Chapter 16

Stabilization of multi-input networked control systems over additive white Gaussian noise channels

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Abstract

In this paper, we study stabilization of multi-input networked control systems over additive white Gaussian noise (AWGN) channels. Different from the single-input case, which is available in the literature and boils down to a typical $\mathcal{H}_2$ optimal control problem, the multi-input case involves a judicious allocation of the total capacity among the input channels in addition to the design of the feedback controller. With this channel-controller codesign, we successfully show that a networked multi-input system over AWGN channels can be stabilized by state feedback under channel resource allocation, if and only if the total channel capacity is greater than the topological entropy of the plant. A numerical example is given to demonstrate our result.

16.1 Introduction

The networked control systems (NCSs) have received great attention recently. They are feedback systems in which the plant and controller communicate through the shared network. Such systems have many applications, including mobile sensor networks [1], multiagent systems [2], aerial space technologies [3], etc. Many papers on this topic have been published in technical journals and conferences. See the special issues [4], [5], and the references therein.

One fundamental issue studied in the context of NCS is stabilization under information constraints due to communication channels. These constraints take various forms, such as quantization [6,7], packet drop [8], data rate constraint [9], signal-to-noise ratio (SNR) [10] constraint, and so on. Numerous results for stabilization of NCSs under information constraints are reported in the literature. For single-input NCSs, logarithmic quantization of the control inputs was considered in References 6 and 7 that show that the coarsest quantization density ensuring closed-loop stabilizability is given in terms of the Mahler measure of the plant, that is, the absolute product of the unstable poles. The multiplicative stochastic input channel has been studied in Reference 8 that states that the networked feedback system can be mean-square stabilized by state feedback, if and only if the mean-square capacity of the multiplicative channel exceeds the topological entropy of the plant, that is, the logarithm of the Mahler measure. For multi-input NCSs, the authors of Reference 11 model the information constraint in the input channels as general sector uncertainties including the logarithmic quantization as a special case. Their main contribution lies in introducing the channel resource allocation and solving the networked stabilization problem. Specifically, they assume that the allowable information constraint is determined by the total network resource available to the channels that can be allocated by the controller designer. Thanks to the additional design freedom gained by the channel resource allocation, an analytical solution has been obtained, which states that the largest overall uncertainty bound rendering stabilization is given again in terms of the Mahler measure. In Reference 12, the multi-input NCSs over multiplicative stochastic channels are studied. With the help of channel resource allocation, its authors extend the stabilizability condition in Reference 8 to the multi-input case. These results shed some light on the significance and role of channel resource allocation in NCSs, entailing the idea of channel-controller codesign, that is, the control designer should participate in the channel design rather than passively taking the given channels. This idea will bring us substantially more freedom and flexibility in designing NCSs and is envisioned to be common practice in future engineering applications. Later, one can see that our main result in this paper can be obtained by allocating the channel resource judiciously.

Another line of work [10], that is most pertinent to our work in this paper, models the information constraint for a single-input NCS as the SNR constraint in an additive white Gaussian noise (AWGN) channel. The technique of $\mathcal{H}_2$ optimal control is used to design the stabilizing controller. A nice analytic solution is obtained for the minimum channel capacity required to stabilize the NCS, which is also given in terms of the topological entropy of the plant. The authors in References 13–15 have studied further the disturbance attenuation issue for NCS over an AWGN channel. These papers show that the requirement for the channel capacity greater than the topological entropy of the plant remains to be necessary for feedback stabilization, even if nonlinear time-varying communication and control laws are used. One interesting observation from the literature is that the NCS stabilization problem over an AWGN channel is closely related to some nonstandard $\mathcal{H}_2$ optimal control problem. This fact will be seen in this paper when we derive our result later. For the multi-input NCSs over the AWGN channels, unfortunately, the existing results remain to be quite incomplete. An investigation is carried out in Reference 16, which assumes that the total transmission power is constrained and can be distributed among different channels, leading to a necessary and sufficient stabilization condition on the transmission power. Different from the result in Reference 10 that is given directly in terms of the topological entropy of the plant, the condition in Reference 16 involves unpleasant computation of the $\mathcal{H}_2$ norm of a transfer function.

