Proceedings of the 7th Asian Control Conference,
Hong Kong, China, August 27-29, 2009

Stabilization of Networked Control Systems with Finite Data Rate

Shuang Wan and Li Qiu

***Abstract*— Networked control system (NCS) is a newly emerging topic within control theory community. In this paper we pay particular attention to the problem concerning data rate constraint. Motivated by many successful results in related research, we study the lowest data rate needed to stabilize an LTI system when the controller structure is limited to be a static state feedback and the channel is modeled as a finite logarithmic quantizer. We prove that the lowest data rate is a function of the system Mahler measure alone. We also give the optimal state feedback controller achieving the lowest data rate and the associated Lyapunov function when proving the closed-loop stability.**

I. INTRODUCTION

Networked control system (NCS) has drawn broad interest within control system community recently. It takes the effects of non-ideal communication channels into consideration when applying system analysis or synthesis. Refer to some survey papers such as [7], [15] for an overview. There are many subtopics concentrating on different aspects of channel effects, e.g. quantization [4], [5], delay [11], signal-to-noise (SNR) constraint [2], disturbance [12], packet drop [16], etc.

However there is still one more aspect which receives much attention, i.e. the stabilization of linear system with data rate constraints. Such problems normally arise when a digital communication channel is employed to transmit control signals. More precisely, a channel at data rate R can only transmit no more than R digits in one time step. Hence it can produce no more than 2^R distinct output values. Thanks to the efforts of many researchers, it is becoming clear that the data rate of the channel has to exceed some specific value so as to make it possible to design a controller stabilizing the system.

Early efforts include the work by Wong and Brockett, i.e. [18], [19]. In both papers they investigate the necessary and sufficient condition for stabilizing a linear system with finite code-word options. The idea of the asymptotic convergence of system state is replaced by a weaker concept of containability which is related to uniform boundedness. The authors show that, the containability is achievable if and only if

$$\tau^2 \leq D,$$

where τ is a value corresponding to the unstable dynamic feature of the system and D is the number of all possible codes, which is related to data rate. This is the first time that the data rate constraint and the unstable dynamics of the system were connected effectively.

Several years later Nair and Evans [13], [14] do some fundamental research on stochastic systems. They consider a stochastic system with noise, and the stability is defined as $\sup_k \mathrm{E}\|X_k\|^2 < \infty$ where X_k is the state variable. They derived that a stabilizing controller for the system exists if and only if the data rate of the channel exceeds the so-called *topological entropy* of the system, i.e.

$$R > H(A) := \sum_{\lambda \in \sigma(A)} \max\{0, \log_2 |\lambda|\},$$

where A is the system matrix of the state-space model for the given system and $\sigma(A)$ is the spectrum of A. This not only is consistent with earlier results but also make the bound more precise and clear.

In fact this is not the only appearance of topological entropy. In the paper of Matveev and Savkin [10], they put their emphasis on a linear system with multiple sensors, and each sensor only partially observes the system. With a similar system and channel setup, they prove that the system is asymptotically stabilizable with a time-variant coding scheme at channel coder, if and only if the data rate of the channel satisfies

$$R > H(A).$$

Similar results also come out in the paper by Tatikonda and Mitter [17], which mainly discuss MIMO systems. They employ a traditionally designed controller, and prove that the system can be stabilized if and only if

$$R > H(A).$$

Li and Baillieul have also shown consistent results [9]. By their virtual system approach for digital finite communication bandwidth (DFCB) control, the absolute lowest data rate is found to be

$$R > \log_2 e \cdot \sum_{i=1}^{n} p_i,$$

where $p_1, \cdots, p_n$ are all the unstable poles of the system. This is the continuous-time version of topological entropy while other results are the discrete-time version.

The topological entropy also shows up in other aspects of NCS research. One of the most related results belongs to quantization research given by Elia and Mitter [4] and is later restated by Fu and Xie [5]. They find that for a single-input linear system, the lowest quantization density for infinite logarithmic quantizers is exactly

$$\rho_{\text{inf}} = \frac{2^{H(A)} - 1}{2^{H(A)} + 1}$$

This work is supported by Hong Kong Research Grants Council

S. Wan is a Ph.D. candidate with the Department of Electronic and Computer Engineering, the Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR, China swan@ust.hk

L. Qiu is with the Department of Electronic and Computer Engineering, the Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong SAR, China eeqiu@ust.hk

