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# State Estimation Over a Network: Packet-dropping Analysis and Design

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*Abstract*— **In this paper, we consider discrete-time state estimation over a network. Two scenarios are investigated. In the first scenario, we assume the sensor sends its measurement packet across a packet-dropping network, and we find minimum packet arrival rate that guarantees certain performance at the remote estimator. In the second scenario, we assume the network does not drop any data packet but the sensor has the freedom to send its current measurement data or not. We look for the optimal measurement data sending rate at the sensor which provides the optimal tradeoff between the sending cost at the sensor and the error at the estimator. Examples are provided to illustrate the theories and algorithms developed.**

## I. INTRODUCTION

The last decade has seen a growing interests in the analysis and design of networked control systems from both the control community and the network and communication community. When compared with classical feedback control systems, networked control systems offer many advantages such as reducing the system wiring, making the system easy to operate and maintain and later diagnose in case of malfunctioning, and increasing system agility [1].

Although networked control systems provide many advantages, inserting a network in between the plant and the controller can introduce many potential problems at the same time. For instance, zero-delayed sensing and actuation, perfect information and synchronization may not be guaranteed in the new system architecture due to the possible finite bandwidth. In many application, data packet drops and delays may occur due to network traffic conditions. These must be revisited and analyzed before networked control systems become commonplace.

Many researchers have investigated these issues and some significant results were obtained and many are in progress. Tatikonda [2] and Sahai [3] presented some interesting results in the area of control under communication constraints. Specifically, Tatikonda gave a necessary and sufficient condition on the channel data rate such that a noiseless LTI system in the closed loop is asymptotically stable. He also gave rate results for stabilizing a noisy LTI system over a digital channel. Sahai proposed the notion of anytime capacity to deal with real time estimation and control for a networked control system. Elia in [4] considered the problem of stabilization a networked control system over fading channels. The effect of packet drops on state estimation was studied by Sinopoli, et. al. in [5], where the authors showed that there exists a critical rate of packet arrivals below which the modified Kalman filter diverges, and converges otherwise. The authors extended their result from estimation to closed loop control in [6] where stability region of packet arrival rates are provided. Following the spirit of [5], Liu and Goldsmith [7] extended the idea to the case where there are multiple sensors and the packets arriving from different sensors are dropped independently. They provided similar bounds on the packet arrival rate for a stable estimate, again in the expected sense. [8], [9] characterize packet losses as a Markov chain and give some sufficient and necessary stability conditions under the notion of peak covariance stability. Different from [5], [8], [9], Shi et al. [10] investigates the stability of the Kalman filter via a probabilistic approach. A scheme based on multi-description coding for packet dropping networks, but limited to the estimation, is considered in [11]. Gupta et al. [12] studied the problem of LQG control across packet-dropping networks and showed that it is optimal to let the sensor preprocess the measurement data and sends its local state estimate to the remote estimator over a packet-dropping network. The readers are referred to [13] and references therein for some recent results in the area of networked control systems.

In this paper, we consider discrete-time state estimation over a network. Two scenarios are investigated. In the first scenario, we assume the sensor sends its measurement packet across a packet-dropping network, and we look for the minimum packet arrival rate such that certain performance is guaranteed at the remote estimator. We call this first problem as "*Packet-dropping Analysis*". In the second scenario, we assume the network does not drop any data packet but the sensor has the freedom to send its current measurement data or not. Sending less measurement data leads to a smaller cost at the sensor (e.g., less energy or less bandwidth used). On the other hand, receiving less measurement data leads to a higher estimation error at the estimator. Thus we look for the optimal measurement data sending rate at the sensor such that the cost at the sensor is small yet the error at the estimator is also small. We call this second problem as "*Packet-dropping Design*".

The contribution of this paper is summarized as follows.

1) In "*Packet-dropping Analysis*", we show that the average estimation error is decreases monotonically with the packet arrival rate. We then propose an efficient binary search algorithm to find the minimum packet arrival rate such that certain performance is guaranteed at the remote estimator.
2) In "*Packet-dropping Design*", for scalar systems, we obtain the unique sending rate which provides the optimal tradeoff between the cost at the sensor and the

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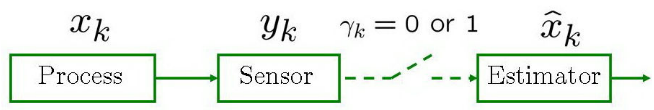


Fig. 1. System Block Diagram

estimation error at the estimator; for vector systems, we provide a numerical method to find the optimal sending rate.