Motivated by these existing results, we study further stabilization of a multi-input NCS over the AWGN channels in this paper. Instead of assuming the constrained total transmission power, we assume that the total capacity of the input channels are constrained and can be allocated among different channels. By allocating the channel resource, we successfully derive the minimum total capacity required for stabilization given also by the topological entropy of the plant.

The remainder of this paper is organized as follows. Section 16.2 formulates the NCS problem to be studied in this paper, and section 16.3 provides some preliminary results on $\mathcal{H}_2$ optimal control. The main result is stated and proved in section 16.4. A numerical example is worked out in section 16.5 to illustrate our main result. The paper is concluded in section 16.6. The notation of this paper is more or less standard and will be made clear as we proceed.

16.2 Problem formulation

We consider a discrete-time system described by state-space equation

$$x(k+1) = Ax(k) + Bu(k)$$

where $u(k) \in \mathbb{R}^m$ and $x(k) \in \mathbb{R}^n$. We will denote this system by $[A|B]$ for simplicity. Assume that $[A|B]$ is stabilizable and the state variable $x(k)$ is available for feedback control. As shown in Figure 16.1, we are interested in stabilizing $[A|B]$ by a constant state feedback controller F over a communication network that is modeled as m parallel AWGN input channels. Here, by parallel, we mean that each component of the controller output is separately sent through an independent AWGN channel to the actuator. One of these AWGN channels is depicted in Figure 16.2, where the transmitted signal v_i and the noise d_i are zero mean Gaussian random processes with variances $\tilde{\sigma}_i^2$ and σ_i^2, respectively. Different from the classical setup in LQG control where the noise comes from outside with fixed power, the noise considered here is generated internally from the transmission process. The noise power is

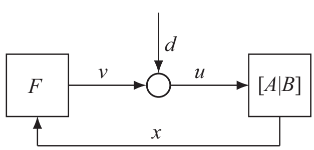

Figure 16.1 Networked control system over AWGN channels

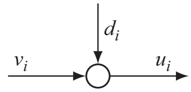

Figure 16.2 An additive Gaussian channel

proportional to the transmission power with proportional coefficient given by the SNR of the channel [17]:

$$\mathrm{SNR}_i = \frac{\tilde{\sigma}_i^2}{\sigma_i^2} \tag{16.1}$$

We will come back to this point when we do the channel resource allocation to derive the main result in section 16.4. The capacity of the channel in Figure 16.2 is given by

$$\mathfrak{C}_i = \frac{1}{2}\log(1 + \mathrm{SNR}_i)$$

Then the total capacity of the communication network is as follows:

$$\mathfrak{C} = \mathfrak{C}_1 + \cdots + \mathfrak{C}_m$$

Clearly, the larger capacity, or equivalently the larger SNR, implies that more reliable information can be transmitted through the channel. Therefore, the capacity $\mathfrak{C}_i$ measures the information constraint of the ith channel and the total capacity $\mathfrak{C}$ measures the information constraint of the communication network.

Assume that all the signals in Figure 16.1 are wide sense stationary and the closed-loop system has reached its steady state. According to our setup, the total noise d is a vector white Gaussian noise with covariance

$$\Sigma^2 = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}$$

The closed-loop transfer function from the noise d to the controller output v is the complimentary sensitivity function

$$T(z) = F(zI - A - BF)^{-1}B$$

Then the power spectrum density of v_i is given by

$$\{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii}$$

and the mean power of v_i is

$$\frac{1}{2\pi}\int_0^{2\pi} \{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii} d\omega$$

where $\{\cdot\}_{ii}$ stands for the ith diagonal element of the matrix. In view of (16.1), the SNR of channel i is expressed as

$$\begin{aligned}\mathrm{SNR}_i &= \frac{1}{2\pi}\int_0^{2\pi} \{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii} d\omega / \sigma_i^2 \\ &= \frac{1}{2\pi}\int_0^{2\pi} \{\Sigma^{-1} T(e^{j\omega})\Sigma^2 T(e^{j\omega})^* \Sigma^{-1}\}_{ii} d\omega\end{aligned}$$