978-89-956056-9-1/09/©2009 ACA

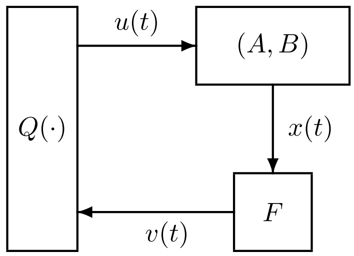

Fig. 1. The Closed-loop System (A, B, F, Q)

or its highest quantization sector bound coincides the Mahler measure of system matrix A

$$\delta_{\text{sup}}^{-1} = M(A) := \prod_{\lambda \in \sigma(A)} \max\{1, |\lambda|\} = 2^{H(A)}.$$

Again something determined by topological entropy.

Motivated by the successful results above, we wonder whether the topological entropy still serves as the minimum data rate if a finite logarithmic quantizer is used to quantize input signals. We concern this since most results in the data rate literature ([17] etc.) achieve the topological entropy by directly quantizing the state with uniform quantizers, then it is natural to further ask whether this could also be true if the input signal is quantized instead. On the other hand, infinite logarithmic quantizers have proved their advantage in quantizing input signals, but meanwhile they are not realistic and require infinite data rate. Hence we truncate infinite logarithmic quantizers to obtain finite ones, and hope it still allows the least admissible data rate to be as low as uniform quantizers.

To answer the question we consider the following setup. Given a discrete time linear system (A, B) or

$$x(t+1) = Ax(t) + Bu(t),$$

with system state $x(t) \in \mathbb{R}^n$ and input signal $u(t) \in \mathbb{R}$. We assume that all eigenvalues of A are unstable, (A, B) is stabilizable and $x(t)$ is available for state feedback. In the feedback loop first $x(t)$ meets the controller F and is mapped to a temporary control signal $v(t) = Fx(t)$. Then $v(t)$ becomes the input of the channel $Q(\cdot)$ while the output is $u(t)$. At last $u(t)$ goes back to (A, B) as an input and the loop is completed. We denote the closed-loop system as (A, B, F, Q), which is depicted in Fig. 1.

Now we state our goal in this paper. For a closed-loop system (A, B, F, Q), find the lowest value of data rate R such that there exists a static state feedback controller F stabilizing (A, B). We will state the answer to our question in the next section, which turns out to be not so good as expected: the lowest admissible data rate is higher than $H(A)$ for time-invariant quantizers. After the answer we will present the derivation of the optimal Lyapunov function and controller, which are used in the proof of our main result. Finally all the proofs are collected in the appendix.

II. THE OPTIMAL DATA RATE

In this section we will show that the lowest data rate required for stability is actually a function of system Mahler measure or topological entropy only.

The channel model we adopt here is a time-invariant finite logarithmic quantizer, i.e.

$$u(t) = Q(v(t)) := \begin{cases} \rho^l u_0, & \text{if } \frac{\rho^l u_0}{1+\delta} < v \leq \frac{\rho^l u_0}{1-\delta} \\ 0, & \text{if } |v| \leq \frac{\rho^N u_0}{1-\delta} \\ -Q(-v(t)), & \text{if } v < -\frac{\rho^N u_0}{1-\delta} \end{cases}$$

where $u_0 > 0, 0 < \rho < 1, \delta = \frac{1-\rho}{1+\rho}$ and $l = 0, 1, 2, \ldots, N$. The quantizer fails if $|v(t)| > \frac{u_0}{1-\delta}$.

Note that the quantizer $Q(\cdot)$ defined above has $2N+1$ levels in all. Hence to work properly it requires a data rate satisfying

$$R \geq \log_2(2N+1).$$

We will try to find the optimal N first, and then obtain the optimal R by this inequality.

Since the asymptotic stability is not possible with time-invariant finite quantizers in the closed-loop system [3], we have to define a weaker stability first. Denote a system Lyapunov function candidate to be $V(x) = x'Px$, where $P \in \mathbb{R}^{n \times n}$ is a well selected positive definite matrix, and given $a > 0$, also denote that $\Omega_a = \{x | V(x) < a\}$, then here comes the definition.

Definition 1: A closed-loop system (A, B, F, Q) is said to be *practically stable w.r.t.* r_1 *and* r_2 if there exists a Lyapunov function candidate $V(x)$ such that, for any compact set $\mathcal{C}$ with the origin as its interior point and $\Omega_{r_1} \supset \mathcal{C} \supset \Omega_{r_2}$ holds, we have $V(x(t+1)) < V(x(t))$ when $x(t) \in \mathcal{C} \setminus \Omega_{r_2}$, and $x(t+1) \in \Omega_{r_2}$ when $x(t) \in \Omega_{r_2}$.