The rest of the paper is organized as follows. In Section II, we provide the mathematical problem setup. Some preliminaries are provided in Section III. Packet-dropping analysis is then presented in Section IV, followed by packet-dropping design in Section V. Examples are given in Section VII and concluding remarks are given in the end.

## II. Problem Formulation

We consider the problem of discrete-time state estimation over a communication network as shown in Fig. 1.

The process dynamics and sensor measurement equation are given as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + w_k, &(1)\\ y_k &= Cx_k + v_k. &(2)\end{aligned}$$

In the above equations, $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the observation vector, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero mean, Gaussian random vectors with $\mathbb{E}[w_k w_j'] = \delta_{kj} Q \geq 0$, $\mathbb{E}[v_k v_j'] = \delta_{kj} R > 0$, $\mathbb{E}[w_k v_j'] = 0 \; \forall j, k$, where $\delta_{kj} = 0$ if $k \neq j$ and $\delta_{kj} = 1$ otherwise. We assume that $A$ is stable, $(A, C)$ is observable, and $(A, \sqrt{Q})$ is controllable.

We consider two scenarios in this paper. In the first scenario, the network introduces packet drops. An i.i.d Bernoulli random variable $\lambda_k$ with $\mathbb{E}[\lambda_k] = \lambda$ is used to indicate whether $y_k$ is dropped or not, i.e., $\lambda_k = 0$ if $y_k$ is dropped by the network and $\lambda_k = 1$ if $y_k$ is received by the estimator.

In the second scenario, the network always successfully deliver the packet, but there is a unit cost $\Delta$ when the sensor sends $y_k$. The unit cost could represent the energy or the bandwidth used by the sensor. Thus sending less measurement data leads to a smaller cost at the sensor. However, less measurement data leads to a higher estimation error at the estimator. We use $\lambda_k$ to indicate whether $y_k$ is sent by the sensor or not. Similar to the first scenario, we assume $\lambda_k$ is an i.i.d Bernoulli random variable with $\mathbb{E}[\lambda_k] = \lambda$, i.e., we consider a stochastic decision scheme here. It is easy to note that $\lambda$ characterizes the tradeoff between the cost at the sensor and the estimation error at the estimator.

In the first scenario, $\lambda$ denotes the packet arrival rate at the estimator. In the second scenario, $\lambda$ denotes the packet sending rate at the sensor. For both scenarios, we define the following state quantities at the remote state estimator:

$$\begin{aligned} \hat{x}_k &\triangleq \mathbb{E}[x_k | \text{all data packets up to } k], \\ P_k &\triangleq \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)' | \text{all data packets up to } k]. \end{aligned}$$

As we will see shortly in the next section, $P_k$ is a random quantity due to the randomness of $\gamma_k$. Since $P_k$ reflects how well $\hat{x}_k$ approximates $x_k$, we are interested in properties of $P_k$. In particular, we are interested in the following problem in the first scenario:

*Problem 1 (Packet-dropping Analysis):* Find the minimum packet arrival rate $\lambda$ such that

$$\lim_{k \to \infty} \mathbb{E}[P_k] \leq P_{\text{desired}} \qquad (3)$$

for a given $P_{\text{desired}} \geq 0$.

In the second scenario, we are interested in the following problem:

*Problem 2 (Packet-dropping Design):* Find the optimal packet sending rate $\lambda$ such that

$$\mathrm{Tr}\{\lim_{k \to \infty} \mathbb{E}[P_k]\} + \lambda \Delta$$

is minimized.

The first term in the cost function corresponds to the average error at the estimator and the second term corresponds to the average cost at the sensor.

## III. Preliminaries

### A. *Definitions*

The following terms that are frequently used in subsequent sections are defined in this section. It is assumed that $(A, C, Q, R)$ are the same as they appear in Section II. $\mathbb{S}_+^n$ is the set of $n$ by $n$ positive semidefinite matrices. When $X \in \mathbb{S}_+^n$, we simply write $X \geq 0$; when $X$ is positive definite, we write $X > 0$. We define the function $h: \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$h(X) \triangleq AXA' + Q. \qquad (4)$$

As we shall see shortly, applying $h$ to the previous error covariance matrix corresponds to the time update of the standard Kalman filter.