Consequently, the capacity of channel i is given by

$$\mathfrak{C}_i = \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}$$

Finally, the total channel capacity is

$$\begin{aligned}\mathfrak{C} &= \mathfrak{C}_1 + \cdots + \mathfrak{C}_m \\ &= \frac{1}{2}\log\prod_{i=1}^{m}\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii}\end{aligned}$$

Our objective is to find the smallest total channel capacity such that the NCS over AWGN channels can be stabilized by a constant state feedback controller, that is, to find

$$\inf_{F:A+BF \text{ is stable}} \mathfrak{C} \tag{16.2}$$

with given $[A|B]$ and $\sigma_1, \ldots, \sigma_m > 0$. This is a difficult problem. However, by judiciously allocating the channel resource, we are able to mitigate this difficulty and derive the same nice analytic solution as in Reference 10 derived for the single-input case. For this purpose, instead of imposing the information constraint on the input channels specified *a priori*, we assume that the channel capacities $\mathfrak{C}_i$, $i = 1, \ldots, m$, can be allocated with a given total capacity $\mathfrak{C}$. This assumption is quite legitimate and can be utilized as an extra design freedom for the NCS. For example, the total channel resource or budget for the m input communication channels is often fixed. Allocating more resource to a certain channel will increase its reliability. How to allocate the channel resource appropriately for control of NCS can be considered as a case of channel-controller codesign. The controller designer should simultaneously design the controller and channels to stabilize the closed-loop feedback system. Applying this channel-controller codesign gives rise to the following minimization problem:

$$\inf_{\sigma_1,\ldots,\sigma_m > 0}\ \inf_{F:A+BF \text{ is stable}} \mathfrak{C} \tag{16.3}$$

that is, the infimum of the total channel capacity required for networked stabilization with channel resource allocation. At first sight, this problem looks even harder than problem (16.2). However, surprisingly, it can be analytically solved, as shown in the remainder of this paper.

Before proceeding, let us recall two notions that were introduced to dynamical systems theory long time ago but only appeared in the control literature recently. One is the Mahler measure [18] of an $n \times n$ matrix A, denoted by $M(A)$, which is simply the absolute value of the product of the unstable eigenvalues of A, that is, $M(A) = \Pi_{i=1}^{n} \max\{1, |\lambda_i(A)|\}$. The second is the topological entropy [19] of A, denoted by $h(A)$, which is simply the logarithm of $M(A)$, that is, $h(A) = \log M(A)$.

16.3 Preliminary on $\mathcal{H}_2$ optimal control

As discussed in the previous section, the NCS stabilization problem over AWGN channels is closely related to some nonstandard $\mathcal{H}_2$ optimal control problem. To find the solution to (16.3), the following lemma on optimal complementary sensitivity will be needed.

Lemma 16.1 *Assume that $[A|B]$ is stabilizable. Then*

$$\inf_{F:A+BF \text{ is stable}} \frac{1}{2}\log\det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega})d\omega\right) = h(A) \tag{16.4}$$

Furthermore, if A has no eigenvalues on the unit circle, then the unique optimal controller F is given by

$$F = -B'X(I+BB'X)^{-1}A$$

where X is the unique stabilizing solution of Riccati equation

$$A'X(I+BB'X)^{-1}A = X \tag{16.5}$$

Proof Consider the feedback system shown in Figure 16.3, where F is a state feedback gain and $d_0\delta(k+1)$ is an impulse with an arbitrary direction d_0 that stimulates the system. So the initial state is given by $x_0 = Bd_0$.