Moreover, a system (A, B) is said to be *practically stabilized by* F *w.r.t. quantizer* $Q(\cdot)$ if (A, B, F, Q) is practically stable w.r.t. some r_1 and r_2. If such an F exists for (A, B) and $Q(\cdot)$ then (A, B) is said to be *practically stabilizable w.r.t.* $Q(\cdot)$.

Similar definitions also showed up in many other papers, e.g. [4], [19], etc. Such definitions relax the restriction that the state has to converge to origin, and only require the convergence to a uniformly bounded neighborhood of origin.

Now we are ready to present our main result in this paper.

Theorem 1: In our system setup, the system (A, B) is practically stabilizable w.r.t. $Q(\cdot)$ if and only if the quantizer $Q(\cdot)$ satisfies

$$R > \log_2 \left(2 \log_{\frac{M(A)+1}{M(A)-1}} M(A) + 1 \right).$$

Proof: See appendix. ■

The result seems to be fine since only $M(A)$ is involved, however it is not so good as expected. We wish to find a consistent result to others, i.e. $H(A)$ or $\log_2 M(A)$ serves as the minimum data rate, but the result is more complicated and in fact higher than $H(A)$. Hence we may conclude that time-invariant finite logarithmic quantizers require higher data rate for stabilization. But things could be better if time-variant quantizers are used instead. A recent paper by Fu, Xie and Su [6] reports that for a first order system, i.e. $x(t) \in \mathbb{R}$, a lower bound consistent with ours is found. And furthermore, they apply the control action only once every m time steps, and as $m \to \infty$, the ultimate lowest data rate is exactly $H(A)$. This suggests that maybe it also holds in our case. Further research is to be carried out.

III. THE OPTIMAL LYAPUNOV FUNCTION AND CONTROLLER

One crucial fact used in the proof to *Theorem 1* is the optimal value of the Lyapunov function and the corresponding controller given by an algebraic Riccati equation (ARE). We will give explicit derivation of the optimal solutions in the following, which can be written in analytic form as shown in [8]:

$$P = \left[(1-\delta^2) \sum_{t=1}^{\infty} A^{-t} B B' A'^{-t} \right]^{-1}.$$

Note that this is well defined since (A, B) is assumed to be stabilizable. Moreover with the optimal P at hand we also know the optimal controller to be

$$F = -[I + (1-\delta^2) B' P B]^{-1} B' P A.$$

In fact the optimal Lyapunov function is derived for infinite quantizers. However since the only difference between infinite quantizers and finite ones is that the later have zero-control region, which merely gives extra parameters to be designed, they share the same argument on this issue. Hence it suffices to study the case with infinite quantizers and the conclusion will also work for finite ones. We will only study infinite quantizers in this part.

The model for infinite quantizers is a bit different from the one for the finite ones. According to [8], such a quantizer can be modeled as a unity transfer function with an additive uncertainty with norm bound, or specifically

$$u(t) = Q(v(t)) = v(t) + \Delta(v(t)), \|\Delta\|_\infty \le \delta.$$

Now for the closed-loop system (A, B, F, Q) where Q adopts the infinite logarithmic quantizer model above, we defined the modified quadratic stability below:

Definition 2: A closed-loop system (A, B, F, Q) is *quadratically stable w.r.t.* δ if its Lyapunov function $V(x)$ satisfies $V(x(t+1)) - V(x(t)) < 0$ for all $x(t) = x \in \mathbb{R}^n \setminus \{0\}$ and $\|\Delta\|_\infty \le \delta$.

There are also other important definitions of stability. One of them is robust stability. Many important conclusions are linked to it, and we will make use of this. Considering the robust sense in our definition of quadratic stability, we are to show that the two are in fact equivalent, and the optimal Lyapunov function and controller emerge during the proof. The proof will be divided into two cases, i.e. analysis problem and synthesis problem.

