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The Stability Robustness Determination of State Space Models with Real Unstructured Perturbations*

L. Qiu† and E. J. Davison†

Abstract. This paper considers the robust stability of a linear time-invariant state space model subject to real parameter perturbations. The problem is to find the distance of a given stable matrix from the set of unstable matrices. A new method, based on the properties of the Kronecker sum and two other composite matrices, is developed to study this problem; this new method makes it possible to distinguish real perturbations from complex ones. Although a procedure to find the exact value of the distance is still not available, some explicit lower bounds on the distance are obtained. The bounds are applicable only for the case of real plant perturbations, and are easy to compute numerically; if the matrix is large in size, an iterative procedure is given to compute the bounds. Various examples including a 46th-order spacecraft system are given to illustrate the results obtained. The examples show that the new bounds obtained can have an arbitrary degree of improvement over previously reported ones.

Key words. Robust stability, Real unstructured perturbations, Stability radius, Composite matrices, State space models.

1. Introduction

In the past decade a great deal of research has been done on the robust stability problem. However, most of the results obtained are based on the transfer function representation of a system and use frequency-domain arguments. Some attention, however, has been paid to the time-domain approach of the robust stability problem, e.g., [BG], [HM], [HP], [L], [PT], [QD1], [M3], [V], and [Y]. Two major methods are used in these papers. One is based on Lyapunov's stability theory [PT], [Y]; the other uses basically the frequency-domain stability criterion [BG], [HM], [HP], [L], [QD1], [M3], [V].

This paper develops a new method for the stability robustness analysis of a state space model subject to real perturbations. Specifically, it is desired to determine the distance of a given stable matrix $A \in \mathbb{R}^{n \times n}$ from the set of all unstable matrices in $\mathbb{R}^{n \times n}$, where the distance in $\mathbb{R}^{n \times n}$ is defined by the spectral norm. This problem has been previously studied, e.g., [BG], [HM], [HP], [L], [PT], [QD1], [M3], and

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† Department of Electrical Engineering, University of Toronto, Toronto, Ontario, Canada M5S 1A4.

[V], and some lower bounds on the distance have been obtained. These bounds are derived without assuming that the matrix space is real; therefore they are applicable for both real and complex perturbations. If only real perturbations are present, however, the bounds obtained are conservative. In this paper lower bounds are obtained assuming that only real perturbations are present. The approach used is based on some properties of the Kronecker sum and two other composite matrices. The new bounds are easy to compute numerically if A is modest in size and the computations required are numerically well defined. If the size of A is large, the new bounds can be computed using an iterative procedure with no excessive complexity required. Examples show that the new bounds obtained are less conservative than previously reported ones.

The structure of this paper is as follows. Section 2 formally defines the stability robustness measure to be studied and summarizes some existing results on this measure. Section 3 contains some preliminary results on properties of the Kronecker product and sum. A new lower bound on the robustness measure is given in Section 4 in terms of the singular values of the Kronecker sum. Section 5 discusses various special cases; it is shown that the new bound becomes exact in certain cases. Section 6 relates the Kronecker product and sum of matrices to operators in a matrix space, which leads to some useful properties of the Kronecker sum used in Section 4. Two additional composite matrices are defined in Section 7, and their properties are described. In Section 8 two new lower bounds on the stability robustness measure are obtained in terms of the composite matrices defined in Section 7. Computational aspects of the problem, when the dimension of the matrix is large, are considered in Section 9. Some numerical examples are given in Section 10.

The following notation is used throughout this paper. For an $m \times n$ matrix A, A' is the transpose of A and A^* is the conjugate transpose of A. $\sigma_i(A)$, $i = 1, 2, \ldots, \min(m, n)$, denotes the ith singular value of A with order $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min(m,n)}(A)$; in particular, $\sigma_1(A)$ and $\sigma_{\min(m,n)}(A)$ are denoted by $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively. $\|A\|_s$ denotes the spectral norm of A and $\|A\|_F$ denotes the Frobenius norm of A, so that

$$\|A\|_s = \bar{\sigma}(A), \qquad \|A\|_F = \left[\sum_{i=1}^{\min(m,n)} \sigma_i^2(A)\right]^{1/2}.$$

If A is square, the trace and spectrum of A are denoted by $\mathrm{tr}(A)$ and $\mathrm{sp}(A)$, respectively, and the ith eigenvalue of A is denoted by $\lambda_i(A)$ with no specific order imposed.

2. Development

Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Let $\mathbf{C}^-$ be the open left half of the complex plane, i.e., $\mathbf{C}^- = \{s \in \mathbb{C}, \mathrm{Re}(s) < 0\}$. A matrix $A \in \mathbb{F}^{n \times n}$ is said to be stable if $\mathrm{sp}(A) \subset \mathbf{C}^-$; if this is not the case A is said to be unstable. It is desired to find the distance of a given stable matrix $A \in \mathbb{F}^{n \times n}$ from the set of all unstable matrices in $\mathbb{F}^{n \times n}$, which is defined by

$$\mu_{\mathbb{F}}(A) := \inf\{\|\Delta A\|_s : \Delta A \in \mathbb{F}^{n \times n} \text{ and } \mathrm{sp}(A + \Delta A) \not\subset \mathbf{C}^-\}. \tag{2.1}$$

The focus of this paper is on $\mu_{\mathbb{R}}(A)$ for $A \in \mathbb{R}^{n\times n}$, while $\mu_{\mathbb{C}}(A)$ is introduced for comparison purposes.

Let the boundary of $\mathbb{C}^-$ be denoted by $\partial\mathbb{C}^-$, i.e., $\partial\mathbb{C}^- = \{j\omega : \omega \in \mathbb{R}\}$. Then simple continuity arguments show that

$$\mu_{\mathbb{F}}(A) = \inf\{\|\Delta A\|_s : \Delta A \in \mathbb{F}^{n\times n} \text{ and } \mathrm{sp}(A + \Delta A) \cap \partial\mathbb{C}^- \neq \varnothing\}. \tag{2.2}$$

An immediate consequence of (2.1)–(2.2) is that $\mu_{\mathbb{R}}(A) \geq \mu_{\mathbb{C}}(A)$ if $A \in \mathbb{R}^{n\times n}$. Some other facts and previously obtained results about $\mu_{\mathbb{F}}(A)$ are summarized in the following theorem:

Theorem 2.1. *If* $A \in \mathbb{F}^{n\times n}$ *and* $\mathrm{sp}(A) \subset \mathbb{C}^-$, *then*

(a) $\mu_{\mathbb{F}}(A) \leq \min\{-\mathrm{Re}\,\lambda_i(A), i = 1, 2, \ldots, n\}$,
(b) $\mu_{\mathbb{F}}(A) \leq \underline{\sigma}(A)$,
(c) $\mu_{\mathbb{F}}(A^*) = \mu_{\mathbb{F}}(A)$,
(d) $\mu_{\mathbb{F}}(\alpha A) = \alpha\mu_{\mathbb{F}}(A)$ *for any* $\alpha > 0$,
(e) $\mu_{\mathbb{F}}(A) = \mu_{\mathbb{F}}(U^*AU)$ *for any* $U \in \mathbb{F}^{n\times n}$ *with* $U^*U = I$,
(f) $\mu_{\mathbb{F}}(A) \geq 1/\bar{\sigma}(P)$, *where P satisfies the Lyapunov equation* $A^*P + PA = -2I$,
(g) $\mu_{\mathbb{C}}(A) = \inf_{\omega\in\mathbb{R}} \underline{\sigma}(j\omega I - A)$ *and* $\mu_{\mathbb{R}}(A) \geq \inf_{\omega\in\mathbb{R}} \underline{\sigma}(j\omega I - A)$,
(h) *if A is normal*, $\mu_{\mathbb{F}}(A) = \min\{-\mathrm{Re}\,\lambda_i(A), i = 1, 2, \ldots, n\}$.