Assume temporarily that A has no eigenvalues on the unit circle. Solving the minimum energy control problem that is a special $\mathcal{H}_2$ optimal control problem with cost function $J = \|v\|_2^2$, we get the minimum energy [20]

$$J^* = x_0'Xx_0 = d_0'B'XBd_0 \tag{16.6}$$

where X is the stabilizing solution to Riccati equation (16.5). The optimal state feedback gain is given by $F = -B'X(I+BB'X)^{-1}A$. Since $v(z) = T(z)d_0$, we have

$$\begin{aligned} J &= \frac{1}{2\pi}\int_0^{2\pi} v(e^{j\omega})^* v(e^{j\omega})d\omega \\ &= d_0'\left(\frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega})d\omega\right)d_0 \end{aligned} \tag{16.7}$$

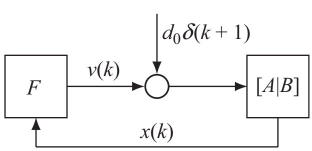

Figure 16.3 Minimum energy control

Comparing (16.6) with (16.7) implies that the partially ordered set

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega}) d\omega : A + BF \text{ is stable} \right\}$$

has an infimum that is given by

$$\inf_{F:A+BF \text{ is stable}} \frac{1}{2\pi} \int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega}) d\omega = B'XB$$

Without loss of generality, we can assume that

$$[A|B] = \left[\begin{array}{cc|c} A_s & 0 & B_s \\ 0 & A_u & B_u \end{array} \right]$$

where A_s is stable and A_u is anti-stable. Then by the existence and uniqueness of solution to (16.5), the solution satisfies

$$X = \begin{bmatrix} 0 & 0 \\ 0 & X_u \end{bmatrix}$$

where X_u is the stabilizing solution to

$$A_u' X_u (I + B_u B_u' X_u)^{-1} A_u = X_u \tag{16.8}$$

Moreover, $X_u > 0$ and has a closed form expression

$$X_u = \left(\sum_{k=1}^{\infty} A_u^{-k} B_u B_u' A_u'^{-k} \right)^{-1}$$

Taking determinant on both sides of (16.8), we get

$$\begin{aligned} &\det(A_u' X_u (I + B_u B_u' X_u)^{-1} A_u) \\ &\quad = \det(A_u' A_u) \det(X_u) \det(I + B_u B_u' X_u)^{-1} \\ &\quad = \det(X_u) \end{aligned}$$

Since $X_u > 0$, it follows that

$$\det(I + B_u B_u' X_u) = \det(A_u' A_u) = M(A_u)^2$$

Therefore,

$$
\begin{aligned}
&\inf_{F:A+BF \text{ is stable}} \frac{1}{2}\log\ \det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})^* T(e^{j\omega})d\omega\right)\\
&\quad=\frac{1}{2}\log\ \det(I+B'XB)\\
&\quad=\frac{1}{2}\log\ \det(I+B_{\mathrm{u}}'X_{\mathrm{u}}B_{\mathrm{u}})\\
&\quad=\frac{1}{2}\log\ \det(I+B_{\mathrm{u}}B_{\mathrm{u}}'X_{\mathrm{u}})\\
&\quad=h(A_{\mathrm{u}})=h(A)
\end{aligned}
$$

If A has eigenvalues on the unit circle, let $A(\epsilon)=(1+\epsilon)A$ with $\epsilon>0$ such that $A(\epsilon)$ has the same number of eigenvalues inside the unit circle as A but no eigenvalues on the unit circle. Applying the above procedure and taking limit $\epsilon\to 0$ shows that eigenvalues on the unit circle do not affect the infimum. □

In our application, however, we are more interested in a performance index with the order of $T(e^{j\omega})$ and $T(e^{j\omega})^*$ in (16.4) reversed. See the following lemma.

Lemma 16.2 *Assume that $[A|B]$ is stabilizable. Then*

$$
\inf_{F:A+BF \text{ is stable}} \frac{1}{2}\log\ \det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})T(e^{j\omega})^* d\omega\right)\ge h(A) \tag{16.9}
$$

Proof For an arbitrary F such that $A+BF$ is stable, the matrix $A'+F'B'$ is also stable. This implies that the system $[A'|F']$ is stabilizable and, moreover, B' is a stabilizing state feedback gain. In this case, the complementary sensitivity function of system $[A'|F']$ is $T'(z)=B'(zI-A'-F'B')^{-1}F'$. According to Lemma 16.1,

$$
\begin{aligned}
&\frac{1}{2}\log\ \det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T'(e^{j\omega})^* T'(e^{j\omega})d\omega\right)\\
&\quad=\frac{1}{2}\log\ \det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T(e^{-j\omega})T(e^{-j\omega})^* d\omega\right)\\
&\quad=\frac{1}{2}\log\ \det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})T(e^{j\omega})^* d\omega\right)\\
&\quad\ge h(A)
\end{aligned}
$$