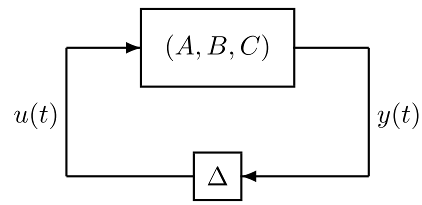

Fig. 2. Analysis Problem Setup

A. Analysis Problem

Consider the following situation shown in Fig 2. Given a discrete time linear system

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n, C' \in \mathbb{R}^n$. Suppose that A is stable. The output $y(t)$ and the input $u(t)$ are connected by a nonlinear memoryless disturbance block, i.e. $u(t) = \Delta(y(t))$. Then for the analysis problem we need to prove the following theorem.

Theorem 2: The closed-loop system (A, B, C, Δ) is quadratically stable w.r.t. δ if and only if the small gain condition is satisfied. In other words, there exists P such that $V(x(t+1)) - V(x(t)) < 0$ for all t if and only if $\|G(z)\| = \|C(zI - A)^{-1}B\| < \delta^{-1}$. Moreover P is the solution to the following ARE related to (A, B, C):

$$P = A'PA + \delta^2 C'C + A'PB(I - B'PB)^{-1}B'PA \quad (1)$$

with $P \ge 0, I - B'PB > 0$.

Proof: See appendix. ■

B. Synthesis Problem

For synthesis problem we consider another setup. Given a discrete time linear system $x(t+1) = Ax(t) + Bu(t)$, where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^n$, suppose that (A, B) is stabilizable, and $x(t)$ is available for state feedback. Then we put it into a close loop for stabilization, as shown at page bottom.

The loop consists of the system (A, B), a controller F, and a channel. The controller F is presumed to be a static linear gain. The channel is modeled as introducing a multiplicative uncertainty Δ which is bounded in the sense that $\|\Delta\|_\infty \le \delta$. We aim to prove the result below:

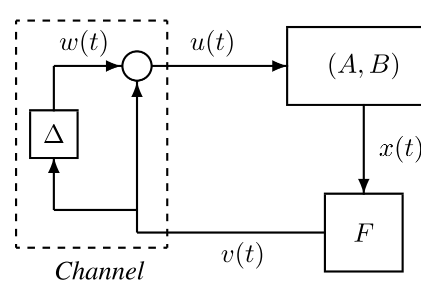

Fig. 3. Synthesis Problem Setup

Theorem 3: The closed-loop system (A, B, F, Q) is quadratically stable if and only if $A + BF$ stable and $\|F(zI - A - BF)^{-1}B\|_\infty < \delta^{-1}$. Moreover one admissible F is given by $F = -[I + (1 - \delta^2)B'PB]^{-1}B'PA$, where P is the solution to the synthesis Riccati equation

$$P = A'P[I + (1 - \delta^2)BB'P]^{-1}A \tag{2}$$

with $P \geq 0$ and $I - \delta^2 B'PB > 0$, which also establishes the Lyapunov function defining the quadratic stability, i.e. $V(x) = x'Px$.

Proof: See appendix. ■

It is quite intuitively obvious but in fact non-trivial that the proofs for analysis and synthesis problem are equivalent in some sense. Indeed by tedious but straight forward calculation we may find that if given (A, B) in synthesis problem, the analysis ARE and the synthesis ARE share the same solution by setting the system in analysis problem to be (A_c, B, C) where $C = F = -[I + (1 - \delta^2)B'PB]^{-1}B'PA, A_c = A+BF$. Hence we may conclude that the optimal Lyapunov function is exactly the solution of the AREs, which is given earlier in this part.

This goes farther than what we used in the previous part, and we believe that the result may be meaningful not only to the deduction in this paper, but also to related research in this topic.

However one may find that the optimal controller is different from the one appeared in many papers, e.g. [4], [5]. The reason is that we found the optimal controller in different sense. The controller in [4] is in fact an $\mathcal{H}_2$ optimal controller, and the related ARE in the paper is also for $\mathcal{H}_2$ optimization. But our ARE is in $\mathcal{H}_\infty$ sense, and so is the optimal controller. Nevertheless, the results on ultimate bound of quantizer parameters are identical. This confirms that we are all correct, and implies that the given bound is quite fundamental as well.

IV. CONCLUSION

In this paper the stabilization of a discrete-time linear system with data rate constraint is considered. The channel is modeled as a finite logarithmic quantizer. Its only difference from the infinite quantizer is that it has upper bound and zero-control region. With such quantizers a linear system cannot achieve asymptotic stability, so the concept of practical stability is also introduced. The lowest data rate which assures the existence of a stabilizing controller is given by a function only determined by the Mahler measure or equivalently the topological entropy of the system.