For functions $f, f_1, f_2 : \mathbb{S}_+^n \to \mathbb{S}_+^n$, $f_1 \circ f_2$ is defined as

$$f_1 \circ f_2(X) \triangleq f_1\big(f_2(X)\big). \qquad (5)$$

Define the function $\tilde{g} : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$\tilde{g}(X) \triangleq X - XC'[CXC' + R]^{-1}CX, \qquad (6)$$

and function $g : \mathbb{S}_+^n \to \mathbb{S}_+^n$ as

$$g(X) \triangleq \tilde{g} \circ h(X). \qquad (7)$$

Denote $\hat{P}_\lambda \geq 0$ as the unique solution to the following equation[1]

$$X = (1 - \lambda) h(X) + \lambda g(X). \qquad (8)$$

Therefore $\hat{P}_\lambda$ satisfies

$$\hat{P}_\lambda = (1 - \lambda) h(\hat{P}_\lambda) + \lambda g(\hat{P}_\lambda).$$

When $\lambda = 0$ or 1, we obtain $\hat{P}_0$ and $\hat{P}_1$ which satisfy

$$\begin{aligned} \hat{P}_0 &= h(\hat{P}_0), &(9)\\ \hat{P}_1 &= g(\hat{P}_1). &(10)\end{aligned}$$

[1]The existence of $\hat{P}_\lambda$ is due to the assumptions that $A$ is stable, $(A, C)$ is observable and $(A, \sqrt{Q})$ controllable.

$\hat{P}_0$ is the steady-state error covariance matrix when $\gamma_k = 0$ for all $k \geq 1$, and $\hat{P}_1$ is the steady-state error covariance matrix when $\gamma_k = 1$ for all $k \geq 1$.

### B. Kalman Filtering

Consider the case when $\gamma_k = 1$. It is well known that the optimal linear estimator for the system described by Eqn (1) and (2) is a standard Kalman filter, denoted as **KF**. We write $(\hat{x}_k, P_k)$ in compact form as

$$(\hat{x}_k, P_k) = \mathbf{KF}(\hat{x}_{k-1}, P_{k-1}, y_k),$$

which represents the follow set of equations:

$$\left\{ \begin{array}{lcl} \hat{x}_k^- & = & A\hat{x}_{k-1}, \\ P_k^- & = & AP_{k-1}A' + Q, \\ K_k & = & P_k^- C'[CP_k^- C' + R_k]^{-1}, \\ \hat{x}_k & = & A\hat{x}_{k-1} + K_k(y_k - CA\hat{x}_{k-1}), \\ P_k & = & (I - K_k C)P_k^-. \end{array} \right.$$

With some manipulation, $P_k$ can be shown to satisfy

$$P_k = g(P_{k-1}).$$

Therefore $\hat{P}_1$ defined in Eqn (10) is the steady-state error covariance matrix of the Kalman filter.

Now consider the case when $\gamma_k$ may be equal to 0. Sinopoli et al. [5] showed that the Kalman filter is still the optimal linear estimator in this setting. There is a slight change to the standard Kalman filter in that only the time update is performed when $\gamma_k = 0$. When $\gamma_k = 1$, both the time and measurement update steps are performed. The filtering equations are thus the same as **KF** except that

$$\hat{x}_k = \hat{x}_k^- + \gamma_k K_k (y_k - C\hat{x}_k^-), \quad (11)$$

$$P_k = P_k^- - \gamma_k K_k C P_k^-. \quad (12)$$

Unlike the standard Kalman filtering setting where $P_k$ is a deterministic quantity (given an initial value $P_0$), the randomness of $\gamma_k$ makes $P_k$ a random variable as well.