Since the choice of stabilizing F is arbitrary, it follows that

$$
\inf_{F:A+BF \text{ is stable}} \frac{1}{2}\log\ \det\left(I+\frac{1}{2\pi}\int_0^{2\pi} T(e^{j\omega})T(e^{j\omega})^* d\omega\right)\ge h(A)
$$

which concludes the proof. □

One can observe that when $T(e^{j\omega})$ is normal, that is, $T(e^{j\omega})\ T(e^{j\omega})^*=T(e^{j\omega})^*\ T(e^{j\omega})$ for all $\omega\in[0,2\pi)$, the left-hand side of (16.9) is the same as that of (16.4), and therefore the equality in (16.9) holds. It is natural to ask whether the equality holds in general. At this moment, we are not sure about this. Nevertheless, our guess is that the answer is negative.

In the single-input case, the left-hand sides of (16.4) and (16.9) are the same and they are equivalent to a standard $\mathcal{H}_2$ optimization problem.

Lemma 16.3 *When* $m = 1$,

$$\inf_{F:A+BF \text{ is stable}} \|T(z)\|_2 = [M(A)^2 - 1]^{1/2}$$

Proof The proof follows directly from Lemma 16.1. □

16.4 Main result

The main result of this paper is presented in the following theorem.

Theorem 16.1.

$$\inf_{\sigma_1,\ldots,\sigma_m > 0} \inf_{F:A+BF \text{ is stable}} \mathfrak{C} = h(A)$$

Proof To simplify the proof, we assume that A has no eigenvalues on the unit circle. This assumption can be removed following the same argument as in [10]. Without loss of generality, realization matrices (A, B, F) are assumed to have the following decomposition:

$$A = \begin{bmatrix} A_{\mathrm{s}} & 0 \\ 0 & A_{\mathrm{u}} \end{bmatrix}, \quad B = \begin{bmatrix} B_{\mathrm{s}} \\ B_{\mathrm{u}} \end{bmatrix}, \quad F = [\,F_{\mathrm{s}} \quad F_{\mathrm{u}}\,]$$

with compatible partition, where A_{s} is stable and A_{u} is anti-stable. As in the single-input case, $F_{\mathrm{s}} = 0$ can be taken in minimizing the capacity [10], and thus

$$T(z) = F_{\mathrm{u}}(zI - A_{\mathrm{u}} - B_{\mathrm{u}}F_{\mathrm{u}})^{-1}B_{\mathrm{u}}$$

can also be assumed in the proof. Consequently, we simply assume that A is anti-stable.

First, we prove that for a noise with given covariance Σ^2 and a stabilizing state feedback gain F, the total channel capacity $\mathfrak{C} \geq h(A)$. Denote $\tilde{B} = B\Sigma$ and $\tilde{F} = \Sigma^{-1}F$, then $[A|\tilde{B}]$ is stabilizable and $\tilde{F}$ is a stabilizing gain for this system. Let $\tilde{T}(z) = \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B}$. By Lemma 16.2, we have

$$\frac{1}{2}\log\,\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} \tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^* d\omega\right) \geq h(A)$$

which is equivalent to

$$\frac{1}{2}\log\,\det\left(I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1} d\omega\right) \geq h(A)$$

Therefore,

$$
\begin{aligned}
\mathfrak{C} &= \frac{1}{2}\log \prod_{i=1}^{m} \left\{ I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1} T(e^{j\omega})\Sigma^2 T(e^{j\omega})^* \Sigma^{-1} d\omega \right\}_{ii} \\
&\geq \frac{1}{2}\log \det\left(I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1} T(e^{j\omega})\Sigma^2 T(e^{j\omega})^* \Sigma^{-1} d\omega \right) \\
&\geq h(A)
\end{aligned}
$$

where the first inequality follows directly from Hadamard's inequality [21]: for any $m \times m$ positive definite matrix $Q = [q_{ij}]$, $\det(Q) \leq \Pi_{i=1}^{m} q_{ii}$. The equality holds if and only if Q is diagonal.

