Tensor Product Methods in Stability Robustness Analysis*

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Abstract

One of the methods recently utilized in the stability robustness analysis uses various matrix compositions. This paper gives an overview of this method using a tensor product theoretic perspective. The applications of this method to the stability robustness of matrices with unstructured uncertainties, one-parameter uncertainties or double-parameter uncertainties are then discussed.

1 Introduction

The problem of quantitative stability robustness analysis concerns the amount of uncertainties a stable object (matrix, polynomial, real rational matrix, *etc.*) can tolerate in order to maintain stability. This problem has attracted a considerable amount of research in recent years. Since the stability robustness study of a variety of objects, *e.g.* matrices, polynomials, real rational matrices is desired, and since the uncertainties may enter the objects in different ways, different mathematical tools have been employed for different objects/uncertainty models. In the stability robustness study of real matrices under real uncertainties, one of the methods recently utilized uses properties of matrix compositions (Kronecker product, bialternate product, *etc.*), *e.g.* see Fu and Barmish [6], Genesio and Tesi [10], Qiu and Davison [17, 18, 19], Saydy, Tits and Abed [20], Tesi and Vicino [21]. In these papers it is shown that this method can be used to treat several class of matrix uncertainty models. The purpose of this paper is to give an overview of the mathematical tools used in this method, and to discuss some applications of the method.

The Kronecker product of matrices is a familiar matrix composition. It appears in many standard linear algebra references. A dedicated treatment is given in Graham [8]. Some of its applications to circuit and system theory is discussed in Brewer [5]. Its application in matrix stability problems can be found in Barnett and Storey [3] and Fuller [7]. The so-called "bialternate product" of matrices is not commonly seen in the literature; its definition and properties may be traced to a paper by Stéphanos [22] published early this century. Further discussion can be found in Bellman [4], Fuller [7], Jury [12] and MacDuffee [14]. This product is used in [7] and [12] to study the matrix stability problem. Another matrix composition, called the "Lyapunov matrix", was originally defined in Lyapunov [13]. Further discussion of this matrix can be found in Barnett and Storey [3], Fuller [7], Jury [12] and MacFarlane [14]. It is used in [3], [12], [14] in solving the Lyapunov equation and is used in [7] to analyze matrix stability. All of the above literature introduces these matrix compositions as separate entities, and the relationship between them has not been emphasized.

In this paper, we will give the definition, and some of the properties, of a class of matrix compositions, which includes the ones mentioned above, in the frame-work of modern multilinear algebra, see, *e.g.* Marcus [14], Greub [9]. It will be seen that the matrix compositions mentioned above are related to various tensor products of linear spaces and maps; the relationship between them will then become clear and the proofs of many of their properties can then be simplified.

The multilinear algebra is rooted in the tensor product of linear spaces. It is of interest to note however that the definition of the tensor product of spaces varies in different references. The different definitions which have been made may be represented by the following group of references: i)Greub [9], Marcus [14]; ii)Atkison [2], Halmos [11], Waerden [23]; iii)Jacobson [12]; iv)Akivis and Goldberg [1]. We adopt the definition in [2]. Multilinear algebra is a mature branch of mathematics, but its application in control theory has been scare. This paper may provide some perspective for potential future research.

The structure of this paper is as follows. Section 2 rigorously gives the theory of tensor products needed in the stability robustness analysis. The material is obtained mainly from [2], [14]. Section 3 shows how the tensor product enters into the stability robustness analysis. It is shown that the key role which is played by the tensor product is that it can be used to reduce a robust stability problem to a singularity problem, which leads to a significant simplification. Section 4 studies the stability robustness of matrices with unstructured uncertainties for both the continuous time case and the discrete time case. In the continuous time case, we study additive uncertainties; the results given are directly obtained from [17]. In the discrete time case, we study multiplicative uncertainties; the results obtained are new. Section 5 considers the stability robustness of matrices with one-parameter uncertainties. The results are obtained from [10], [20], [21]. Section 6 considers the stability robustness of matrices with two-parameter uncertainties; it is shown that the problem can be reduced to a problem involving the solution of a set of two polynomial equations in two variables, and that tensor products can be used to solve the set of equations.

Although we are primarily interested in the stability robustness analysis of matrices, the results obtained will be stated more conveniently, if we consider matrices as linear maps between finite-dimensional linear spaces. We denote by $\mathbf{F}$ the fields of real numbers $\mathbf{R}$ or the field of complex numbers $\mathbf{C}$; we use script letters $\mathcal{X}$, $\mathcal{Y}$, ... to denote finite-dimensional linear spaces over $\mathbf{F}$. The set of all linear maps on $\mathcal{X}$ is denoted by $\mathrm{L}(\mathcal{X})$. If $A \in \mathrm{L}(\mathcal{X})$, the set of all eigenvalues of A (including multiplicities) is denoted by $\Lambda(A)$. The trace and the determinant of A, $\mathrm{tr}(A)$ and $\det(A)$, are the sum and the product of all eigenvalues of A respectively. In the case when $\mathcal{X}$ is an inner product space with $\dim(\mathcal{X}) = n$ and when $A \in \mathrm{L}(\mathcal{X})$, $\sigma_i(A)$, $i = 1, 2, \ldots, n$, denotes the i-th singular value of A with ordering $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A)$; in particular, $\sigma_1(A)$ and $\sigma_n(A)$ are denoted by $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ respectively. The spectral norm of A is denoted by $\|A\|_s$, which has the property that $\|A\|_s = \overline{\sigma}(A)$.

*This work has been supported by the National Sciences and Engineering Research Council of Canada under grant no. A4396.