## IV. Packet-dropping Analysis

In this section, we consider Problem 1 for the first scenario. Notice that we can write $P_k$ as

$$P_k = \left\{ \begin{array}{ll} h(P_{k-1}), & \text{if } \lambda_k = 0, \\ g(P_{k-1}), & \text{if } \lambda_k = 1. \end{array} \right.$$

It turns out that the exact value of $\mathbb{E}[P_k]$ is difficult to find even for scalar systems [5]. However, we can bound $\mathbb{E}[P_k]$ as follows.

$$\begin{array}{rcl} \mathbb{E}[P_k] & = & \mathbb{E}[\mathbb{E}[P_k|P_{k-1}]] \\ & = & \mathbb{E}[(1-\lambda)h(P_{k-1}) + \lambda g(P_{k-1})] \\ & = & \mathbb{E}[(1-\lambda)h(P_{k-1})] + \lambda \mathbb{E}[\tilde{g}\big(h(P_{k-1})\big)] \\ & \leq & (1-\lambda)h\big(\mathbb{E}[P_{k-1}]\big) + \lambda \tilde{g}\big(h(\mathbb{E}[P_{k-1}])\big) \\ & = & (1-\lambda)h\big(\mathbb{E}[P_{k-1}]\big) + \lambda g\big(\mathbb{E}[P_{k-1}]\big), \end{array}$$

where the inequality is due to the concaveness of $\tilde{g}$ (Lemma 3) and Jensen's Inequality. Use induction, it is easy to see that

$$\lim_{k\to\infty} \mathbb{E}[P_k] \leq \hat{P}_\lambda.$$

Therefore we look for the minimum $\lambda$ such that

$$\hat{P}_\lambda \leq P_{\text{desired}},$$

which will guarantee

$$\lim_{k\to\infty} \mathbb{E}[P_k] \leq P_{\text{desired}}.$$

Before we state the main result of finding such minimum $\lambda$, we introduce two lemmas. The first lemma introduces some properties of the functions $h, \tilde{g}$ and $g$ defined earlier in the previous section.

*Lemma 3:* For any $0 \leq X \leq Y$ and any $\alpha \in [0, 1]$,

1)
$$h(X) \leq h(Y), \ g(X) \leq g(Y), \ g(X) \leq h(X).$$

2) $\tilde{g}(X)$ is concave in $X$, i.e.,
$$\tilde{g}\big(\alpha X + (1-\alpha)Y\big) \geq \alpha\tilde{g}(X) + (1-\alpha)\tilde{g}(Y).$$

*Proof:*

1) $h(X) \leq h(Y)$ is easy to verify from the definition of $h$. From Lemma 1-c in [5], $\tilde{g}(X) \leq \tilde{g}(Y)$, therefore
$$g(X) = \tilde{g}\big(h(X)\big) \leq \tilde{g}\big(h(Y)\big) = g(Y).$$
Lastly, $g(X) = \tilde{g}\big(h(X)\big) \leq h(X)$ as
$$\tilde{g}(X) = X - XC'[CXC' + R]^{-1}CX \leq X.$$

2) The proof can be found in Lemma 1-e in [5].

■

The second lemma shows that $\hat{P}_\lambda$ is a monotonically decreasing function of $\lambda$.

*Lemma 4:* For any $\lambda_1, \lambda_2 \in [0, 1]$

$$\hat{P}_{\lambda_1} \geq \hat{P}_{\lambda_2} \text{ iff } \lambda_1 \leq \lambda_2.$$

*Proof:* Notice that

$$\hat{P}_\lambda = \lim_{k\to\infty} \hat{P}_{\lambda,k}$$

with

$$\hat{P}_{\lambda,k+1} = (1-\lambda)h(\hat{P}_{\lambda,k}) + \lambda g(\hat{P}_{\lambda,k})$$

for any $\hat{P}_{\lambda,0} \geq 0$. We use induction to prove $\hat{P}_{\lambda_1} \geq \hat{P}_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$. Let $\hat{P}_{\lambda_1,0} = \hat{P}_{\lambda_2,0}$. Then

$$\begin{array}{rcl} \hat{P}_{\lambda_1,1} & = & (1-\lambda_1)h(\hat{P}_{\lambda_1,0}) + \lambda_1 g(\hat{P}_{\lambda_1,0}) \\ & = & (1-\lambda_1)h(\hat{P}_{\lambda_2,0}) + \lambda_1 g(\hat{P}_{\lambda_2,0}) \\ & \geq & (1-\lambda_2)h(\hat{P}_{\lambda_2,0}) + \lambda_2 g(\hat{P}_{\lambda_2,0}) \\ & = & \hat{P}_{\lambda_2,1}, \end{array}$$