The proof of (a)–(e) is trivial. (a) and (b) provides two trivial upper bounds for $\mu_{\mathbb{F}}(A)$. (f) is proved in [PT] for the case $\mathbb{F} = \mathbb{R}$; the case $\mathbb{F} = \mathbb{C}$ is similar. For the proof of (g) and (h), see [HP], [HM], [L], [QD1], [M3], and [V]. It is also shown in [M3] that the infima in (g) can be taken from the subset $|\omega| < \bar{\sigma}(A) + \underline{\sigma}(A)$ of $\mathbb{R}$. Since (g) gives the exact expression for $\mu_{\mathbb{C}}(A)$ but (f) gives only a lower bound for $\mu_{\mathbb{C}}(A)$, the bound in (g) is tighter than the bound in (f) for both the complex and real case. This is proved in [QD1] and [BG] using another approach. It is also shown in [BG] that the bound in (g) is applicable to nonlinear time-varying perturbations if the perturbation ΔA is considered as a nonlinear operator on $\mathbb{F}^n$ and its norm is defined properly. It is shown in [PT] that the bound in (f) is applicable to linear time-varying perturbations but not to nonlinear perturbations. Therefore the bound given in (g) is in general superior to the bound given in (f). The only advantage of (f) over (g) is that the former is easier to compute than the latter. The existing techniques to compute $\inf_{\omega\in\mathbb{R}} \underline{\sigma}(j\omega I - A)$ involve some numerical difficulties [B3]. We also note that some lower bounds for $\mu_{\mathbb{F}}(A)$ which are only applicable for special classes of stable matrices are obtained in [PT], [L], and [Y] by using the diagonal form, polar decomposition, and symmetric part of A, respectively. An interesting fact is that in the case when A is normal all of these lower bounds coincide with the exact value of $\mu_{\mathbb{F}}(A)$ given in (h).

The exact expression for $\mu_{\mathbb{R}}(A)$ for general real matrices has not yet been obtained. The lower bounds given in Theorem 2.1(f) and (g) share a disadvantage in that they cannot distinguish between real and complex perturbations; this is because the methods used to derive them are not able to make the distinction. In order to reduce this conservatism, a new method which can make the distinction has to be developed. In this paper such a method is established using the properties of the Kronecker sum and other matrix compositions. Lower bounds of $\mu_{\mathbb{R}}(A)$ are found. The new bounds are applicable only to the real matrix space and linear time-invariant perturbations; hence it is expected that they would be less conservative to apply than the ones in Theorem 2.1(f) and (g). Examples show that the new bounds obtained for $\mu_{\mathbb{R}}(A)$ can have an arbitrary degree of improvement over the ones given by Theorem 2.1(f) and (g).

3. Preliminaries

Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $B = [b_{ij}] \in \mathbb{R}^{p\times q}$. Then the Kronecker product of A and B, denoted by $A \otimes B$, is defined as follows:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{F}^{mp\times nq}. \tag{3.1}$$

If $m = n$ and $p = q$, the Kronecker sum of A and B, denoted by $A \oplus B$, is defined by

$$A \oplus B = A \otimes I_p + I_m \otimes B \in \mathbb{F}^{mp\times mp}. \tag{3.2}$$

The following proposition gives a list of properties of the Kronecker product and sum, which is used in the development.

Proposition 3.1 [G].

(a) *If* $\alpha, \beta \in \mathbb{F}$, *then*

$$A \otimes (\alpha B + \beta C) = \alpha(A \otimes B) + \beta(A \otimes C),$$

$$(\alpha A + \beta B) \otimes C = \alpha(A \otimes C) + \beta(B \otimes C).$$

(b) $(A \otimes B)^* = A^* \otimes B^*$.
(c) $(A \otimes B)(D \otimes C) = AD \otimes BC$.
(d) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, *if* A, B *are nonsingular.*
(e) *If* $A \in \mathbb{F}^{m\times m}$, $B \in \mathbb{F}^{n\times n}$, $\mathrm{sp}(A \oplus B) = \{\lambda_i(A) + \lambda_j(B), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$.

The following result can be easily developed from Proposition 3.1.

Proposition 3.2.

(a) *If* $U, V \in \mathbb{F}^{n\times n}$ *are unitary matrices, then so is* $U \otimes V$.
(b) *If* $A, B \in \mathbb{F}^{n\times n}$ *have singular value decompositions* $A = U_1 S_1 V_1^*$ *and* $B = U_2 S_2 V_2^*$, *then* $A \otimes B$ *has a singular value decomposition*

$$A \otimes B = (U_1 \otimes U_2)(S_1 \otimes S_2)(V_1 \otimes V_2)^*. \tag{3.3}$$

(c) $\|A \otimes B\|_s = \|A\|_s \|B\|_s$.

The norm equality in Proposition 3.2(c) is actually a special case of the general theory of norms of tensor products [LF].

4. A Robustness Bound

In what follows, it is always assumed that $A \in \mathbb{R}^{n \times n}$ and that A is stable, i.e., $\mathrm{sp}(A) \subset \mathbf{C}^-$. Since only real matrix spaces are considered, we write $\mu(A)$ for $\mu_{\mathbb{R}}(A)$. To rule out trivial situations, it is assumed that $n \geq 2$.

It is desired to find

$$\mu(A) := \inf\{\|\Delta A\|_s : \Delta A \in \mathbb{R}^{n \times n}, \mathrm{sp}(A + \Delta A) \cap \partial\mathbf{C}^- \neq \varnothing\}. \tag{4.1}$$

Let

$$\mu_1(A) := \inf\{\|\Delta A\|_s : \Delta A \in \mathbb{R}^{n \times n}, 0 \in \mathrm{sp}(A + \Delta A)\}, \tag{4.2}$$

$$\mu_2(A) := \inf\{\|\Delta A\|_s : \Delta A \in \mathbb{R}^{n \times n}, \mathrm{sp}(A + \Delta A) \cap (\partial\mathbf{C}^- \sim \{0\}) \neq \varnothing\}, \tag{4.3}$$

where "$\sim$" means the difference of two sets. Then it is clear that

$$\mu(A) = \min\{\mu_1(A), \mu_2(A)\}. \tag{4.4}$$

$\mu_1(A)$ can be easily obtained as

$$\mu_1(A) = \underline{\sigma}(A). \tag{4.5}$$

The following analysis will therefore focus on $\mu_2(A)$. Two lemmas are required.