2 Tensor Products

Let $\mathcal{X}$, $\mathcal{Y}$ be finite-dimensional linear spaces over F with $\dim(\mathcal{X}) = n$ and $\dim(\mathcal{Y}) = m$. A function $\phi : \mathcal{X} \times \mathcal{Y} \to \mathsf{F}$ is called a *bilinear form* on $\mathcal{X} \times \mathcal{Y}$ if it is linear in each of its arguments when the other is held fixed, i.e.

$$\begin{aligned}\phi(\alpha_1 x_1 + \alpha_2 x_2, y) &= \alpha_1\phi(x_1, y) + \alpha_2\phi(x_2, y)\\ \phi(x, \alpha_1 y_1 + \alpha_2 y_2) &= \alpha_1\phi(x, y_1) + \alpha_2\phi(x, y_2),\end{aligned}$$

for all $x, x_i \in \mathcal{X}$, $y, y_i \in \mathcal{Y}$ and $\alpha_i \in \mathsf{F}$ for $i = 1, 2$. The set of all bilinear forms on $\mathcal{X} \times \mathcal{Y}$ is a linear space with the addition and multiplication by scalars defined in the obvious way. It is easy to see that the dimension of this linear space is nm, the product of the dimensions of $\mathcal{X}$ and $\mathcal{Y}$.

Let $\mathcal{X}'$, $\mathcal{Y}'$ be the dual space of $\mathcal{X}$, $\mathcal{Y}$ respectively. The *tensor product* of the space $\mathcal{X}$ and $\mathcal{Y}$, denoted by $\mathcal{X} \otimes \mathcal{Y}$, is the linear space of all bilinear forms on $\mathcal{X}' \times \mathcal{Y}'$. Since $\mathcal{X}'$ and $\mathcal{X}$ have the same dimensions, we have $\dim(\mathcal{X} \otimes \mathcal{Y}) = nm$.

One type of element in $\mathcal{X} \otimes \mathcal{Y}$ is of particular interest. Let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Assign for any $f \in \mathcal{X}'$ and $g \in \mathcal{Y}'$ the scalar $f(x)g(y)$. This is obviously a bilinear form on $\mathcal{X}'$ and $\mathcal{Y}'$ and hence an element in $\mathcal{X} \otimes \mathcal{Y}$. This element is denoted by $x \otimes y$. Elements of $\mathcal{X} \otimes \mathcal{Y}$ which can be written as $x \otimes y$ for some $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are called *decomposable tensors*. It can be shown that the set of all decomposable tensors spans the whole $\mathcal{X} \otimes \mathcal{Y}$.

Let $A \in \mathbf{L}(\mathcal{X})$ and $B \in \mathbf{L}(\mathcal{Y})$. The *tensor product* of A and B, denoted by $A \otimes B$, is an element in $\mathbf{L}(\mathcal{X} \otimes \mathcal{Y})$ and is defined to be the linear extension of the following map of decomposable tensors

$$(A \otimes B)(x \otimes y) = Ax \otimes By. \tag{1}$$

Since the decomposable tensors in $\mathcal{X} \otimes \mathcal{Y}$ span $\mathcal{X} \otimes \mathcal{Y}$, the extension is uniquely determined.

Furthermore, let $C \in \mathbf{L}(\mathcal{X})$ and $D \in \mathbf{L}(\mathcal{Y})$. It is easy to see from (1) that

$$(C \otimes D)(A \otimes B) = CA \otimes DB.$$

The following theorem, regarding the eigenvalues of a combination of tensor products of linear maps, is a key result in our development.

Theorem 1 *Let $A \in \mathbf{L}(\mathcal{X})$, $B \in \mathbf{L}(\mathcal{Y})$ and let $\Lambda(A) = \{\lambda_i : i = 1, 2, \ldots, n\}$, $\Lambda(B) = \{\mu_j : j = 1, 2, \ldots, m\}$. Then*

$$\begin{aligned}&\Lambda\Big(\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k\Big)\\ &= \Big\{\sum_{h,k=0}^{l} \gamma_{hk}\lambda_i^h \mu_j^k : i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\Big\}.\end{aligned}$$

Now let us restrict our attention to bilinear forms on $\mathcal{X} \times \mathcal{X}$. A bilinear form ϕ on $\mathcal{X} \times \mathcal{X}$ is said to be *symmetric* if $\phi(x_1, x_2) = \phi(x_2, x_1)$ for all $x_1, x_2 \in \mathcal{X}$. It is said to be *skew-symmetric* if $\phi(x_1, x_2) = -\phi(x_2, x_1)$ for all $x_1, x_2 \in \mathcal{X}$. The set of all symmetric bilinear forms on $\mathcal{X}' \times \mathcal{X}'$ is called the *symmetric tensor product* of $\mathcal{X}$ with itself and is denoted by $\mathcal{X} \vee \mathcal{X}$. The set of all skew-symmetric bilinear forms on $\mathcal{X}' \times \mathcal{X}'$ is called the *skew-symmetric tensor product* of $\mathcal{X}$ with itself and is denoted by $\mathcal{X} \wedge \mathcal{X}$. Apparently, $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$ are subspaces of $\mathcal{X} \otimes \mathcal{X}$. Let $\phi \in \mathcal{X} \otimes \mathcal{X}$. Define bilinear forms ϕ_1, ϕ_2 on $\mathcal{X}' \times \mathcal{X}'$ by

$$\begin{aligned}\phi_1(f, g) &= \frac{1}{2}[\phi(f, g) + \phi(g, f)]\\ \phi_2(f, g) &= \frac{1}{2}[\phi(f, g) - \phi(g, f)].\end{aligned}$$

Then ϕ_1 is symmetric, ϕ_2 is skew-symmetric and $\phi = \phi_1 + \phi_2$. This shows that

$$(\mathcal{X} \vee \mathcal{X}) + (\mathcal{X} \wedge \mathcal{X}) = \mathcal{X} \otimes \mathcal{X}.$$