where the inequality is from the fact that $g(X) \leq h(X)$ for any $X \geq 0$. Now assume $\hat{P}_{\lambda_1,m} \geq \hat{P}_{\lambda_2,m}$ for some $m \geq 1$. Then

$$\begin{array}{rcl} \hat{P}_{\lambda_1,m+1} & = & (1-\lambda_1)h(\hat{P}_{\lambda_1,m}) + \lambda_1 g(\hat{P}_{\lambda_1,m}) \\ & \geq & (1-\lambda_1)h(\hat{P}_{\lambda_2,m}) + \lambda_1 g(\hat{P}_{\lambda_2,m}) \\ & \geq & (1-\lambda_2)h(\hat{P}_{\lambda_2,m}) + \lambda_2 g(\hat{P}_{\lambda_2,m}) \\ & = & \hat{P}_{\lambda_2,m+1}, \end{array}$$

where the first inequality is from Lemma 3. The induction is thus complete. The only if part is easy to prove by contradiction. ■

Since $\hat{P}_\lambda$ is monotonically decreasing in $\lambda$ with $\lambda \in [0,1]$, for any given $P_{\text{desired}} \geq 0$, the minimum value of $\lambda$ such that

$$\hat{P}_\lambda \leq P_{\text{desired}}$$

can be efficiently found using Algorithm 1 for any given $\epsilon \in (0,1)$.

**Algorithm 1** BINARY SEARCH ALGORITHM

$t := 0$, $L := 0$, $U := 1$, $\lambda_0 := \frac{L+U}{2}$, $\delta_0 := 1$.
**while** $\delta_t > \epsilon$ **do**
  **if** $\hat{P}_{\lambda_t} \leq P_{\text{desired}}$ **then**
    $t := t+1$, $U := \lambda_{t-1}$, $\lambda_t := \frac{L+U}{2}$.
    $\delta_t := |\lambda_t - \lambda_{t-1}|$.
  **else**
    $t := t+1$, $L := \lambda_{t-1}$, $\lambda_t := \frac{L+U}{2}$.
    $\delta_t := |\lambda_t - \lambda_{t-1}|$.
  **end if**
**end while**

Let $\lambda^*$ denote the minimum $\lambda \in [0,1]$ such that

$$\hat{P}_\lambda \leq P_{\text{desired}}.$$

Then it is easy to verify that

$$|\lambda_t - \lambda^*| \leq \left(\frac{1}{2}\right)^t.$$

Therefore

$$\begin{aligned}\delta_t &= |\lambda_t - \lambda_{t-1}| \\ &\leq |\lambda_t - \lambda^*| + |\lambda^* - \lambda_{t-1}| \\ &\leq \left(\frac{1}{2}\right)^t + \left(\frac{1}{2}\right)^{t-1} = 3\left(\frac{1}{2}\right)^t.\end{aligned}$$

Consequently, Algorithm 1 stops after at most $\log_2 3 - \log_2 \epsilon$ steps. For example, when $\epsilon = 10^{-5}$, the maximum number of steps running by Algorithm 1 is 19.

## V. PACKET-DROPPING DESIGN

We now consider Problem 2 in this section. Similar to the previous section, instead of minimizing

$$\operatorname{Tr}\{\lim_{k\to\infty} \mathbb{E}[P_k]\} + \lambda\Delta,$$

we minimize

$$\operatorname{Tr}\{\hat{P}_\lambda\} + \lambda\Delta.$$

We first study scalar systems and we show that a unique $\lambda^*$ that minimizes $\operatorname{Tr}\{\hat{P}_\lambda\} + \lambda\Delta$ exists.

### A. Scalar Systems

When $A = a \in (0,1)$, $\operatorname{Tr}\{\hat{P}_\lambda\} = \hat{P}_\lambda$. Before we state the main result, we introduce the following lemma.