Lemma 4.1. *Given a real matrix $B \in \mathbb{R}^{n \times n}$, assume $\mathrm{sp}(B) \cap (\partial\mathbf{C}^- \sim \{0\}) \neq \varnothing$, then $\mathrm{rank}(B \oplus B) \leq n^2 - 2$.*

Proof. Since B is real, if $\mathrm{sp}(B) \cap (\partial\mathbf{C}^- \sim \{0\}) \neq \varnothing$, this implies that B must have at least one pair of imaginary eigenvalues $\pm j\omega$ for some $\omega \in \mathbb{R} \sim \{0\}$. By Proposition 3.1(e), $B \oplus B$ has two eigenvalues at the origin. Let v_1, v_2 be eigenvectors of B corresponding to eigenvalues $j\omega$ and $-j\omega$, respectively; then v_1, v_2 are linearly independent and it follows easily that $v_1 \otimes v_2$ and $v_2 \otimes v_1$ are linearly independent eigenvectors of $B \oplus B$ corresponding to the two eigenvalues at the origin. ∎

Lemma 4.2 [HJ]. *If $B \in \mathbb{R}^{n \times n}$, then, for any nonnegative integer $r \leq n$,*

$$\min\{\|\Delta B\|_s : \Delta B \in \mathbb{R}^{n \times n}, \mathrm{rank}(B + \Delta B) \leq r\} = \sigma_{r+1}(B).$$

A lower bound on $\mu_2(A)$ can then be obtained.

Theorem 4.1. *Given a stable matrix $A \in \mathbb{R}^{n \times n}$, then*

$$\mu_2(A) \geq \tfrac{1}{2}\sigma_{n^2-1}(A \oplus A). \tag{4.6}$$

Proof. If $\|\Delta A\|_s < \frac{1}{2}\sigma_{n^2-1}(A \oplus A)$, then

$$\begin{aligned}\|\Delta A \oplus \Delta A\|_s &= \|\Delta A \otimes I + I \otimes \Delta A\|_s \leq \|\Delta A \otimes I\|_s + \|I \otimes \Delta A\|_s \\ &= 2\|\Delta A\|_s < \sigma_{n^2-1}(A \oplus A).\end{aligned}$$

From Lemma 4.2, we know that

$$\mathrm{rank}[(A + \Delta A) \oplus (A + \Delta A)] = \mathrm{rank}[(A \oplus A) + (\Delta A \oplus \Delta A)] > n^2 - 2.$$

It follows from Lemma 4.1, therefore, that $A + \Delta A$ has no imaginary eigenvalues. Therefore, if $\mathrm{sp}(A + \Delta A) \cap (\partial \mathbb{C}^- \sim \{0\}) \neq \varnothing$, $\|\Delta A\|_s$ is greater than or equal to $\frac{1}{2}\sigma_{n^2-1}(A \oplus A)$. ■

The following theorem, an immediate consequence of (4.4)–(4.6), gives the first bound on the stability robustness of real matrices in this paper.

Theorem 4.2. *Given a stable matrix $A \in \mathbb{R}^{n \times n}$, then*

$$\mu(A) \geq \min\{\underline{\sigma}(A), \tfrac{1}{2}\sigma_{n^2-1}(A \oplus A)\}. \tag{4.7}$$

The bound (4.7) takes a simple form that can be easily computed using standard software. Experience shows that for a matrix A of moderate size, computing (4.7) is in fact faster than computing the previous bounds given by Theorem 2.1(f) and (g), and for a large number of examples (4.7) is tighter as well. In the next section we show that the bound (4.7) is exact in some special cases, in particular, for the case when $A \in \mathbb{R}^{2 \times 2}$; it is to be noted that the previous bounds given by Theorem 2.1(f) and (g) are in general not exact for arbitrary 2×2 real matrices.

5. Discussion of Special Cases

A question which naturally arises is whether or not the bound given by Theorem 4.2 is exact, i.e., whether or not the inequality in (4.7) is actually an equality. The answer to this question is negative for arbitrary stable matrices, and an even tighter lower bound is obtained in Section 8. However, the bound (4.7) is exact for some special classes of matrices.

From Theorem 2.1(b) and Theorem 4.2, it is observed that $\mu(A) = \underline{\sigma}(A)$ if $\underline{\sigma}(A) \leq \frac{1}{2}\sigma_{n^2-1}(A \oplus A)$; in this case, $\mu(A)$ is obtained exactly. The exact $\mu_{\mathbb{R}}(A)$ can also be obtained in some other cases.

Theorem 5.1. *If $A \in \mathbb{R}^{n \times n}$ is a stable normal matrix, then*

$$\begin{aligned}\mu(A) &= \min\{\underline{\sigma}(A), \tfrac{1}{2}\sigma_{n^2-1}(A \oplus A)\} \\ &= \min\{-\mathrm{Re}\,\lambda_i(A), i = 1, 2, \ldots, n\}.\end{aligned} \tag{5.1}$$

Proof. Let U be a unitary matrix such that $U^*AU = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i = \lambda_i(A)$, $i = 1, 2, \ldots, n$, with $\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2) \geq \cdots \geq \mathrm{Re}(\lambda_n)$. Since $U \otimes U$ is also a unitary matrix, $A \oplus A$ has the same singular values as $(U \otimes U)^{-1}(A \oplus A)(U \otimes U) = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \oplus \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Thus the singular values of $A \oplus A$ are $\{|\lambda_i + \lambda_j|, i, j = 1, 2, \ldots, n\}$. If λ_1 is real, then $\underline{\sigma}(A) = |\lambda_1|$ and $\frac{1}{2}\sigma_{n^2-1}(A \oplus A) \geq |\lambda_1|$. If λ_1 is not real, then $\underline{\sigma}(A) \geq -\mathrm{Re}(\lambda_1)$ and $\frac{1}{2}\sigma_{n^2-1}(A \oplus A) = -\mathrm{Re}(\lambda_1)$. In both cases, $\min\{\underline{\sigma}(A), \frac{1}{2}\sigma_{n^2-1}(A \oplus A)\} = -\mathrm{Re}(\lambda_1)$. By applying Theorem 2.1(a) and Theorem 4.2, we obtain

$$\begin{aligned}\mu(A) &\leq \min\{-\mathrm{Re}\,\lambda_i(A), i = 1, 2, \ldots, n\} = -\mathrm{Re}(\lambda_1) \\ &= \min\{\underline{\sigma}(A), \tfrac{1}{2}\sigma_{n^2-1}(A \oplus A)\} \leq \mu(A).\end{aligned}$$

This ends the proof. ■

Although the exact value of $\mu(A)$ for a normal matrix A has been previously obtained as stated in Theorem 2.1(h), this theorem shows that the new bound (4.7) also is exact if A is normal.

Theorem 5.2. *If A is a 2×2 real stable matrix, then*

$$\sigma_3(A \oplus A) = -\operatorname{tr}(A) \tag{5.2}$$

and

$$\mu(A) = \min\{\underline{\sigma}(A), -\tfrac{1}{2}\operatorname{tr}(A)\}. \tag{5.3}$$

Proof. The proof of (5.2) involves an elementary but tedious calculation. It is given in [QD2] and is omitted here. If (5.2) is true, then $\mu(A) \geq \min\{\underline{\sigma}(A), -\frac{1}{2}\operatorname{tr}(A)\}$. On the other hand, if $\Delta A = -\frac{1}{2}\operatorname{tr}(A)I$, then $A + \Delta A$ is unstable and $\|\Delta A\| = -\frac{1}{2}\operatorname{tr}(A)$. This together with Theorem 2.1(b) implies $\mu(A) \leq \min\{\underline{\sigma}(A), -\frac{1}{2}\operatorname{tr}(A)\}$. This proves (5.3). ■

The 2×2 case has also been studied in [HM] where it has been shown that $\mu(A) = \min\{\underline{\sigma}(A), -\frac{1}{2}\operatorname{tr}(A)\}$, but no previous general bounds, when applied to the 2×2 case, give the exact answer.