Without loss of generality, assume that $[A|B]$ has the following Wonham decomposition [22]:

$$
A = \begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix}
$$

where each pair $[A_i|b_i]$ is stabilizable with state dimension n_i. Now we show that for any $\epsilon > 0$, if the total capacity constraint is given by $h(A) + \epsilon$, then one can find an allocation of this constraint among the input channels in the form $\left\{h(A_1) + \frac{\epsilon}{m}, \ldots, h(A_m) + \frac{\epsilon}{m}\right\}$ and simultaneously design a state feedback gain F such that the closed-loop system is stable and each channel capacity satisfies the constraint $\mathfrak{C}_i < h(A_i) + \frac{\epsilon}{m}$. The allocation of channel capacities is done indirectly here by choosing the noise covariance matrix. Specifically, let

$$
\Sigma = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \delta & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \delta^{m-1} \end{bmatrix}
$$

with δ a small real number. Define

$$
S = \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \\ 0 & \delta I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \delta^{m-1} I_{n_m} \end{bmatrix}
$$

Then

$$\begin{aligned}\tilde{T}(z) &= \tilde{F}(zI - A - \tilde{B}\tilde{F})^{-1}\tilde{B} \\ &= \tilde{F}S(zI - S^{-1}AS - S^{-1}\tilde{B}\tilde{F}S)^{-1}S^{-1}\tilde{B}\end{aligned}$$

where

$$S^{-1}AS = \begin{bmatrix} A_1 & o(\delta) & \cdots & o(\delta) \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\delta) \\ 0 & \cdots & 0 & A_m \end{bmatrix}, \quad S^{-1}\tilde{B} = \begin{bmatrix} b_1 & o(\delta) & \cdots & o(\delta) \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & o(\delta) \\ 0 & \cdots & 0 & b_m \end{bmatrix}$$

and $\frac{o(\delta)}{\delta}$ approaches to a finite constant as $\delta \to 0$.

For any given total capacity constraint $h(A) + \epsilon$, we can always find an allocation of the total constraint in the form $\left\{h(A_1) + \frac{\epsilon}{m}, \ldots, h(A_m) + \frac{\epsilon}{m}\right\}$. By Lemma 16.3, for each $[A_i|b_i]$, we can design a stabilizing state feedback gain f_i such that $\|T_i(z)\|_2^2 = M(A_i)^2 - 1$, where $T_i(z) = f_i(zI - A_i - b_i f_i)^{-1} b_i$. Now let $\tilde{F}S = \mathrm{diag}\{f_1, f_2, \ldots, f_m\}$, then

$$\begin{aligned}\mathfrak{C}_i &= \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}d\omega\right\}_{ii} \\ &= \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi} \tilde{T}(e^{j\omega})\tilde{T}(e^{j\omega})^* d\omega\right\}_{ii} \\ &= \frac{1}{2}\log(1 + \|T_i(z)\|_2^2) + o(\delta) \\ &= \frac{1}{2}\log M(A_i)^2 + o(\delta) \\ &= h(A_i) + o(\delta)\end{aligned}$$

By choosing a sufficiently small $\delta > 0$, we can make the actual channel capacities satisfy the constraints $\mathfrak{C}_i < h(A_i) + \frac{\epsilon}{m}$, $i = 1, \ldots, m$. Apparently, the total capacity satisfies $\mathfrak{C} < h(\mathrm{A}) + \epsilon$. □

Remark 16.1. *From the lines of the above proof, we can see that the channel resource allocation is done indirectly here by choosing the noise power covariance matrix. One may question the tenability of this with the argument that the noise power cannot be allocated. This doubt actually originated from the conventional setting in LQG control that the noise comes from outside with given power. However, recall that in our setup, the noise is generated internally from the transmission process with power proportional to the transmission power. Therefore, although it looks on the surface that we are choosing the noise power, we are in fact distributing the transmission power. With this being clarified, the aforementioned doubt will vanish away. So, for any given total capacity constraint greater* *than the topological entropy of the plant, under channel resource allocation, we can design a state feedback gain to stabilize the system such that each channel satisfies its capacity constraint.*

Finally, we solve the problem as formulated in (16.3) and obtain a necessary and sufficient condition for stabilization of the multi-input NCS over AWGN channels with the help of channel resource allocation. The minimum total channel capacity required for stabilization is equal to the topological entropy of the plant that is the same as that needed for the single-input case. Once again, we witness the benefits brought by the channel-controller codesign. With the additional design freedom gained by channel resource allocation, the closed-loop stabilization becomes easier.