*Lemma 5:* If $A = a \in (0,1)$, then $\hat{P}_\lambda$ is convex in $\lambda$, i.e., for any $\alpha \in [0,1]$,

$$\alpha\hat{P}_{\lambda_1} + (1-\alpha)\hat{P}_{\lambda_2} \geq \hat{P}_{\alpha\lambda_1 + (1-\alpha)\lambda_2}.$$

*Proof:* To show $\hat{P}_\lambda$ is convex in $\lambda$, it suffices to show that

$$\frac{d^2\hat{P}_\lambda}{d\lambda^2} \geq 0.$$

Consider the equation

$$x = (1-\lambda)h(x) + \lambda g(x)$$

which can be rewritten as

$$\begin{aligned} x &= (1-\lambda)h(x) + \frac{\lambda h(x) r}{h(x)+r} \\ &= h(x) - \frac{\lambda h(x)^2}{h(x)+r}, \end{aligned}$$

where $r = \frac{R}{C}$. We write $h(x) = h$ in the remaining proof for simplicity. Taking derivative on both sides with respect to $\lambda$ leads to

$$\begin{aligned} \frac{dx}{d\lambda} &= \frac{\partial h}{\partial x}\frac{dx}{d\lambda} - \frac{h^2}{h+r} - \lambda\frac{\partial}{\partial h}\left(\frac{h^2}{h+r}\right)\frac{\partial h}{\partial x}\frac{dx}{d\lambda} \\ &= \left(a^2 - \frac{\lambda a^2(h^2+2hr)}{(h+r)^2}\right)\frac{dx}{d\lambda} - \frac{h^2}{h+r}. \end{aligned} \tag{13}$$

Let

$$M = 1 - a^2 + \frac{\lambda a^2(h^2+2hr)}{(h+r)^2}, \; N = -\frac{h^2}{h+r}.$$

Then $\frac{dx}{d\lambda}$ can be written as

$$\frac{dx}{d\lambda} = \frac{N}{M}. \tag{14}$$

Notice that $M > 0$ and $N < 0$, therefore $\frac{dx}{d\lambda} < 0$, which also proves that $\hat{P}_\lambda$ is decreasing in $\lambda$ as shown in Lemma 4. Taking derivative on both sides of Eqn (14) with respect to $\lambda$ leads to the following

$$\begin{aligned} \frac{d^2x}{d\lambda^2} &= \frac{1}{M}\frac{\partial N}{\partial h}\frac{\partial h}{\partial x}\frac{dx}{d\lambda} - \frac{N}{M^2}\frac{\partial M}{\partial h}\frac{\partial h}{\partial x}\frac{dx}{d\lambda} \\ &= \frac{1}{M}\frac{\partial h}{\partial x}\frac{dx}{d\lambda}\left[\frac{\partial N}{\partial h} - \frac{N}{M}\frac{\partial M}{\partial h}\right] \\ &= \frac{a^2x'}{M(h+r)^2}\left[-h^2 - 2hr - 2\frac{\lambda a^2 r^2 N}{(h+r)M}\right], \end{aligned}$$

where we write $\frac{dx}{d\lambda} = x'$. Since $M > 0$ and $x' < 0$, in order to show $\frac{d^2x}{d\lambda^2} \geq 0$, it suffices to show

$$h^2 + 2hr \geq -2\frac{\lambda a^2 r^2 N}{(h+r)M}.$$

Since $0 < a < 1$, we only need to show

$$h^2 + 2hr \geq -2\frac{\lambda r^2 N}{(h+r)M},$$

or

$$h^2 + 2hr \geq \frac{2\lambda h^2 r^2}{(1-a^2)(h+r)^2 + (h^2+2hr)\lambda a^2}$$

which holds as

$$\begin{aligned} & 2hr[(1-a^2)(h+r)^2 + (h^2+2hr)\lambda a^2] \\ \geq\ & 2hr[\lambda(1-a^2)(h+r)^2 + (h^2+2hr)\lambda a^2] \\ \geq\ & 2hr[2\lambda(1-a^2)hr + 2hr\lambda a^2] \\ =\ & 4\lambda h^2 r^2 \geq 2\lambda h^2 r^2. \end{aligned}$$

Therefore $\frac{d^2 x}{d\lambda^2} \geq 0$ and $\hat{P}_\lambda$ is convex in $\lambda$. ■

We are now ready to present the main result for scalar systems.