On the other hand, if $\phi \in (\mathcal{X} \vee \mathcal{X}) \cap (\mathcal{X} \wedge \mathcal{X})$, then $\phi(f, g) = \phi(g, f) = -\phi(f, g)$ for any $f, g \in \mathcal{X}'$, which means $\phi(f, g) = 0$ for all $f, g \in \mathcal{X}'$. This shows that

$$(\mathcal{X} \vee \mathcal{X}) \cap (\mathcal{X} \wedge \mathcal{X}) = 0.$$

Therefore $\mathcal{X} \otimes \mathcal{X}$ is the direct sum of $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$. The following two projections can then be defined: $P_\vee : \mathcal{X} \otimes \mathcal{X} \to \mathcal{X} \vee \mathcal{X}$ is the projection on $\mathcal{X} \vee \mathcal{X}$ along $\mathcal{X} \wedge \mathcal{X}$ and $P_\wedge \mathcal{X} \otimes \mathcal{X} \to \mathcal{X} \wedge \mathcal{X}$ is the projection on $\mathcal{X} \wedge \mathcal{X}$ along $\mathcal{X} \vee \mathcal{X}$. Since $\mathcal{X} \otimes \mathcal{X}$ is spanned by the decomposable tensors, it is of interest to determine the effects of $P_\vee$ and $P_\wedge$ on the decomposable tensor. A moment's thought leads to

$$\begin{aligned}P_\vee(x_1 \otimes x_2) &= \frac{1}{2}(x_1 \otimes x_2 + x_2 \otimes x_1)\\ P_\wedge(x_1 \otimes x_2) &= \frac{1}{2}(x_1 \otimes x_2 - x_2 \otimes x_1).\end{aligned}$$

The tensors of the above forms are called *decomposable symmetric tensors* and *decomposable skew-symmetric tensors* respectively, and are denoted by $x_1 \vee x_2$ and $x_1 \wedge x_2$ respectively.

Let $A, B \in \mathbf{L}(\mathcal{X})$. The *symmetric tensor product* of A and B, denoted by $A \vee B$, is defined to be $P_\vee(A \otimes B)|\mathcal{X} \vee \mathcal{X}$. The *skew-symmetric tensor product* of A and B, denoted by $A \wedge B$, is defined to by $P_\wedge(A \otimes B)|\mathcal{X} \wedge \mathcal{X}$.

Another key result, regarding the eigenvalues of a combination of symmetric and skew-symmetric tensor products of linear maps, is given in the following.

Theorem 2 *Let $A \in \mathbf{L}(\mathcal{X})$ and let $\Lambda(A) = \{\lambda_i : i = 1, \ldots, n\}$. Then*

$$\begin{aligned}\Lambda\Big(\sum_{h,k=0}^{l} \gamma_{hk} A^h \vee A^k\Big) &= \Big\{\frac{1}{2}\sum_{h,k=0}^{l} \gamma_{hk}(\lambda_i^h\lambda_j^k + \lambda_i^k\lambda_j^h) :\\ &\qquad i = 1, \ldots, n;\ j = i, \ldots, n\Big\}\\ \Lambda\Big(\sum_{h,k=0}^{l} \gamma_{hk} A^h \wedge A^k\Big) &= \Big\{\frac{1}{2}\sum_{h,k=0}^{l} \gamma_{hk}(\lambda_i^h\lambda_j^k + \lambda_i^k\lambda_j^h) :\\ &\qquad i = 1, \ldots, n-1;\ j = i+1, \ldots, n\Big\}.\end{aligned}$$

The set of coefficients $\{\gamma_{hk}\}_{h,k=0}^{l}$ is said to be symmetric if $\gamma_{hk} = \gamma_{kh}$ for all $0 \le h, k \le l$. It is of interest to note that in the case when $\{\gamma_{hk}\}_{h,k=0}^{l}$ is symmetric, the direct union of $\Lambda(\sum_{h,k=0}^{l} \gamma_{hk} A^h \vee A^k)$ and $\Lambda(\sum_{h,k=0}^{l} \gamma_{hk} A^h \wedge A^k)$ is exactly equal to $\Lambda(\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes A^k)$. This fact is a consequence of the following more general result.

Theorem 3 *If $\{\gamma_{hk}\}_{h,k=0}^{l}$ is a set of symmetric coefficients, then $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$ are reducing subspaces of $\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k$.*

Theorem 3 implies that if $\mathcal{X} \otimes \mathcal{X}$ is decomposed as the direct sum of $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$, then the map $\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k$ has the following "diagonal" structure:

$$\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k = \begin{bmatrix} \sum_{h,k=0}^{l} \gamma_{hk} A^h \vee B^k & 0 \\ 0 & \sum_{h,k=0}^{l} \gamma_{hk} A^h \wedge B^k \end{bmatrix}.$$

Now suppose that $\mathcal{X}$ and $\mathcal{Y}$ are inner product spaces. Then $\mathcal{X} \otimes \mathcal{Y}$ becomes an inner product space if we let the inner product in $\mathcal{X} \otimes \mathcal{X}$ be the sesquilinear extension of the following inner product of decomposable tensors

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle = \langle x_1, x_2\rangle\langle y_1, y_2\rangle.$$

With this inner product structure, we can talk about singular values of linear maps.

Theorem 4 *Let $\mathcal{X}$, $\mathcal{Y}$ be inner product spaces and $A \in \mathrm{L}(\mathcal{X})$ and $B \in \mathrm{L}(\mathcal{Y})$. Then the set of singular values of $A \otimes B$ is $\{\sigma_i(A)\sigma_j(B) : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$.*

This theorem implies that $\overline{\sigma}(A \otimes B) = \overline{\sigma}(A)\overline{\sigma}(B)$ and $\underline{\sigma}(A \otimes B) = \underline{\sigma}(A)\underline{\sigma}(B)$.