During the derivation, a property of the optimal Lyapunov function is also used which is proved in the later part. The analytic solution of the optimal Lyapunov function and the optimal controller is also derived.

The work by Fu, Xie and Su [6] has shed some light on the possibility to find even closer relationship between the data rate bound and the topological entropy. It is believed that a more general and rigorous result will be found on this direction, which is currently under our research.

V. APPENDIX

A. Proof to Theorem 1

A lemma is needed before the proof begins. We need the restatement of practical stability which is easier to handle. Denote the zero-control region of quantizer $Q(\cdot)$ to be $\Omega_0 = \{x|Q(Fx) = 0\} = \{x||Fx| \leq \frac{\rho^N u_0}{1-\delta}\}$, then

Lemma 1: (A, B, F, Q) is practically stable w.r.t. r_1 and r_2 if and only if for any $x \in \Omega_{r_1} \cap \Omega_0$ which also satisfies $x'A'PAx \geq x'Px$, we have $Ax \in \Omega_{r_2}$.

Proof: We prove the necessity and sufficiency separately. Note that in the definition of practical stability, $\mathcal{C}$ is arbitrary as long as $\Omega_{r_1} \supset \mathcal{C} \supset \Omega_{r_2}$, we may always set $\mathcal{C} = \Omega_{r_1}$ without loss of generality.

Necessity: If (A, B, F, Q) is practically stable then for $x(t) \in \Omega_{r_1} \cap \Omega_0 - \Omega_{r_2}$, $V(x(t+1)) < V(x(t))$; for $x(t) \in \Omega_{r_2} \cap \Omega_0$, $x(t+1) = Ax(t) \in \Omega_{r_2}$.

Sufficiency: If $x \in \Omega_{r_1} - \Omega_0$, then under the nonzero quantized feedback control, $V(x)$ is assured to drop, the same as the infinite quantizer case. Especially if $x \in \Omega_{r_2} - \Omega_0$, $V(x(t+1)) < V(x(t)) < r_2$, i.e. $x(t+1) \in \Omega_{r_2}$.

If $x \in \Omega_{r_1} \cap \Omega_0 - \Omega_{r_2}$, we have $V(x(t+1)) < V(x(t))$ otherwise by the proposition $x \in \Omega_{r_2}$, a contradiction. If $x \in \Omega_{r_2} \cap \Omega_0$, there are two situations: $V(x(t)) > V(x(t+1))$ and $V(x(t)) \leq V(x(t+1))$. For the first situation $V(x(t+1)) < V(x(t)) < r_2$; for the second situation $V(x(t+1)) < r_2$ by the proposition.

Conclusively if $x \in \Omega_{r_1} - \Omega_{r_2}$ then $V(x(t+1)) < V(x(t))$, and if $x \in \Omega_{r_2}$ then $V(x(t+1)) < r_2$. This is exactly the definition of practical stability. ■

Note that $Q(\cdot)$ is merely a finite proportion of an infinite logarithmic quantizer, but it has been proved that [8] an infinite logarithmic quantizer with proper density assures that $V(x)$ always drops, as long as the optimal Lyapunov function and controller are used. Then this also happens in our quantizer when input does not fall into the zero-control region. Hence (A, B) is practically stabilized by the optimal controller w.r.t. $Q(\cdot)$ if and only if the upper bound of zero-control region satisfies some constraint, which gives the limit to R described in the theorem.

Proof to Theorem 1: Look into *Lemma 1* and we may find out that the maximum upper bound of the zero-control region (or equivalently the maximum lower bound of the quantization region) can be further concluded into the following optimization problem: find

$$\max_{x'Px \leq x'A'PAx \leq r_2} Fx,$$

where F is the optimal controller.

If $x'Px = 0$ then the stability is trivial; if $x'Px = kr_2 \neq 0$ where $0 < k \leq 1$ also satisfies $Ax \in \Omega_{r_2}$, then

$$\begin{aligned} r_2 &\geq x'A'PAx \\ &= x'Px \cdot \frac{x'A'PAx}{x'Px} \\ &= kr_2 \cdot \frac{x'A'PAx}{x'Px}. \end{aligned}$$

Hence

$$k \leq \frac{x'Px}{x'A'PAx} \text{ for all such } x,$$

or

$$\begin{aligned}\tilde{k} = \max k &= \left(\max_{x\in\Omega_{kr_2}} \frac{x'A'PAx}{x'Px}\right)^{-1} \\ &= \left(\max_{x} \frac{x'A'PAx}{x'Px}\right)^{-1}.\end{aligned}$$