*Theorem 6:* Consider the scalar system with $A = a \in (0,1)$. Then $\lambda^*$ which minimizes $\hat{P}_\lambda + \lambda\Delta$ is obtained by solving the following two equations:

$$\hat{P}_{\lambda^*} = h - \lambda^* \frac{h^2}{h+r}, \tag{15}$$

$$\Delta = \left(a^2 - \frac{\lambda^* a^2 (h^2+2hr)}{(h+r)^2}\right)\Delta + \frac{h^2}{h+r}, \tag{16}$$

where $h = h(\hat{P}_{\lambda^*}) = a^2\hat{P}_{\lambda^*} + q$.

*Proof:* Eqn (15) holds from the definition of $\hat{P}_{\lambda^*}$. Since $\hat{P}_\lambda$ is convex in $\lambda$ and $\lambda\Delta$ is linear in $\lambda$, $\hat{P}_\lambda + \lambda\Delta$ is also convex in $\lambda$. Therefore $\lambda^*$ that minimizes $\hat{P}_\lambda + \lambda\Delta$ is given by

$$\frac{d\hat{P}_\lambda}{d\lambda}|_{\lambda^*} = -\Delta.$$

From Eqn (13), $\lambda^*$ thus satisfies Eqn (16). ■

### B. Vector Systems

For a general $n$ dimensional system, showing $\mathrm{Tr}\{\hat{P}_\lambda\} + \lambda\Delta$ is convex in $\lambda$ is difficult though from simulation the statement seems to be valid. One possible way to find $\lambda^*$ is through a numerical method, e.g., we first get the plot of $\mathrm{Tr}\{\hat{P}_\lambda\} + \lambda\Delta$ as a function of $\lambda$, and then we determine $\lambda^*$ from the plot. An example will be given in Section VI-B.

## VI. Examples

In this section, we provide a few examples to illustrate the theories and algorithms developed in the previous sections. We start with an example for the *Packet-dropping Analysis*.

### A. Packet-dropping Analysis

Consider system (1) and (2) with

$$A = \begin{bmatrix} 0.9 & 0 & 0.5 & 0 \\ 0 & 0.9 & 0 & 0.5 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix}, \ C' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$Q = \mathrm{diag}(0.2, 0.2, 0.2, 0.2), \ R = \mathrm{diag}(0.2, 0.2).$$

Fig. 2 plots $\mathrm{Tr}(\hat{P}_\lambda)$ as a function of $\lambda$ as well as the Monte Carlo simulation of $\mathbb{E}[P_k]$ when $k \to \infty$. As the figure demonstrates, $\mathrm{Tr}(\hat{P}_\lambda)$ provides a tight approximation of $\mathbb{E}[P_k]$ when $k \to \infty$. $\mathrm{Tr}(\hat{P}_\lambda)$ is also a monotonically decreasing function of $\lambda$.

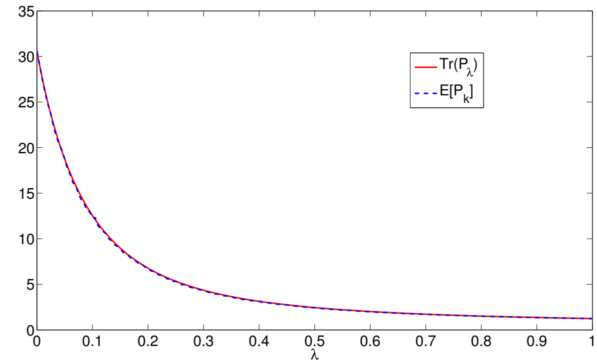


Fig. 2. Average Error Covariance and Packet-arrival Rate

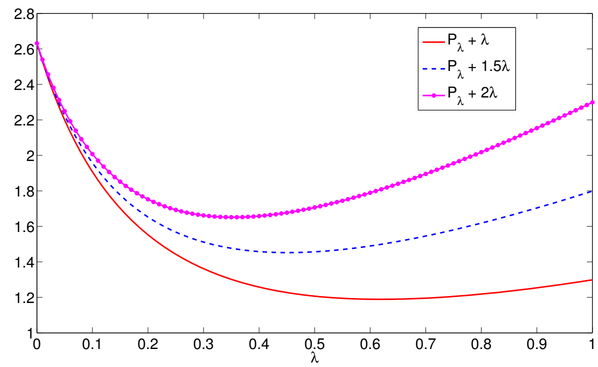


Fig. 3. Tradeoff between Sensor Cost and Estimation Error: Scalar System

Assume $\epsilon = 0.01$. After 6 steps, Algorithm 1 returns $\lambda = 0.1484$ when $P_{\text{desired}} = 4 * I_4$, returns $\lambda = 0.2734$ when $P_{\text{desired}} = 2 * I_4$, and returns $\lambda = 0.4609$ when $P_{\text{desired}} = I_4$.