6. Some Properties of Matrices $A \otimes A$ and $A \oplus A$

The bound developed in Section 4 requires the singular values of the matrix $A \oplus A$ to be determined. Such a calculation may be difficult to carry out if the size of A is large. Two possible directions can be pursued to reduce the computational complexity. One method is to find other operations on matrices which can be used to analyze $\mu(A)$ and which have smaller dimension than $A \oplus A$. The other method is to compute the singular values of $A \oplus A$ without actually constructing $A \oplus A$. This is the theme of the coming sections. This section gives some background knowledge.

Let $A \in \mathbb{F}^{n \times n}$. Consider $A \otimes A$ and $A \oplus A$ as linear operators on the Hilbert space F^{n^2}, mapping $x \in \mathbb{F}^{n^2}$ to $(A \otimes A)x$ and $(A \oplus A)x \in \mathbb{F}^{n^2}$, respectively. The inner product on $\mathbb{F}^{n^2}$ is defined in the usual way, i.e., $\langle x, y\rangle = x^*y$, for all $x, y \in \mathbb{F}^{n^2}$. The norm induced by this inner product is the 2-norm $\|\cdot\|_2$.

The $n \times n$ matrix space $\mathbb{F}^{n \times n}$ is also an n^2-dimensional vector space over $\mathbb{F}$. It becomes a Hilbert space, if we define an inner product on it by $\langle X, Y\rangle = \operatorname{tr}(X^*Y)$, for all $X, Y \in \mathbb{F}^{n \times n}$. The norm in $\mathbb{F}^{n \times n}$ induced by this inner product is the Frobenius norm $\|\cdot\|_F$. Now define a linear operator Vec: $\mathbb{F}^{n \times n} \to \mathbb{F}^{n^2}$ by

$$\operatorname{Vec}\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = [x_{11} \quad \cdots \quad x_{n1} \quad x_{12} \quad \cdots \quad x_{n2} \quad \cdots \quad x_{nn}]'. \tag{6.1}$$

We need two properties of Vec to proceed.

Lemma 6.1 [G]. *Let* $X, Y, Z \in \mathbb{F}^{n \times n}$, *then*

(a) $\operatorname{tr}(X^* Y) = [\operatorname{Vec}(X)]^* \operatorname{Vec}(Y)$,
(b) $\operatorname{Vec}(XYZ) = (Z' \otimes X) \operatorname{Vec}(Y)$.

Lemma 6.1(a) implies Vec is an isometric isomorphism from the Hilbert space $\mathbb{F}^{n \times n}$ onto the Hilbert space $\mathbb{F}^{n^2}$. Under this isomorphism the operator $A \otimes A$ on $\mathbb{F}^{n^2}$ becomes the operator K mapping $X \in \mathbb{F}^{n \times n}$ to $K(X) = AXA'$. Similarly, under isomorphism Vec, the operator $A \oplus A$ on $\mathbb{F}^{n^2}$ becomes the operator L mapping $X \in \mathbb{F}^{n \times n}$ to $L(X) = AX + XA'$. The operator L is usually called the Lyapunov transformation.

Let $\mathrm{S}_1 \subset \mathbb{F}^{n \times n}$ be the subspace of all symmetric matrices, and let $\mathrm{S}_2 \subset \mathbb{F}^{n \times n}$ be the subspace of all skew-symmetric matrices; $\mathrm{S}_1 = \{X \in \mathbb{F}^{n \times n}: X = X\}$ and $\mathrm{S}_2 = \{X \in \mathbb{F}^{n \times n}: X' = -X\}$. The following two easily proved propositions are required in the later development.

Proposition 6.1.

$$\mathrm{S}_1 \perp \mathrm{S}_2 \quad \textit{and} \quad \mathrm{S}_1 \dotplus \mathrm{S}_2 = \mathbb{F}^{n \times n}.$$

Proposition 6.1 states that S_1 and S_2 are orthogonal complements to each other.

Proposition 6.2.

$$K(\mathrm{S}_1) \subset \mathrm{S}_1,$$
$$K(\mathrm{S}_2) \subset \mathrm{S}_2,$$
$$L(\mathrm{S}_1) \subset \mathrm{S}_1,$$
$$L(\mathrm{S}_2) \subset \mathrm{S}_2.$$

Proposition 6.2 states that S_1 and S_2 are reducing subspaces of $\mathbb{F}^{n \times n}$ for the operators K and L.

Since the operator Vec is an isomorphism from $\mathbb{F}^{n \times n}$ to $\mathbb{F}^{n^2}$, and since $A \otimes A$ and $A \oplus A$ are the induced operators of K and L under Vec, respectively, the following two corollaries can be easily obtained.

Corollary 6.1.

$$\operatorname{Vec}(\mathrm{S}_1) \perp \operatorname{Vec}(\mathrm{S}_2) \quad \textit{and} \quad \operatorname{Vec}(\mathrm{S}_1) \dotplus \operatorname{Vec}(\mathrm{S}_2) = \mathbb{F}^{n^2}.$$

Corollary 6.2.

$$(A \otimes A) \operatorname{Vec}(\mathrm{S}_1) \subset \operatorname{Vec}(\mathrm{S}_1),$$
$$(A \otimes A) \operatorname{Vec}(\mathrm{S}_2) \subset \operatorname{Vec}(\mathrm{S}_2),$$
$$(A \oplus A) \operatorname{Vec}(\mathrm{S}_1) \subset \operatorname{Vec}(\mathrm{S}_1),$$
$$(A \oplus A) \operatorname{Vec}(\mathrm{S}_2) \subset \operatorname{Vec}(\mathrm{S}_2).$$

7. Two Other Composite Matrices

The Kronecker product can be considered as a composition of two matrices. Two other compositions of matrices are now introduced. These compositions have similar properties as the Kronecker product, but have smaller dimension, and stability robustness bounds can be obtained in terms of these compositions.

Let $A = [a_{ij}] \in \mathbb{F}^{n\times n}$, $B = [b_{ij}] \in \mathbb{F}^{n\times n}$, $n \geq 2$. Let (i_1, i_2) be the ith pair of integers in the sequence

$$(1, 1), (1, 2), \ldots, (1, n), (2, 2), \ldots, (2, n), (3, 3), \ldots, (n-1, n), (n, n). \tag{7.1}$$

Definition 7.1.

$$A \mathbin{\bar{\otimes}} B := [c_{ij}] \in \mathbb{F}^{(1/2)n(n+1)\times(1/2)n(n+1)},$$

where

$$c_{ij} := \begin{cases} a_{i_1 j_1} b_{i_1 j_1} & \text{if } i_1 = i_2 \text{ and } j_1 = j_2, \\ \frac{1}{2}(a_{i_1 j_1} b_{i_2 j_2} + a_{i_1 j_2} b_{i_2 j_1} + a_{i_2 j_1} b_{i_1 j_2} + a_{i_2 j_2} b_{i_1 j_1}) & \text{if } i_1 \neq i_2 \text{ and } j_1 \neq j_2, \\ \frac{\sqrt{2}}{2}(a_{i_1 j_1} b_{i_2 j_2} + a_{i_2 j_2} b_{i_1 j_1}) & \text{otherwise.} \end{cases} \tag{7.2}$$

Let (r_1, r_2) be the rth pair of integers in the sequence

$$(1, 2), (1, 3), \ldots, (1, n), (2, 3), \ldots, (2, n), (3, 4), \ldots, (n-1, n). \tag{7.3}$$