Let $x_1, x_2, x_3, x_4 \in \mathcal{X}$. Then

$$\begin{aligned}
&\langle x_1 \vee x_2, x_3 \wedge x_4 \rangle \\
&= \langle \frac{1}{2}(x_1 \otimes x_2 + x_2 \otimes x_1), \frac{1}{2}(x_3 \otimes x_4 - x_4 \otimes x_3) \rangle \\
&= \frac{1}{4}(\langle x_1, x_3 \rangle \langle x_2, x_4 \rangle - \langle x_1, x_4 \rangle \langle x_2, x_3 \rangle \\
&\qquad + \langle x_2, x_3 \rangle \langle x_1, x_4 \rangle - \langle x_2, x_4 \rangle \langle x_1, x_3 \rangle) \\
&= 0.
\end{aligned}$$

Since the decomposable tensors span $\mathcal{X} \otimes \mathcal{X}$, the symmetric and skew-symmetric decomposable tensors span $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$ respectively. Therefore $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$ are orthogonal complements of each other and $P_\vee$ and $P_\wedge$ are orthogonal projections. By the fact that $\overline{\sigma}(P_\vee) = 1$ and $\overline{\sigma}(P_\wedge) = 1$, we obtain $\overline{\sigma}(A \vee B) \leq \overline{\sigma}(A)\overline{\sigma}(B)$ and $\overline{\sigma}(A \wedge B) \leq \overline{\sigma}(A)\overline{\sigma}(B)$ for any $A, B \in \mathrm{L}(X)$.

Let $\{x_i : i = 1, 2, \ldots, n\}$ be a basis of $\mathcal{X}$ and $\{y_j : j = 1, 2, \ldots, m\}$ be a basis of $\mathcal{Y}$. Let the matrix representations of $A \in \mathrm{L}(\mathcal{X})$ and $B \in \mathrm{L}(\mathcal{Y})$ under these bases be $[A] = [a_{ij}]$ and $[B] = [b_{ij}]$ respectively. It is easy to verify that

$$\{x_i \otimes y_j : i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\} \tag{2}$$

forms a basis of $\mathcal{X} \otimes \mathcal{Y}$; by saying this we also mean that the basis vectors $x_i \otimes y_j$ are ordered lexicographically, *i.e.* $x_{i_1} \otimes y_{j_1}$ preceeds $x_{i_2} \otimes y_{j_2}$ if $mi_1 + j_1 < mi_2 + j_2$. The matrix representation of $A \otimes B$ under the basis (2) is given by

$$[A \otimes B] = \begin{bmatrix} a_{11}[B] & \cdots & a_{1n}[B] \\ \vdots & & \vdots \\ a_{n1}[B] & \cdots & a_{nn}[B] \end{bmatrix}.$$

This is just the Kronecker product of the matrices $[A]$ and $[B]$.

Now let $\{x_i : i = 1, 2, \ldots, n\}$ be a basis of $\mathcal{X}$ and let $[A] = [a_{ij}]$, $[B] = [b_{ij}]$ be the matrix representations of $A, B \in \mathrm{L}(\mathcal{X})$ respectively. Then a basis of $\mathcal{X} \vee \mathcal{X}$ is given by

$$\{\alpha_{ij} x_i \vee x_j : i = 1, 2, \ldots, n; j = i, i+1, \ldots, n\}, \tag{3}$$

where

$$\alpha_{ij} = \begin{cases} \frac{1}{2} & \text{if } i = j \\ \frac{1}{\sqrt{2}} & \text{if } i \neq j, \end{cases}$$

and a basis of $\mathcal{X} \wedge \mathcal{X}$ is given by

$$\{\frac{1}{\sqrt{2}} x_i \vee x_j : i = 1, 2, \ldots, n-1; j = i+1, i+2, \ldots, n\}. \tag{4}$$

Here it is also assumed that the basis vectors in (2) and (3) are ordered lexicographically. The matrix representation $[A \vee B]$ of $A \vee B$ under basis (2) can be given by the following way: Let (p_1, p_2) and (q_1, q_2) be the p-th and q-th pairs of integers respectively in the lexicographically ordered sequence $\{(i, j) : i = 1, 2, \ldots, n; j = i, i+1, \ldots, n\}$. Then

$$[A \vee B] = [c_{pq}] \in \mathrm{F}^{\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)},$$

where

$$c_{pq} = \begin{cases} a_{p_1q_1} b_{p_1q_1} & \text{if } p_1 = p_2 \text{ and } q_1 = q_2 \\ \frac{1}{2}(a_{p_1q_1} b_{p_2q_2} + a_{p_1q_2} b_{p_2q_1} & \\ \quad + a_{p_2q_1} b_{p_1q_2} + a_{p_2q_2} b_{p_1q_1}) & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2 \\ \frac{\sqrt{2}}{2}(a_{p_1q_1} b_{p_2q_2} + a_{p_2q_2} b_{p_1q_1}) & \text{otherwise.} \end{cases} \tag{5}$$

The so-called "Lyapunov matrix" is the matrix representation of $A \vee I + I \vee A$ under a different basis of $\mathcal{X} \vee \mathcal{X}$.

The matrix representation of $A \wedge B$ under basis (4) is given by the following way: Let (r_1, r_2) and (s_1, s_2) be the r-th and s-th pairs of integers respectively in the lexicographically ordered sequence $\{(i, j) : i = 1, 2, \ldots, n-1; j = i+1, i+2, \ldots, n\}$. Then

$$[A \wedge B] = [d_{rs}] \in \mathrm{F}^{\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)},$$

where

$$d_{rs} = \frac{1}{2}(a_{r_1s_1} b_{r_2s_2} - a_{r_1s_2} b_{r_2s_1} - a_{r_2s_1} b_{r_1s_2} + a_{r_2s_2} b_{r_1s_1}). \tag{6}$$

This is the "bialternate product" of matrices $[A]$ and $[B]$.