This is equivalent to say that

$$\max_{x'Px\leq x'A'PAx\leq r_2} Fx = \max_{x'Px\leq \tilde{k}r_2} Fx,$$

then F practically stabilizes (A,B) w.r.t. $Q(\cdot)$ if and only if $u_0\rho^N \leq \max_{x'Px\leq\tilde{k}r_2} Fx$. Therefore the maximum ratio between the upper and lower bounds of the quantization region is given by

$$\max \rho^N = \frac{\max_{x'Px\leq\tilde{k}r_2} Fx}{\max_{x'Px\leq r_1} Fx} = \sqrt{\frac{\tilde{k}r_2}{r_1}}.$$

Now we try to find $\tilde{k}$. It's easy to see that the calculation formula of $\tilde{k}$ is a generalized Rayleigh quotient. Hence

$$\begin{aligned}\tilde{k} &= \left(\max_{x} \frac{x'A'PAx}{x'Px}\right)^{-1} \\ &= \left(\max_{y} \frac{y'(P^{-\frac{1}{2}})'A'PAP^{-\frac{1}{2}}y}{y'y}\right)^{-1} \\ &= \left[\lambda_{\max}(P^{-\frac{1}{2}}A'PAP^{-\frac{1}{2}})\right]^{-1}.\end{aligned}$$

It is shown in Part 3 that the Lyapunov function P is given by the following ARE

$$P = A'PA - (1-\delta^2)A'PB[I + (1-\delta^2)B'PB]^{-1}B'PA,$$

so we have

$$\begin{aligned}P^{-\frac{1}{2}}A'PAP^{-\frac{1}{2}} = I + (1-\delta^2)P^{-\frac{1}{2}}A'PB[I + \\ (1-\delta^2)B'PB]^{-1}B'PAP^{-\frac{1}{2}}.\end{aligned}$$

Note that we are discussing SI systems, hence $B'PB$ is in fact scalar, and $B'PA$ is a vector. This implies that

$$P^{-\frac{1}{2}}A'PAP^{-\frac{1}{2}} = I + zz',$$

where $z = (1-\delta^2)^{\frac{1}{2}}P^{-\frac{1}{2}}A'PB[I + (1-\delta^2)B'PB]^{-\frac{1}{2}}$ is a column vector. Considering that $\text{rank}(zz') = 1$, it is not hard to find that

$$\begin{aligned}\lambda_{\max}(I+zz') &= \det(I+zz') \\ &= \det(P^{-\frac{1}{2}}A'PAP^{-\frac{1}{2}}) \\ &= \det(A'A) = M(A)^2.\end{aligned}$$

Hence $\tilde{k} = M(A)^{-2}$. Considering the fact that normally $r_1 > r_2$, $\sup \rho^N = 1/M(A)$. Then finally push ρ to its optimal value and we have

$$\inf N = \log_{\frac{M(A)+1}{M(A)-1}} M(A).$$

Finally the optimal data rate is given by

$$\inf R = \log_2\left(2\log_{\frac{M(A)+1}{M(A)-1}} M(A) + 1\right),$$

which finishes the proof.

B. Proof to Theorem 2

We start from the small gain condition first. By *bounded real lemma* [1], given (A,B,C), $\|G(z)\|_\infty = \|C(zI - A)^{-1}B\|_\infty < \delta^{-1}$ if and only if there exists an X such that $X \geq 0, I - \delta^2 BXB > 0$ and

$$X = A'XA + C'C + \delta^2 A'XB(I - \delta^2 B'XB)^{-1}B'XA.$$

Multiply δ^2 on both sides and denote $P = \delta^2 X$, and we can rewrite the ARE into (1), i.e.

$$P = A'PA + \delta^2 C'C + A'PB(I - B'PB)^{-1}B'PA$$

with $P \geq 0, I - B'PB > 0$. Hence the small gain condition is equivalent to the existence of P.