16.5 An illustrative example

In this section, we provide an example to illustrate the result in section 16.4. For the sake of numerical computation, we take the logarithm with base 2 in our example.

Consider an unstable system $[A|B]$ with

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Clearly, it is stabilizable. However, it is easy to verify that $[A|B_i]$ is not stabilizable, where B_i denotes the ith column of B. This means that it is impossible to stabilize the closed-loop system by using only one input channel. Both input channels have to be used to accomplish stabilization. The topological entropy of the plant is

$$h(A) = \log_2 2 + \log_2 4 + \log_2 8 = 6$$

We solve the minimal energy control problem for the following two single-input systems:

$$\left[\begin{array}{cc|c} 2 & 0 & 1 \\ 0 & 4 & 1 \end{array}\right] \quad \text{and} \quad [8|1]$$

The optimal state feedback gains for the two inputs are given by

$$f_1 = \begin{bmatrix} \frac{21}{16} & -\frac{105}{16} \end{bmatrix} \quad \text{and} \quad f_2 = -\frac{63}{8}$$

respectively. Now let $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix}$ and design the state feedback gain to be

$$F = \begin{bmatrix} \frac{21}{16} & -\frac{105}{16} & 0 \\ 0 & 0 & -\frac{63}{8} \end{bmatrix} \tag{16.10}$$

Table 16.1 Simulation results

δ	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}$
10^{-1}	3.75	3	6.75
10^{-2}	$3 + 1.3 \times 10^{-2}$	3	$6 + 1.3 \times 10^{-2}$
10^{-3}	$3 + 1.3 \times 10^{-4}$	3	$6 + 1.3 \times 10^{-4}$
10^{-4}	$3 + 1.3 \times 10^{-6}$	3	$6 + 1.3 \times 10^{-6}$

Under the above state feedback controller, the numerical results on the channel capacities for different δ are summarized in Table 16.1.

We can see that as $\delta \to 0$, the total capacity $\mathfrak{C} \to h(A)$. In other words, for any $\epsilon > 0$, when the total channel capacity constraint is given by $h(A) + \epsilon$, we can always simultaneously design a state feedback gain F and find an allocation of the capacities among input channels to make the closed-loop system stable. To demonstrate more clearly how the channel resource allocation is done, let the total capacity constraint be specifically given by $6 + 4 \times 10^{-2}$. Then we allocate this constraint among the two input channels as $\{3 + 2 \times 10^{-2}, 3 + 2 \times 10^{-2}\}$. Now we choose $\delta = 10^{-2}$ and use the state feedback gain (16.10). Under this channel-controller codesign, the channel capacities $\mathfrak{C}_1 = 3 + 1.3 \times 10^{-2} < 3 + 2 \times 10^{-2}$, $\mathfrak{C}_2 = 3 < 3 + 2 \times 10^{-2}$ as shown in Table 16.1. The total capacity satisfies the constraint $\mathfrak{C} = 6 + 1.3 \times 10^{-2} < h(A) + \epsilon$.

16.6 Conclusion

In this paper, we study stabilization of multi-input NCS over AWGN channels. Different from the single-input case, that is available in the literature and boils down to a typical $\mathcal{H}_2$ optimal control problem, the multi-input case involves an allocation of the total capacity among the input channels in addition to the design of the feedback controller. With this channel-controller codesign, we successfully show that a multi-input NCS over AWGN channels can be stabilized by state feedback control under channel resource allocation, if and only if the total channel capacity is greater than the topological entropy of the plant. A numerical example is given to demonstrate our result.

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