Note that if $\mathcal{X}$, $\mathcal{Y}$ are inner product spaces and $\{x_i\}$, $\{y_j\}$ are orthonormal bases of $\mathcal{X}$, $\mathcal{Y}$ respectively, then (2), (3), (4) are orthonormal bases of $\mathcal{X} \otimes \mathcal{Y}$, $\mathcal{X} \vee \mathcal{Y}$, $\mathcal{X} \wedge \mathcal{X}$ respectively. This property is important regarding singular values.

To complete this section, we outline the idea of the proof of the key results Theorems 1-2. It is known that for any $A \in \mathrm{L}(\mathcal{X})$, there exists a basis $\{x_i\}$ such that $[A]$ is an upper triangular matrix. Such a basis is called a triangular basis for A and under this basis the eigenvalues of A are just the diagonal elements of $[A]$. Similarly, for any $B \in \mathrm{L}(\mathcal{Y})$, there exists a triangular basis $\{y_j\}$ of $\mathcal{Y}$ for B such that $[B]$ is an upper triangular matrix with the eigenvalues of B as its diagonal elements. It is easy to check that the basis (2) of $\mathcal{X} \otimes \mathcal{Y}$ formed from the triangular basis of $\mathcal{X}$ for A and that of $\mathcal{Y}$ for B is a triangular basis for the map $\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k$. Therefore the diagonal elements of $[\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k]$, which are $\sum_{h,k=0}^{l} \gamma_{hk} \lambda_i^h \mu_j^k$, are the eigenvalues of $\sum_{h,k=0}^{l} \gamma_{hk} A^h \otimes B^k$. Similarly, the bases (3) and (4) of $\mathcal{X} \vee \mathcal{X}$ and $\mathcal{X} \wedge \mathcal{X}$ formed from the triangular basis of $\mathcal{X}$ for A are triangular bases for $\sum_{h,k=0}^{l} \gamma_{hk} A^h \vee A^k$ and $\sum_{h,k=0}^{l} \gamma_{hk} A^h \wedge A^l$ respectively. Therefore the diagonal elements of $[\sum_{h,k=0}^{l} \gamma_{hk} A^h \vee A^k]$, which are $\frac{1}{2} \sum_{h,k=0}^{l} \gamma_{hk} (\lambda_i^h \lambda_j^k + \lambda_i^k \lambda_j^h)$, are the eigenvalues of $\sum_{h,k=0}^{l} \gamma_{hk} A^h \vee A^k$ and the diagonal elements of $[\sum_{h,k=0}^{l} \gamma_{hk} A^h \wedge A^k]$, which are $\frac{1}{2} \sum_{h,k=0}^{l} \gamma_{hk} (\lambda_i^h \lambda_j^k + \lambda_i^k \lambda_j^h)$, are the eigenvalues of $\sum_{h,k=0}^{l} \gamma_{hk} A^h \wedge A^k$.

3 Preliminary Stability Robustness Results

The main purpose of this paper is to use the tensor product concept as a tool to study the stability robustness of real matrices. To make what follows conform with the concept introduced in last section, we can consider a matrix in $\mathrm{F}^{n \times n}$ as a map on F^n. We will study the stability of matrices both with respect to continuous time systems and with respect to discrete time systems. A matrix $A \in \mathbf{R}^{n \times n}$ is stable with respect to continuous time systems if all of its eigenvalues are contained in the open left half of the complex plane. The set of all such stable matrices is denoted by S_c. A matrix $A \in \mathbf{R}^{n \times n}$ is stable with respect to discrete time systems if all its eigenvalues are contained in the open unit disk of the complex plane. The set of all such stable matrices is denoted by S_d. A basic problem considered in the stability robustness of matrices is as follows. Given a connected[1] set of matrices in $\mathbf{R}^{n \times n}$, determine if all the elements in this set are stable. The remaining sections of this paper simply consider this problem for different forms of sets. The following fundamental results serve as the starting point of the study of all of the problems in the remaining sections.

Theorem 5 *Let U be a connected subset of $\mathbf{R}^{n \times n}$. Assume that $U \cap S_c$ is not empty. Then the following statements are equivalent.*

(a) $U \subset S_c$.

[1]The connectedness is a topological concept. Here we assume that the topology in $\mathbf{R}^{n \times n}$ is the usual finite-dimensional linear space topology.

(b) $\det(A\otimes I+I\otimes A)\neq 0$ *for all* $A\in U$.

(c) $\det(A)\neq 0$ *and the rank of* $A\otimes I+I\otimes A$ *is greater than* n^2-2 *for all* $A\in U$.

(d) $\det(A\vee I+I\vee A)\neq 0$ *for all* $A\in U$.

(e) $\det(A)\neq 0$ *and* $\det(A\wedge I+I\wedge A)\neq 0$ *for all* $A\in U$.

Theorem 6 *Let* V *be a connected subset of* $\mathbf{R}^{n\times n}$. *Assume that* $V\cap S_d$ *is not empty. Then the following statements are equivalent.*

(a) $V\subset S_d$.

(b) $\det(I-A\otimes A)\neq 0$ *for all* $A\in V$.

(c) $\det(I-A)\neq 0$, $\det(I+A)\neq 0$ *and the rank of* $\det(I-A\otimes A)\neq 0$ *is greater than* n^2-2 *for all* $A\in V$.

(d) $\det(I-A\vee A)\neq 0$ *for all* $A\in V$.

(e) $\det(I-A)\neq 0$, $\det(I+A)\neq 0$ *and* $\det(I-A\wedge A)\neq 0$ *for all* $A\in V$.

The proofs of Theorems 5-6 can be constructed by using simple continuity arguments and Theorems 1-2 in the last section.