Now for the other part. Given a P solving (1), set $V(x) = x'Px$, and we notice that

$$\begin{aligned}& V(x(t+1)) - V(x(t)) \\ =\ & (Ax+Bu)'P(Ax+Bu) - x'Px \\ =\ & x'(A'PA - P)x + u'B'PAx \\ & \quad + x'A'PBu + u'B'PBu \\ \overset{(1)}{=}\ & -\delta^2 y'y - x'A'PB(I - B'PB)^{-1}B'PBAx \\ & \quad + u'B'PAx + x'A'PBu + u'B'PBu \\ =\ & -\delta^2\|y\|_2^2 - \|(I - B'PB)^{-1/2}B'PAx \\ & \quad - (I - B'PB)^{1/2}u\|_2^2 + \|u\|_2^2\end{aligned}$$

Since $\|u\|_2/\|y\|_2 \leq \|\Delta\|_\infty \leq \delta$, we have $V(x(t+1)) - V(x(t)) < 0$ for all x. Therefore $V(x(t)) = x'(t)Px(t)$ is an appropriate Lyapunov function for (A,B,C) as long as P exists for (1).

Conversely it is obvious that the (modified) definition of quadratic stability assures the system stability despite the uncertainty, which implies robust stability.

Hence the small gain condition is equivalent to quadratic stability for (A,B,C) w.r.t. δ, with the Lyapunov function constructed.

C. Proof to Theorem 3

Let's look into sufficiency first.

By [8], there exists an F such that $A + BF$ is stable and $\|F(zI - A - BF)^{-1}B\|_\infty < \delta^{-1}$ if and only if there exists a stabilizing solution P to (2) with $P \geq 0$ and $I - \delta^2 B'PB > 0$, and one admissible $F = -[I + (1-\delta^2)B'PB]^{-1}B'PA$. Note that (2) is equivalent to

$$P = A'PA - (1-\delta^2)A'PB[I + (1-\delta^2)B'PB]^{-1}B'PA$$

Denote $\mathcal{R} = I + (1-\delta^2)B'PB$. If the solution to (2) exists then we can calculate

$$\begin{aligned}
& V(x(t+1)) - V(x(t)) \\
= \;& (Ax + Bv + Bw)'P(Ax + Bv + Bw) - x'Px \\
= \;& x'(A'PA - P)x + (v+w)'B'PAx \\
& \quad + x'A'PB(v+w) + (v+w)'B'PB(v+w) \\
= \;& x'(1-\delta^2)A'PB\mathcal{R}^{-1}B'PAx \\
& \quad + (v+w)'B'PAx + x'A'PB(v+w) \\
& \quad + (v+w)'B'PB(v+w) \\
= \;& x'A'PB\mathcal{R}^{-1}B'PAx + v'B'PAx + x'A'PBv \\
& \quad + v'B'PBv - \delta^2 x'A'PB\mathcal{R}^{-1}B'PAx \\
& \quad + w'B'PAx + x'A'PBw + w'B'PBw \\
& \quad + v'B'PBw + w'B'PBv
\end{aligned}$$

$$\begin{aligned}
= \;& (\mathcal{R}^{-1}B'PAx + v)'\mathcal{R}(\mathcal{R}^{-1}B'PAx + v) \\
& - (\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w)'\mathcal{R} \\
& \times (\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w) - v'(I - \delta^2 B'PB)v \\
& + v'B'PBw + w'B'PBv + \delta^{-2}w'(I + B'PB)w \\
= \;& (\mathcal{R}^{-1}B'PAx + v)'\mathcal{R}(\mathcal{R}^{-1}B'PAx + v) \\
& - (\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w)'\mathcal{R} \\
& \times (\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w) + \delta^{-2}w'w - v'v \\
& + (\delta v + \delta^{-1}w)'B'PB(\delta v + \delta^{-1}w) \\
= \;& \|\mathcal{R}^{-1}B'PAx + v\|_{\mathcal{R}}^2 - \|\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w\|_{\mathcal{R}}^2 \\
& + \|\delta v + \delta^{-1}w\|_{B'PB}^2 + \|\delta^{-1}w\|^2 - \|v\|^2
\end{aligned}$$

where given vector x and compatible matrix P, $\|x\|_P^2 = x'Px, \|x\|^2 = \|x\|_I^2 = x'x$. By *Theorem 3.7* in [1, pp. 83-84], the optimal control is given by $v = Fx = -\mathcal{R}^{-1}B'PAx$. If we stick to this controller then we can further extent the result above

$$\begin{aligned}
& V(x(t+1)) - V(x(t)) \\
= \;& -\|\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w\|_{\mathcal{R}}^2 \\
& + \| - \delta\mathcal{R}^{-1}B'PAx + \delta^{-1}w\|_{B'PB}^2 \\
& + \|\delta^{-1}w\|^2 - \|v\|^2 \\
= \;& -\|\delta\mathcal{R}^{-1}B'PAx - \delta^{-1}w\|_{I-\delta^2 B'PB}^2 \\
& + \|\delta^{-1}w\|^2 - \|v\|^2
\end{aligned}$$