Definition 7.2.

$$A \mathbin{\bar{\bar{\otimes}}} B := [d_{rs}] \in \mathbb{F}^{(1/2)n(n-1)\times(1/2)n(n-1)},$$

where

$$d_{rs} := \tfrac{1}{2}(a_{r_1 s_1} b_{r_2 s_2} - a_{r_1 s_2} b_{r_2 s_1} - a_{r_2 s_1} b_{r_1 s_2} + a_{r_2 s_2} b_{r_1 s_1}). \tag{7.4}$$

The operations $\bar{\otimes}$ and $\bar{\bar{\otimes}}$ are studied in two categories of literature. One category involves multilinear algebra [M2] in which $A \mathbin{\bar{\otimes}} B$ and $A \mathbin{\bar{\bar{\otimes}}} B$ are considered as operators on the symmetric and skew-symmetric tensor product spaces, respectively. This point of view is theoretically elegant and provides a clear and complete picture on the relationship among the Kronecker product and operations $\bar{\otimes}$ and $\bar{\bar{\otimes}}$, but it needs mathematical tools which are not commonly used in control iterature. The other category [B1], [F], [J2], [M1], [S2] studies these operations from a pure matrix point of view which is easier to follow but is tedious and incomplete. In the following the properties of $\bar{\otimes}$ and $\bar{\bar{\otimes}}$ are studied using an alternative method which we believe to be a tradeoff between the methods used in the two categories of literature.

The corresponding sum operations of $\bar{\otimes}$ and $\bar{\bar{\otimes}}$ can be defined as follows:

Definition 7.3.

$$A \mathbin{\bar{\oplus}} B := A \mathbin{\bar{\otimes}} I_n + I_n \mathbin{\bar{\otimes}} B \in \mathbb{F}^{(1/2)n(n+1)\times(1/2)n(n+1)}, \tag{7.5}$$

$$A \mathbin{\bar{\bar{\oplus}}} B := A \mathbin{\bar{\bar{\otimes}}} I_n + I_n \mathbin{\bar{\bar{\otimes}}} B \in \mathbb{F}^{(1/2)n(n-1)\times(1/2)n(n-1)}. \tag{7.6}$$

Unlike the Kronecker product and sum, operations $\bar{\otimes}$, $\bar{\bar{\otimes}}$, $\bar{\oplus}$, and $\bar{\bar{\oplus}}$ are defined only for square matrices with the same size. From Definitions 7.1–7.3, it is easy to see that $\bar{\otimes}$, $\bar{\bar{\otimes}}$, $\bar{\oplus}$, and $\bar{\bar{\oplus}}$ are commutative. So $A \bar{\oplus} B$ can also be written as $(A + B) \bar{\otimes} I$ or $I \bar{\otimes} (A + B)$ and $A \bar{\bar{\oplus}} B$ can also be written as $(A + B) \bar{\bar{\otimes}} I$ or $I \bar{\bar{\otimes}} (A + B)$.

The operations $\bar{\otimes}$ and $\bar{\bar{\otimes}}$ are closely related to the Kronecker product operation. Recall from the last section that the space $\mathbb{F}^{n^2}$ with inner product $\langle x, y \rangle = x^* y$, and the space $\mathbb{F}^{n \times n}$ with inner product $\langle X, Y \rangle = \operatorname{tr}(X^* Y)$, are isomorphic to each other with the isomorphism Vec: $\mathbb{F}^{n \times n} \to \mathbb{F}^{n^2}$ defined as in (6.1). Subspaces $\mathbf{S}_1$ and $\mathbf{S}_2$ are defined as $\mathbf{S}_1 = \{X \in \mathbb{F}^{n \times n}: X' = X\}$, $\mathbf{S}_2 = \{X \in \mathbb{F}^{n \times n}: X' = -X\}$.

Define $E_{ij} \in \mathbb{F}^{n \times n}$ to be a matrix with 1 in the (i, j)th entry and 0 elsewhere.

Let (i_1, i_2) be the ith pair of integers in the sequence (7.1) and let

$$U_i = \begin{cases} E_{i_1 i_1} & \text{if } i_1 = i_2, \\ \dfrac{\sqrt{2}}{2}(E_{i_1 i_2} + E_{i_2 i_1}) & \text{otherwise.} \end{cases} \tag{7.7}$$

Then $\{U_1, U_2, \ldots, U_{(1/2)n(n+1)}\}$ is an orthonormal basis of $\mathbf{S}_1$.

Let (r_1, r_2) be the rth pair of integers in the sequence (7.3) and let

$$V_r = \frac{\sqrt{2}}{2}(E_{r_1 r_2} - E_{r_2 r_1}). \tag{7.8}$$

Then $\{V_1, V_2, \ldots, V_{(1/2)n(n-1)}\}$ is an orthonormal basis of $\mathbf{S}_2$.

Let $u_i = \operatorname{Vec}(U_i)$, $i = 1, 2, \ldots, \frac{1}{2}n(n+1)$, and $v_i = \operatorname{Vec}(V_i)$, $i = 1, 2, \ldots, \frac{1}{2}n(n-1)$. Then $\{u_1, u_2, \ldots, u_{(1/2)n(n+1)}\}$ is an orthonormal basis of $\operatorname{Vec}(\mathbf{S}_1)$, and $\{v_1, v_2, \ldots, v_{(1/2)n(n-1)}\}$ is an orthonormal basis of $\operatorname{Vec}(\mathbf{S}_2)$.

Define

$$T_1 := [u_1 \quad u_2 \quad \cdots \quad u_{(1/2)n(n+1)}] \in \mathbb{F}^{n^2 \times (1/2)n(n+1)}, \tag{7.9}$$

$$T_2 := [v_1 \quad v_2 \quad \cdots \quad v_{(1/2)n(n-1)}] \in \mathbb{F}^{n^2 \times (1/2)n(n-1)}. \tag{7.10}$$

It can be verified that $[T_1 \;\; T_2]$ is a real orthogonal matrix.

Proposition 7.1. *Let $A, B \in \mathbb{F}^{n \times n}$. Then*

$$A \bar{\otimes} B = T_1'(A \otimes B) T_1, \tag{7.11}$$

$$A \bar{\bar{\otimes}} B = T_2'(A \otimes B) T_2. \tag{7.12}$$

Proof. See Appendix 1. ■

From Corollaries 6.1 and 6.2, the following proposition easily follows:

Proposition 7.2. *Let $A \in \mathbb{F}^{n \times n}$. Then*

$$T_1'(A \otimes A) T_2 = 0,$$

$$T_2'(A \otimes A) T_1 = 0,$$

$$T_1'(A \oplus A) T_2 = 0,$$

$$T_2'(A \oplus A) T_1 = 0.$$

Let $T = [T_1 \;\; T_2]$. Propositions 7.1 and 7.2 imply that

$$T'(A \otimes A)T = \begin{bmatrix} A \bar{\otimes} A & 0 \\ 0 & A \bar{\bar{\otimes}} A \end{bmatrix} \tag{7.13}$$

and

$$T'(A \oplus A)T = \begin{bmatrix} A \bar{\oplus} A & 0 \\ 0 & A \bar{\bar{\oplus}} A \end{bmatrix}. \tag{7.14}$$

Various properties of the $\bar{\otimes}$, $\bar{\bar{\otimes}}$-product and $\bar{\oplus}$, $\bar{\bar{\oplus}}$-sum can now be obtained. The properties which are used in our development are listed in Proposition 7.3. Although they can be proved directly from Definitions 7.1 and 7.2, the proof is easier to obtain by using Propositions 7.1 and 7.2.