4 Unstructured Uncertainties

In this section, we study the stability robustness of matrices under unstructured uncertainties. In the continuous time case, we consider uncertainties of the form $A+\Delta A$; such a form of uncertainty is called an *additive* uncertainty. In the discrete time case, we consider uncertainties of the form $A(I+\Delta A)$; such a form of uncertainty is called a *multiplicative* uncertainty.

I Continuous time case

Let $A\in\mathbf{R}^{n\times n}$ be a stable matrix in S_c. Define the (real) stability radius of A by

$$r_c(A)=\inf\{\|\Delta A\|_s:\Delta A\in\mathbf{R}^{n\times n}\text{ and }A+\Delta A\not\subset S_c\}. \quad (7)$$

It is desired to have a method to compute $r_c(A)$. Unfortunately, such a method is not available for general matrices in S_c. The following theorem gives some lower bounds on $r_c(A)$.

Theorem 7 *Suppose that* $A\in\mathbf{R}^{n\times n}$ *is a stable matrix. Then*

$$\begin{aligned} r_c(A) &\geq \min\{\underline{\sigma}(A),\tfrac{1}{2}\sigma_{n^2-1}(A\otimes I+I\otimes A)\}\\ r_c(A) &\geq \tfrac{1}{2}\underline{\sigma}(A\vee I+I\vee A)\\ r_c(A) &\geq \min\{\underline{\sigma}(A),\tfrac{1}{2}\underline{\sigma}(A\wedge I+I\wedge A)\}.\end{aligned}$$

In some special cases, the inequalities in Theorem 7 becomes equalities.

Theorem 8 *Suppose that* $A\in\mathbf{R}^{n\times n}$ *is a stable normal matrix. Then*

$$\begin{aligned} r_c(A) &= \min\{\underline{\sigma}(A),\tfrac{1}{2}\sigma_{n^2-1}(A\otimes I+I\otimes A)\}\\ &= \tfrac{1}{2}\underline{\sigma}(A\vee I+I\vee A)\\ &= \min\{\underline{\sigma}(A),\tfrac{1}{2}\underline{\sigma}(A\wedge I+I\wedge A)\}\\ &= \min\{-\Re(\lambda_i):\lambda_i\in \mathrm{sp}(A)\}.\end{aligned}$$

Theorem 9 *Suppose that* $A\in\mathbf{R}^{2\times 2}$ *is a stable matrix. Then*

$$\sigma_{n^2-1}(A\otimes I+I\otimes A)=\underline{\sigma}(A\wedge I+I\wedge A)=-\mathrm{tr}(A)$$

and

$$r_c(A)=\min\{\underline{\sigma}(A),-\tfrac{1}{2}\mathrm{tr}(A)\}.$$

The proof of Theorems 7-9 are given in [17].

II Discrete time case

Let $A\in\mathbf{R}^{n\times n}$ be a stable matrix in S_d. Define the (real) stability radius of A by

$$r_d(A)=\inf\{\|\Delta A\|_s:\Delta A\in\mathbf{R}^{n\times n}\text{ and }A(I+\Delta A)\not\subset S_d\}. \quad (8)$$

As similar to the continuous time case, a method to compute $r_d(A)$ is not available for general matrices in S_d. The following theorem gives some lower bounds on $r_d(A)$.

Theorem 10 *Suppose that* A *is a stable matrix in* $\mathbf{R}^{n\times n}$. *Then*

$$\begin{aligned} r_d(A) &\geq \min\Big\{\overline{\sigma}^{-1}[(I-A)^{-1}A],\ \overline{\sigma}^{-1}[(I+A)^{-1}A],\\ &\qquad \Big(\sigma_2^{-1}[(I-A\otimes A)^{-1}(A\otimes A)]+1\Big)^{\frac{1}{2}}-1\Big\}\\ r_d(A) &\geq \Big(\overline{\sigma}^{-1}[(I-A\vee A)^{-1}(A\vee A)]+1\Big)^{\frac{1}{2}}-1\\ r_d(A) &\geq \min\Big\{\overline{\sigma}^{-1}[(I-A)^{-1}A],\ \overline{\sigma}^{-1}[(I+A)^{-1}A],\\ &\qquad \Big(\overline{\sigma}^{-1}[(I-A\wedge A)^{-1}(A\wedge A)]+1\Big)^{\frac{1}{2}}-1\Big\}.\end{aligned}$$

In some special cases, the inequalities in Theorem 9 becomes equalities.

Theorem 11 *Suppose that* A *is a stable normal matrix in* $\mathbf{R}^{n\times n}$. *Then*

$$\begin{aligned} r_d(A) &= \min\Big\{\overline{\sigma}^{-1}[(I-A)^{-1}A],\ \overline{\sigma}^{-1}[(I+A)^{-1}A],\\ &\qquad \Big(\sigma_2^{-1}[(I-A\otimes A)^{-1}(A\otimes A)]+1\Big)^{\frac{1}{2}}-1\Big\}\\ &= \Big(\overline{\sigma}^{-1}[(I-A\vee A)^{-1}(A\vee A)]+1\Big)^{\frac{1}{2}}-1\\ &= \min\Big\{\overline{\sigma}^{-1}[(I-A)^{-1}A],\ \overline{\sigma}^{-1}[(I+A)^{-1}A],\\ &\qquad \Big(\sigma_2^{-1}[(I-A\wedge A)^{-1}(A\wedge A)]+1\Big)^{\frac{1}{2}}-1\Big\}\\ &= \min\Big\{\frac{1}{|\lambda_i|}-1:\lambda\in\Lambda(A)\Big\}.\end{aligned}$$

