \subset D(\epsilon_{N-1}) \subset \cdots \subset D(\epsilon_1) \subset D(\epsilon).$$

With the following multiple switching control law,

$$u(t)=\begin{cases} \sigma(f(\epsilon_N)x(t)), & \text{if } x(t) \in D(\epsilon_N) \\ \sigma(f(\epsilon_{N-1})x(t)), & \text{if } x(t) \in D(\epsilon_{N-1}),\ x(t) \notin D(\epsilon_N) \\ \vdots & \vdots \\ \sigma(f(\epsilon)x(t)), & \text{if } x(t) \in D(\epsilon),\ x(t) \notin D(\epsilon_1) \\ \sigma(kf_0x_1(t)), & \text{if } x(t) \notin D(\epsilon) \end{cases}$$

the convergence rate is increased. Applying the above control (N=20) to the previous example with the same initial state, the time response of x_3 is plotted in Fig. 5.

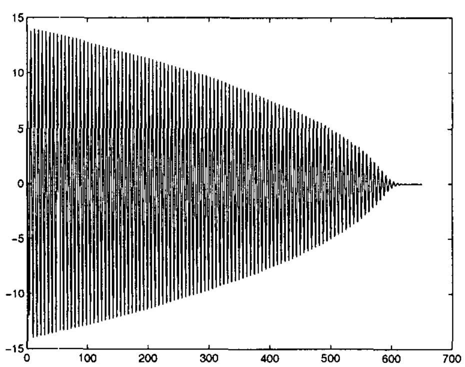

Figure 5: Time-response of x_3 under multiple switching control

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