Proposition 7.3. *Let $A, B, C, D \in \mathbb{F}^{n \times n}$; $\alpha, \beta \in \mathbb{F}$. Then*

(a) $A \bar{\otimes} (\alpha\beta + \beta C) = \alpha(A \bar{\otimes} B) + \beta(A \bar{\otimes} C)$,
$(\alpha A + \beta B) \bar{\otimes} C = \alpha(A \bar{\otimes} C) + \beta(B \bar{\otimes} C)$,
$A \bar{\bar{\otimes}} (\alpha B + \beta C) = \alpha(A \bar{\bar{\otimes}} B) + \beta(A \bar{\bar{\otimes}} C)$,
$(\alpha A + \beta B) \bar{\bar{\otimes}} C = \alpha(A \bar{\bar{\otimes}} C) + \beta(B \bar{\bar{\otimes}} C)$,

(b) $\|A \bar{\otimes} B\|_s \le \|A\|_s \|B\|_s$, $\|A \bar{\bar{\otimes}} B\|_s \le \|A\|_s \|B\|_s$,

(c) $\mathrm{sp}(A \bar{\oplus} A) = \{\lambda_i(A) + \lambda_j(A), i = 1, 2, \ldots, n, j \ge i\}$,
$\mathrm{sp}(A \bar{\bar{\oplus}} A) = \{\lambda_i(A) + \lambda_j(A), i = 1, 2, \ldots, n-1, j > i\}$.

Proof. See Appendix 2. ■

8. Additional Robustness Bounds

In this section the composite matrices introduced in the last section are used to obtain robustness bounds for stable matrices. We again assume throughout the section that $A \in \mathbb{R}^{n \times n}$, $n \ge 2$, and A is stable. Let $\mu(A)$, $\mu_1(A)$, and $\mu_2(A)$ be defined as in (4.1)–(4.3).

If $\mathrm{sp}(A + \Delta A) \cap \partial\mathbf{C}^- \ne \varnothing$, $(A + \Delta A) \bar{\oplus} (A + \Delta A)$ is singular by Proposition 7.3(c); this leads to the second lower bound on $\mu(A)$ in this paper.

Theorem 8.1. *Given a stable matrix $A \in \mathbb{R}^{n \times n}$, then*

$$\mu(A) \ge \tfrac{1}{2}\underline{\sigma}(A \bar{\oplus} A). \tag{8.1}$$

Proof. If $\|\Delta A\|_s < \frac{1}{2}\underline{\sigma}(A \bar{\oplus} A)$,

$$\|\Delta A \bar{\oplus} \Delta A\|_s = \|\Delta A \bar{\otimes} I + I \bar{\otimes} \Delta A\|_s \le \|\Delta A \bar{\otimes} I\|_s + \|I \bar{\otimes} \Delta A\|_s \le 2\|\Delta A\|_s$$
$$< \underline{\sigma}(A \bar{\oplus} A).$$

Thus $A \bar{\oplus} A + \Delta A \bar{\oplus} \Delta A = (A + \Delta A) \bar{\oplus} (A + \Delta A)$ is nonsingular, which implies that

$$\mathrm{sp}(A + \Delta A) \cap \partial\mathbf{C}^- = \varnothing.$$ ■

By Proposition 7.3(c), $(A + \Delta A) \bar{\bar{\oplus}} (A + \Delta A)$ is singular if $\mathrm{sp}(A + \Delta A) \cap (\partial\mathbb{C}^- \sim \{0\}) \neq \varnothing$. This leads to a lower bound on $\mu_2(A)$.

Theorem 8.2. *Given a stable matrix $A \in \mathbb{R}^{n\times n}$, then*

$$\mu_2(A) \geq \tfrac{1}{2}\underline{\sigma}(A \bar{\bar{\oplus}} A). \tag{8.2}$$

The proof of Theorem 8.2 is similar to the proof of Theorem 8.1, so it is omitted. The following main result on the stability robustness of real matrices is then obtained as an immediate consequence of Theorem 8.2 and (4.4)–(4.5).

Theorem 8.3. *Given a stable matrix $A \in \mathbb{R}^{n\times n}$, then*

$$\mu(A) \geq \min\{\underline{\sigma}(A), \tfrac{1}{2}\underline{\sigma}(A \bar{\bar{\oplus}} A)\}. \tag{8.3}$$

It can be shown by using the same technique as the proof of Theorem 5.1 that if A is normal, then

$$\tfrac{1}{2}\underline{\sigma}(A \bar{\oplus} A) = \min\{\underline{\sigma}(A), \tfrac{1}{2}\underline{\sigma}(A \bar{\bar{\oplus}} A)\} = \min\{-\mathrm{Re}\, \lambda_i(A), i = 1, 2, \ldots, n\}, \tag{8.4}$$

so the bounds (8.1) and (8.3) result in the exact value of $\mu(A)$ if A is a normal matrix. If $A \in \mathbb{R}^{2\times 2}$, the definition of $A \bar{\bar{\oplus}} A$ gives that $A \bar{\bar{\oplus}} A = \mathrm{tr}(A)$; thus bound (8.3) also gives the exact value of $\mu(A)$ in the 2×2 case. However, bound (8.1) does not give the exact value of $\mu(A)$ for general 2×2 matrices.

It is of interest to compare the three lower bounds (4.7), (8.1), and (8.3) obtained in this paper. Since $A \bar{\oplus} A$ and $A \bar{\bar{\oplus}} A$ have smaller dimensions than $A \oplus A$, bounds (8.1) and (8.3) are easier to compute. Equation (7.14) shows that the singular values of $A \bar{\oplus} A$ together with those of $A \bar{\bar{\oplus}} A$ are just the singular values of $A \oplus A$; thus either $\underline{\sigma}(A \bar{\oplus} A)$ or $\underline{\sigma}(A \bar{\bar{\oplus}} A)$ must be equal to $\underline{\sigma}(A \oplus A)$. A conjecture, drawn from a large number of examples, is that $\underline{\sigma}(A \bar{\oplus} A) = \underline{\sigma}(A \oplus A)$ for any matrix A. Theorem 5.2 shows that this conjecture is true if A is a 2×2 matrix. Equation (8.4) implies that it is also true if A is normal. However, a proof of this conjecture for general matrices is not available yet. The following proposition provides some information on the relationship between bounds (4.7), (8.1), and (8.3).

Proposition 8.1. *For any $A \in \mathbb{R}^{n\times n}$,*

$$\tfrac{1}{2}\underline{\sigma}(A \bar{\oplus} A) \leq \underline{\sigma}(A). \tag{8.5}$$

Proof. Choose $\Delta A \in \mathbb{R}^{n\times n}$ such that $\|\Delta A\|_s = \underline{\sigma}(A)$ and $A + \Delta A$ is singular. Then $(A + \Delta A) \bar{\oplus} (A + \Delta A) = A \bar{\oplus} A + \Delta A \bar{\oplus} \Delta A$ is singular by Proposition 7.3(c). This can happen only if $\|\Delta A \bar{\oplus} \Delta A\|_s \geq \underline{\sigma}(A \bar{\oplus} A)$. So $\underline{\sigma}(A \bar{\oplus} A) \leq \|\Delta A \bar{\oplus} \Delta A\|_s \leq 2\|\Delta A\|_s = 2\underline{\sigma}(A)$. ■

Proposition 8.1 implies that if the conjecture $\underline{\sigma}(A \oplus A) = \underline{\sigma}(A \bar{\oplus} A)$ is true, then

$$\tfrac{1}{2}\underline{\sigma}(A \bar{\oplus} A) \leq \min\{\underline{\sigma}(A), \tfrac{1}{2}\sigma_{n^2-1}(A \oplus A)\} \leq \min\{\underline{\sigma}(A), \tfrac{1}{2}\underline{\sigma}(A \bar{\bar{\oplus}} A)\}, \tag{8.6}$$

so that of the three bounds (4.7), (8.1), and (8.3) obtained in this paper, bound (8.3) produces the best result and bound (8.1) the worst.