Theorem 12 *Suppose that* $A\in\mathbf{R}^{2\times 2}$ *is a stable matrix. Then*

$$\sigma_2[(I-A\otimes A)^{-1}(A\otimes A)]=\overline{\sigma}[(I-A\wedge A)^{-1}(A\wedge A)]=\frac{\det(A)}{1-\det(A)}$$

and

$$r_d(A)=\min\{\overline{\sigma}^{-1}[(I-A)^{-1}A],\overline{\sigma}^{-1}[(I+A)^{-1}A],\det{}^{-\frac{1}{2}}(A)-1\}.$$

The results for the discrete time case are new. The proofs of Theorems 10-12 are analogous to the proofs of Theorems 7-9. It is seen that Theorems 10-12 are parallel to Theorems 7-9. The additive uncertainty problem for the discrete time case is considered in [17], and it is shown there that the results obtained are not parallel to the continuous time case, *e.g.* for 2×2 stable real matrices, an exact value of the real stability radius for additive uncertainties is not obtained from the lower bound. This is the reason why here we consider multiplicative uncertainties, instead of additive uncertainties, in the discrete time case.

5 One-Parameter Uncertainties

A matrix $A(k)$ with one-parameter uncertainty is of the form

$$A(k)=A_0+kA_1+k^2A_2+\cdots+k^lA_l, \quad (9)$$

where A_i, $i = 0, 1, \ldots, l$, are real matrices and k is a real uncertain parameter. Assume that A_0 is stable (either in the continuous time sense or in the discrete time sense). It is desired to find the largest open interval $(\underline{k}, \overline{k})$, which contains the origin, such that $A(k)$ is stable for all $k \in (\underline{k}, \overline{k})$. This problem is called the one-parameter stability problem. In the continuous time case, it follows from Theorem 5 that the one-parameter stability problem is equivalent to the problem of finding the largest open interval $(\underline{k}, \overline{k})$ such that any of the following is satisfied for all $k \in (\underline{k}, \overline{k})$.

(a) $\det[A(k) \otimes I + I \otimes A(k)] \neq 0$;

(b) $\det[A(k) \vee I + I \vee A(k)] \neq 0$;

(c) $\det[A(k)] \neq 0$ and $\det[A(k) \wedge I + I \wedge A(k)] \neq 0$.

It is easy to verify that all the matrices in (a)-(c) are polynomial matrices of k with degree l. Similarly, in the discrete time case, it follows from Theorem 6 that the one-parameter stability problem is equivalent to the problem of finding the largest open interval $(\underline{k}, \overline{k})$ such that any of the following is satisfied for all $k \in (\underline{k}, \overline{k})$.

(a') $\det[I - A(k) \otimes A(k)] \neq 0$;

(b') $\det[I - A(k) \vee A(k)] \neq 0$;

(c') $\det[I - A(k)] \neq 0$, $\det[I + A(k)] \neq 0$ and $\det[I - A(k) \wedge A(k)] \neq 0$.

It is easy to verify that all the matrices in (a')-(c') are polynomial matrices of k with degree l or $2l$.

Hence, by using Theorems 5-6, one can convert the one-parameter stability problem into the following one-parameter singularity problem: Let

$$B(k) = B_0 + kB_1 + k^2 B_2 + \cdots + k^m B_m, \qquad (10)$$

with B_0 nonsingular. Find the largest open interval $(\underline{k}, \overline{k})$, which contains the origin, such that $B(k)$ is nonsingular for all $k \in (\underline{k}, \overline{k})$. This singularity problem can be easily solved via an ordinary eigenvalue problem.

Formally, we can define

$$\begin{aligned} \underline{k} &= \sup\{k : k < 0 \text{ and } \det[B(k)] = 0\} &(11)\\ \overline{k} &= \inf\{k : k > 0 \text{ and } \det[B(k)] = 0\}. &(12) \end{aligned}$$

We know that

$$\begin{aligned} &\det(B_0^{-1}) \det[B(k)] \\ &\quad = \det(B_0^{-1}) \det(B_0 + kB_1 + k^2 B_2 + \cdots + k^m B_m) \end{aligned}$$

$$= \det\left(I - k \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -B_0^{-1}B_m & -B_0^{-1}B_{m-1} & \cdots & -B_0^{-1}B_1 \end{bmatrix} \right).$$

Let

$$M = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -B_0^{-1}B_m & -B_0^{-1}B_{m-1} & \cdots & -B_0^{-1}B_1 \end{bmatrix}. \qquad (13)$$

Then the equality $\det[B(k)] = 0$ is equivalent to the fact that $\frac{1}{k}$ is an eigenvalue of M. Therefore, the solution of the one-parameter singularity problem is given as follows.

$$\begin{aligned} \underline{k} &= \sup\{k : k < 0 \text{ and } \frac{1}{k} \in \mathrm{sp}(M)\} \\ \overline{k} &= \inf\{k : k > 0 \text{ and } \frac{1}{k} \in \mathrm{sp}(M)\}. \end{aligned}$$

6 Double-Parameter Uncertainties

First in this section, we introduce the double-parameter eigenvalue problem and its solution. Then we will see how it can be used to obtain a solution to the double-parameter uncertainty stability robustness problem.

Let $A_0, A_1, A_2 \in \mathbf{L}(\mathcal{X})$ and $B_0, B_1, B_2 \in \mathbf{L}(\mathcal{Y})$. The double-parameter eigenvalue problem is to find all pairs of real numbers (λ_1, λ_2) such that

$$\left\{ \begin{array}{rcl} \det[A_0 + \lambda_1 A_1 + \lambda_2 A_2] &=& 0 \\ \det[B_0 + \lambda_1 B_1 + \lambda_2 B_2] &=& 0. \end{array} \right. \qquad (14)$$

Obviously, (14) is actually a set of two polynomial equations in two unknowns. It is not hard to show that any set of two polynomial equations of two unknowns can be converted to the form of (14). Conventionally, this set of equations is solved using resultants, see [21]. Here, we show that it can be solved using tensor products.