### B. Packet-dropping Design

We consider two systems for the *Packet-dropping Design* problem. The first one is a scalar system and the other one is the same system considered in the previous example.

*Scalar System:* Consider system (1) and (2) with

$$A = 0.9, \ C = 1, \ Q = 0.5, \ R = 0.5.$$

We consider three different unit sensor cost $\Delta = 1, 1.5$ and 2 respectively. From Theorem 6, the optimal $\lambda^*$ such that $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$ is minimized satisfies Eqn (15) and (16). $\lambda^*$ and $\hat{P}_{\lambda^*}$ are solved as

$$\begin{aligned} \lambda^* &= 0.6190, \ \hat{P}_{\lambda^*} = 0.5700, \ \text{when } \Delta = 1, \\ \lambda^* &= 0.4513, \ \hat{P}_{\lambda^*} = 0.7753, \ \text{when } \Delta = 1.5, \\ \lambda^* &= 0.3535, \ \hat{P}_{\lambda^*} = 0.9446, \ \text{when } \Delta = 2. \end{aligned}$$

We also plot $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$ as a function of $\lambda$. As Fig. 3 demonstrates, $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$ is convex in $\lambda$. The optimal $\lambda$ that minimizes $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$ is easily seen to be the same as $\lambda^*$ calculated from Eqn (15) and (16).

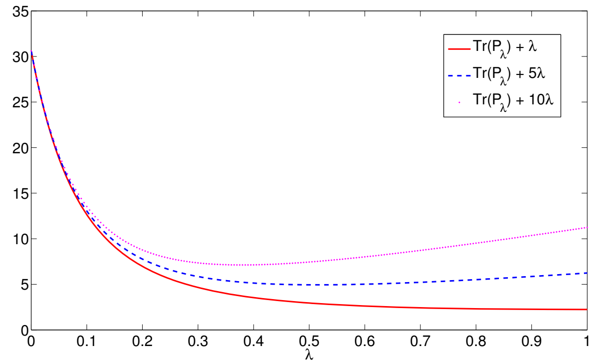


Fig. 4. Tradeoff between Sensor Cost and Estimation Error: Vector System

*Vector System:* We now consider the same system for the *Packet-dropping Analysis*. We also plot $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$ as a function of $\lambda$ numerically. $\Delta = 1, 5$, and 10 are considered. From Fig. 4, we can pick up the optimal $\lambda$ that minimizes $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$. For this example, the following values are found:

$$\begin{aligned}
&\lambda^* = 1,\ \hat{P}_{\lambda^*} = 1.2390,\ \text{when } \Delta = 1,\\
&\lambda^* = 0.5150,\ \hat{P}_{\lambda^*} = 2.3680,\ \text{when } \Delta = 5,\\
&\lambda^* = 0.3800,\ \hat{P}_{\lambda^*} = 3.3252,\ \text{when } \Delta = 10.
\end{aligned}$$

Note that $\mathrm{Tr}(\hat{P}_\lambda) + \lambda\Delta$ seems to be also convex in $\lambda$.

## VII. Conclusion

In this paper, we consider discrete-time state estimation over a network. Two scenarios are investigated. In "*Packet-dropping Analysis*", we show that the average estimation error is a monotonically decreasing function of the packet arrival rate. We then propose an efficient binary search algorithm to find the minimum packet arrival rate such that certain performance is guaranteed at the remote estimator. In "*Packet-dropping Design*", for scalar systems, we obtain the unique sending rate which provides the optimal tradeoff between the cost at the sensor and the estimation error at the estimator; for vector systems, we provide a numerical method to find the optimal sending rate.

There are many future directions along the line of this work. Proving $\mathrm{Tr}(\hat{P}_\lambda)$ is convex in $\lambda$ for vector system seems to be possible; closing the loop and analyzing the overall closed-loop system performance; and finally designing the optimal data sending rate at the controller.

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