9. Computational Aspects for Large Matrices

If matrix A is modest in size, the computation required to obtain lower bounds (4.7), (8.1), and (8.3) is simple and numerically well defined. However, computational difficulties will arise if the $n \times n$ matrix A has a large size, because the composite matrices $A \oplus A$, $A \bar{\oplus} A$, and $A \bar{\bar{\oplus}} A$ have dimensions n^2, $\frac{1}{2}n(n+1)$, and $\frac{1}{2}n(n-1)$, respectively. Therefore it is desired to have an alternative way to determine the required singular values without constructing the composite matrices explicitly. In this section we show this is possible to do. Before presenting the algorithm, some preliminary results must be established.

In Section 6 we have seen that the Hilbert space $\mathbb{F}^{n^2}$ with inner product $\langle x, y\rangle = x^*y$, for all $x, y \in \mathbb{F}^{n^2}$, and the Hilbert space $\mathbb{F}^{n\times n}$ with inner product $\langle X, Y\rangle = \mathrm{tr}(X^*Y)$, for all $X, Y \in \mathbb{F}^{n\times n}$, are isomorphic with isomorphism Vec: $\mathbb{F}^{n\times n} \to \mathbb{F}^{n^2}$ defined in (6.1). If L is the Lyapunov transformation $L(X) = AX + XA'$, for all $X \in \mathbb{F}^{n\times n}$, Lemma 6.1(b) implies that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{F}^{n\times n} & \xrightarrow{\;L\;} & \mathbb{F}^{n\times n} \\ \Big\downarrow \mathrm{Vec} & & \Big\downarrow \mathrm{Vec} \\ \mathbb{F}^{n^2} & \xrightarrow[A \oplus A]{} & \mathbb{F}^{n^2} \end{array}$$

Let $S_1 \subset \mathbb{F}^{n\times n}$ be the subspace of all symmetric matrices, and let $S_2 \subset \mathbb{F}^{n\times n}$ be the subspace of all skew-symmetric matrices. It is shown in Section 6 that S_1 and S_2 are orthogonal complements to each other, and S_1 and S_2 are reducing subspaces of $\mathbb{F}^{n\times n}$ for operator L. Denote the restrictions of L to S_1, S_2 by $L|S_1, L|S_2$, respectively. Equation (7.13) implies that $A \bar{\oplus} A$ is a matrix representation of $L|S_1$ under an orthonormal basis of S_1 and $A \bar{\bar{\oplus}} A$ is a matrix presentation of $L|S_2$ under an orthonormal basis of S_2. This fact is made more explicit in the following:

Define two linear operators Φ: $S_1 \to \mathbb{F}^{(1/2)n(n+1)}$ and Ψ: $S_2 \to \mathbb{F}^{(1/2)n(n-1)}$ by

$$\Phi\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} = [x_{11} \;\; \sqrt{2}x_{12} \;\; \cdots \;\; \sqrt{2}x_{1n} \;\; x_{22} \;\; \sqrt{2}x_{23} \;\; \cdots \;\; \sqrt{2}x_{2n} \;\; \cdots \;\; x_{nn}]' \tag{9.1}$$

and

$$\Psi\begin{bmatrix} 0 & x_{12} & \cdots & x_{1n} \\ -x_{12} & 0 & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ -x_{1n} & -x_{2n} & \cdots & 0 \end{bmatrix} = [\sqrt{2}x_{12} \;\; \sqrt{2}x_{13} \;\; \cdots \;\; \sqrt{2}x_{1n} \;\; \sqrt{2}x_{23} \;\; \cdots \;\; \sqrt{2}x_{2n} \;\; \cdots \;\; \sqrt{2}x_{(n-1)n}]'. \tag{9.2}$$

Two lemmas can now be obtained.

Lemma 9.1. *Let $X, Y \in \mathbf{S}_1$ and $A \in \mathbb{F}^{n\times n}$, then*

(a) $\operatorname{tr}(X'Y) = [\Phi(X)]'\Phi(Y)$,
(b) $\Phi(AX + XA') = (A \bar{\oplus} A)\Phi(X)$.

Proof. Let T_1 be defined as in (7.9). It is easy to check that $\operatorname{Vec}(X) = T_1\Phi(X)$, for all $X \in \mathbf{S}_1$. Then

$$\text{(a)} \quad \begin{aligned} \operatorname{tr}(X'Y) &= [\operatorname{Vec}(X)]' \operatorname{Vec}(Y) \qquad \text{(by Lemma 6.1(a))} \\ &= [\Phi(X)]'T_1'T_1\Phi(Y) \\ &= [\Phi(X)]'\Phi(Y) \qquad \text{(since } T_1 \text{ has orthonormal columns).} \end{aligned}$$

$$\text{(b)} \quad \begin{aligned} \Phi(AX + XA') &= T_1' \operatorname{Vec}(AX + XA') \\ &= T_1'(A \otimes I + I \otimes A) \operatorname{Vec}(X) \qquad \text{(by Lemma 6.1(b))} \\ &= T_1'(A \otimes I + I \otimes A)T_1\Phi(X) \\ &= (A \bar{\otimes} I + I \bar{\otimes} A)\Phi(X) \qquad \text{(by Proposition 7.1)} \\ &= (A \bar{\oplus} A)\Phi(X). \end{aligned}$$ ■

Lemma 9.2. *Let $X, Y \in \mathbf{S}_2$ and $A \in \mathbb{F}^{n\times n}$, then*

(a) $\operatorname{tr}(X'Y) = [\Psi(X)]'\Psi(Y)$,
(b) $\Psi(AX + XA') = (A \bar{\bar{\oplus}} A)\Psi(X)$.