Since $I - \delta^2 B'PB > 0$, we know that $V(x(t+1)) - V(x(t)) < 0$, i.e. there exists at least one stabilizing controller F s.t. (A, B) is quadratically stabilized. Hence the existence of P, or $\|F(zI - A - BF)^{-1}B\|_\infty < \delta^{-1}$, implies the quadratic stability of (A, B).

The reverse part again follows from the definition of quadratic stability.

References

[1] T. Başar and P. Bernhard, $\mathcal{H}_\infty$ *Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach*, Systems and Control: Foundations & Applications, Birkhäuser, Boston, MA, 1991.

[2] J. Braslavsky, R. Middelton, and J. Freudenberg, "Feedback stabilization over signal-to-Noise ratio constrained channels", *IEEE Trans. Automat. Contr.*, vol. 52, no. 8, pp. 1391-1403, August 2007.

[3] D. Delchamps, "Stabilizing a linear system with quantized state feedback", *IEEE Trans. Automat. Contr.*, vol 35, no. 8, pp. 916-924, August 1990.

[4] N. Elia and S. Mitter, "Stabilization of linear systems with limited information", *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1384-1400, September 2001.

[5] M. Fu and L. Xie, "The sector bound approach to quantized feedback control", *IEEE Trans. Automat. Contr.*, vol. 50, no. 11, pp. 1698-1711, November 2005.

[6] M. Fu, L. Xie, and W. Su, "Connections between Quantized Feedback Control and Quantized Estimation", *10th Intl. Conf. on Control, Automation, Robotics and Vision*, 2008.

[7] G. Goodwin, D. Quevedo, and E. Silva, "An introduction to networked control systems", *Proc. The 6th Asian Control Conf.*, Bali, Indonesia, 2006.

[8] G. Gu and L. Qiu, "Networked stabilization of multi-input systems with channel resource allocation", *Reprints of the 17th IFAC World Congress*, 2008.

[9] K. Li and J. Baillieul, "Robust and efficient quantization and coding for control of multidimensional linear systems under data rate constraints", *Int. J. Robust Nonliner Control*, vol. 17, no. 10-11, pp. 898-920, 2007.

[10] A. Matveev and A. Savkin, "Multirate stabilizationi of multiple sensor systems via limited capacity communication channels", *SIAM J. Control Optim.*, vol. 44, no. 2, pp. 584-617, 2005.

[11] R. Middelton and D. Miller, "On the achievable delay margin using LTI control for unstable poles", *IEEE Trans. Automat. Contr.*, vol. 52, no. 7, pp. 1194-1207, July 2007.

[12] N. Martins, M. Dahleh, and J. Doyle, "Fundamental limitations of disturbance attenuation in the presence of side information", *IEEE Trans. Automat. Contr.*, vol. 52, no. 1, pp. 56-66, January 2007.

[13] G. Nair and R. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates", *SIAM J. Control Optim.*, vol. 43, no. 2, pp. 413-436, July 2004.

[14] G. Nair, R. Evans, I. Mareels, and W. Moran, "Topological feedback entropy and nonlinear stabilization", vol. 49, no. 9, pp. 1585-1597, September 2004.

[15] G. Nair, F. Fagnani, S. Sampieri, and R. Evans, "Feedback control under data rate constraints: an overview", *Proc. of IEEE*, vol. 95, pp. 108-137, 2007.

[16] M. Sahebsara, T. Chen and S. Shah, "Optimal $\mathcal{H}_2$ filtering in networked control systems with multiple packet dropout", *IEEE Trans. on Automatic Control*, vol. 52, no. 8, pp. 1508-1513, August 2007.

[17] S. Tatikonda and S. Mitter, "Control under communication constraints", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1056-1068, July 2004.

[18] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints—I: State estimation problems", *IEEE Trans. Automat. Contr.*, vol. 42, no. 9, pp. 1294-1299, September 1997.

[19] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints—II: Stabilization with limited information feedback", *IEEE Trans. Automat. Contr.*, vol. 44, pp. 1049-1053, May 1999.