Let $C_1, C_2, D_1, D_2 \in \mathbf{L}(\mathcal{Z})$. The formal determinant of the map

$$\begin{bmatrix} C_1 & C_2 \\ D_1 & D_2 \end{bmatrix} \in \mathbf{L}(\mathcal{Z} \oplus \mathcal{Z}),$$

where "$\oplus$" means the direct sum, is defined as

$$\mathrm{Det}\begin{bmatrix} C_1 & C_2 \\ D_1 & D_2 \end{bmatrix} = C_1 D_2 - C_2 D_1 \in \mathbf{L}(\mathcal{Z}).$$

Consider the set of equations (14). Denote

$$\begin{aligned} \hat{A}_i &= A_i \otimes I \in \mathbf{L}(\mathcal{X} \otimes \mathcal{Y}) \\ \hat{B}_i &= I \otimes B_i \in \mathbf{L}(\mathcal{X} \otimes \mathcal{Y}) \end{aligned}$$

for $i = 0, 1, 2$. Let

$$\Delta_0 = \mathrm{Det}\begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{B}_1 & \hat{B}_2 \end{bmatrix}$$

$$\Delta_1 = \mathrm{Det}\begin{bmatrix} \hat{A}_0 & \hat{A}_2 \\ \hat{B}_0 & \hat{B}_2 \end{bmatrix}$$

$$\Delta_2 = \mathrm{Det}\begin{bmatrix} \hat{A}_1 & \hat{A}_0 \\ \hat{B}_1 & \hat{B}_0 \end{bmatrix}.$$

Then we have the following theorem:

Theorem 13 *The set of pairs (λ_1, λ_2) which solve the double parameter eigenvalue problem is contained in the set of pairs (λ_1, λ_2) satisfying*

$$\mathrm{Ker}(\Delta_1 - \lambda_1 \Delta_0) \cap \mathrm{Ker}(\Delta_2 - \lambda_2 \Delta_0) \neq 0.$$

Theorem 13 can be derived from the material in [2]. This theorem implies that the double-parameter eigenvalue problem, under a minor condition, can be solved in the following way: (step i) find the set $\Lambda(\Delta_1, \Delta_0)$ of generalized eigenvalues of the pair (Δ_1, Δ_0) and the set $\Lambda(\Delta_2, \Delta_0)$ of all the generalized eigenvalues of the pair (Δ_2, Δ_0); (step ii) for any $\lambda_1 \in \Lambda(\Delta_1, \Delta_0)$ and $\lambda_2 \in \Lambda(\Delta_2, \Delta_0)$, check if (λ_1, λ_2) is the solution of (14). It is easy to see that the minor condition required is that (Δ_1, Δ_0) and (Δ_2, Δ_0) are nondegenerate, *i.e.* they have only finite number of generalized eigenvalues. Generically, this is the case. In fact, if only one of (Δ_1, Δ_0) and (Δ_2, Δ_0) is degenerate, we still can solve the problem by finding the generalized eigenvalues of the nondegenerate pair and then substituting them into (14) to find the other unknown. The "bad" case happens when both pairs are degenerate. In this case, this method fails.

In principle, this technique for the double-parameter eigenvalue problem can be generalized to the multi-parameter eigenvalue problem in which the number of equations and the number of unknowns are more that 2, see [2] for details. Generally, when the number of equations and the number of unknowns are more than a few, the dimensions of the matrices involved become excessive, which causes severe numerical problem.

A matrix $A(k_1,k_2)$ with double-parameter uncertainty is of the form

$$A(k_1,k_2)=\sum_{i=0}^{l}\sum_{j=0}^{m}k_1^i k_2^j A_{ij}, \tag{15}$$

where A_{ij}, $i=0,1,\ldots,l$; $j=0,1,\ldots,m$, are real matrices and k_1, k_2 are the real uncertain parameters. Assume that A_{00} is stable (either in the continuous time sense or in the discrete time sense). It is desired to find the largest positive number r_m such that $A(k_1,k_2)$ is always stable for all (k_1,k_2) with $\left\|\begin{bmatrix}k_1\\k_2\end{bmatrix}\right\| < r_m$. Equivalently, we want to find r_m which is defined as

$$r_m=\inf\left\{\left\|\begin{bmatrix}k_1\\k_2\end{bmatrix}\right\| : A(k_1,k_2)\text{ is unstable}\right\}.$$

By using Theorems 5-6, this stability robustness problem can be converted to the following singularity problem: Find

$$r_m=\inf\left\{\left\|\begin{bmatrix}k_1\\k_2\end{bmatrix}\right\| : \det[B(k_1,k_2)]=0\right\},$$

where

$$B(k_1,k_2)=\sum_{i=0}^{p}\sum_{j=0}^{q}k_1^i k_2^j B_{ij},$$

Now suppose that the norm used is the Hölder 2-norm, *i.e.* $\left\|\begin{bmatrix}k_1\\k_2\end{bmatrix}\right\| = \sqrt{k_1^2+k_2^2}$. By using the Lagrange multiplier method, any pair (k_1,k_2) satisfying $r_m = \left\|\begin{bmatrix}k_1\\k_2\end{bmatrix}\right\|$ has to satisfy

$$\begin{cases}\left(k_2\frac{\partial}{\partial k_1}-k_1\frac{\partial}{\partial k_2}\right)\det[B(k_1,k_2)]=0\\ \det[B(k_1,k_2)]=0.\end{cases} \tag{16}$$

(16) is a set of two polynomial equations in two unknowns. One can convert these equations into the form of (14) and then solve them using the double parameter eigenvalue problem technique.

If the norm used is the Hölder ∞-norm, the problem also leads to solving polynomial equations, see [10] and [21] for details. The double parameter eigenvalue technique can then again be used.

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