Proof. Let T_2 be defined as in (7.10). It is easy to check that $\operatorname{Vec}(X) = T_2\Psi(X)$, for all $\in \mathbf{S}_2$. The rest of the proof is similar to the proof of Lemma 9.1. ■

Lemma 9.1 implies that Φ is an isomorphism and the following diagram commutes:

$$\begin{array}{ccc} \mathbf{S}_1 & \xrightarrow{L|\mathbf{S}_1} & \mathbf{S}_1 \\ \Phi \big\downarrow & & \big\downarrow \Phi \\ \mathbb{F}^{(1/2)n(n+1)} & \xrightarrow[A \bar{\oplus} A]{} & \mathbb{F}^{(1/2)n(n+1)} \end{array}$$

Lemma 9.2 implies that Ψ is an isomorphism and the following diagram commutes:

$$\begin{array}{ccc} \mathbf{S}_2 & \xrightarrow{L|\mathbf{S}_2} & \mathbf{S}_2 \\ \Psi \big\downarrow & & \big\downarrow \Psi \\ \mathbb{F}^{(1/2)n(n-1)} & \xrightarrow[A \bar{\bar{\oplus}} A]{} & \mathbb{F}^{(1/2)n(n-1)} \end{array}$$

From the commutative diagrams it is noted that although $A \oplus A$, $A \bar{\oplus} A$, and $A \bar{\bar{\oplus}} A$ have large dimensions, it is relatively simple to compute the multiplications of these matrices with column vectors or the multiplications of their inverses (if they exist) with column vectors since

$$(A \oplus A)x = \mathrm{Vec}\cdot L\cdot \mathrm{Vec}^{-1}(x), \qquad (A \oplus A)^{-1}x = \mathrm{Vec}\cdot L^{-1}\cdot \mathrm{Vec}^{-1}(x), \tag{9.3}$$

$$(A \bar{\oplus} A)y = \Phi\cdot L\cdot \Phi^{-1}(y), \qquad (A \bar{\oplus} A)^{-1}y = \Phi\cdot L^{-1}\cdot \Phi^{-1}(y), \tag{9.4}$$

$$(A \bar{\bar{\oplus}} A)z = \Psi\cdot L\cdot \Psi^{-1}(z), \qquad (A \bar{\bar{\oplus}} A)^{-1}z = \Psi\cdot L^{-1}\cdot \Psi^{-1}(z), \tag{9.5}$$

for $x \in \mathbb{F}^{n\times n}$, $y \in \mathbb{F}^{(1/2)n(n+1)}$, and $z \in \mathbb{F}^{(1/2)n(n-1)}$, where L^{-1} is the inverse Lyapunov map which can be computed by solving a Lyapunov equation. Equation (9.3)–(9.5) make the iterative methods to compute the singular values of $A \oplus A$, $A \bar{\oplus} A$, or $A \bar{\bar{\oplus}} A$ more favorable, especially when only a few extreme singular values are required. For available iterative methods to compute the singular values of matrices, see [P1], [GLO], and [CW]. The power method and the block power method for the eigenvalue problem of matrices [J1], [P2] can also be adapted to compute the singular values of matrices. All of these methods, accompanied by (9.3)–(9.5), can be used to find the required singular values of $A \oplus A$, $A \bar{\oplus} A$, and $A \bar{\bar{\oplus}} A$. However, in our case, it is unnecessary to transform a vector from column vector form to matrix form using Vec, Φ, or Ψ back and forth in every iteration. It is recognized from the commutative diagrams that the singular values of matrices $A \oplus A$, $A \bar{\oplus} A$, and $A \bar{\bar{\oplus}} A$ are the same as the singular values of operators[1] L, $L|\mathrm{S}_1$, and $L|\mathrm{S}_2$, respectively, and so an iterative procedure can be built to find the singular values of these operators directly. Such an algorithm is given in the following.

Let M be a linear operator on an l-dimensional Hilbert space S. Let M^* be the conjugate of M. Let the inner product in S be $\langle\cdot,\cdot\rangle$ and the induced norm be $\|\cdot\|$. Denote by $\{\sigma_i\}$, $\{u_i\}$, $\{v_i\}$ the singular values and corresponding right and left singular vectors of M with $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_l$. An iterative algorithm for calculating the m ($m \ll l$) dominant singular values of operator M is given as follows.

Algorithm.

Step 1. Randomly choose m initial orthonormal vectors $p_i^{(0)} \in \mathrm{S}$, $i = 1, 2, \ldots, m$.

Step 2. (i) Let

$$\bar{q}_i^{(k)} = M[p_i^{(k)}] \qquad \text{for} \quad i = 1, 2, \ldots, m,$$

orthonormalize $\{\bar{q}_i^{(k)}\}$ and let the result be $\{q_i^{(k)}\}$.

(ii) Let

$$\bar{p}_i^{(k+1)} = M^*[q_i^{(k)}] \qquad \text{for} \quad i = 1, 2, \ldots, m,$$

orthonormalize $\{\bar{p}_i^{(k+1)}\}$ and let the result be $\{p_i^{(k+1)}\}$.

Continue doing (i) and (ii) for $k = 0, 1, 2, \ldots,$ until

$$\left[\sum_{i=1}^{m} \left\| p_i^{(k+1)} - \sum_{j=1}^{m} \langle p_i^{(k+1)}, p_j^{(k)}\rangle p_j^{(k)} \right\|^2\right]^{1/2}$$

is smaller than the error tolerance.

---

[1] For the definition and the properties of the singular values and singular vectors of a compact operator on a Hilbert space, see [GK].

Step 3. Let $\bar{q}_i = Mp_i^{(k+1)}$, for $i = 1, 2, \ldots, m$, and do the following.
(i) For $i = 1, 2, \ldots, m$, let

$$\omega_{ji} = \begin{cases} \langle \bar{q}_i, q_j \rangle, & 0 < j < i \\ 0, & i < j \leq m, \end{cases} \qquad q_i = \bar{q}_i - \sum_{j=1}^{i-1} \omega_{ji} q_i,$$

$$\omega_{ii} = \|q_i\|, \qquad q_i = \frac{q_i}{\omega_{ii}}.$$

(ii) Let $\Omega = [\omega_{ij}] \in \mathbb{R}^{n \times n}$ and let the singular value decomposition of Ω be $\Phi \Sigma \Psi'$, where $\Phi = [\varphi_{ij}]$, $\Sigma = [\sigma_{ij}]$, and $\Psi = [\psi_{ij}]$.

Then, for $i = 1, 2, \ldots, m$, $\sigma_i \approx \sigma_{ii}$, $u_i \approx \sum_{j=1}^{m} \psi_{ji} p_j^{(k+1)}$, and $v_i \approx \sum_{j=1}^{m} \varphi_{ji} q_j$.

The detailed derivation of this algorithm is given in [QD3]. The sketch of the idea is as follows. An initial m-dimensional trial space is chosen in Step 1. Step 2 iterates on trial spaces to obtain an approximation of the subspace spanned by the m right singular vectors corresponding to the m dominant singular values. Step 3 calculates the m dominant singular values and corresponding singular vectors from the approximate singular space obtained in Step 2.

To compute the robustness bounds (4.7), (8.1), and (8.3), we need the bottom singular values of L, $L|S_1$, and $L|S_2$. They can be obtained as the inverses of the dominant singular values of L^{-1}, $L^{-1}|S_1$, and $L^{-1}|S_2$. For a given $Q \in \mathbb{F}^{n \times n}$, $P = L^{-1}(Q)$ can be obtained by solving the Lyapunov equation

$$AP + PA' = Q$$

and $P = (L^{-1})^*(Q)$ can be obtained by solving the Lyapunov equation

$$A'P + PA = Q.$$

Since the Lyapunov equations can be solved more easily if A is a triangular matrix and since unitary similarity transformations of A do not affect the robustness bound, the Schur form of A instead of A itself can be used in the computation.

By using the algorithm given in this section, we can compute the robustness bounds for large matrices which may not be feasible to do using standard QR methods because of the dimensionality problem.

10. Numerical Examples

Several examples are presented to demonstrate the new bounds obtained, and to compare them with previous bounds.

Example 1. The matrix considered is as follows:

$$A = \begin{bmatrix} 0 & 1 & 100 \\ -10 & -1 & 2 \\ -1 & 1 & -110 \end{bmatrix}$$

Previous bounds (Theorem 2.1(f) and (g)) give $\mu_{\mathbb{R}}(A) \geq 0.1626$ and $\mu_{\mathbb{R}}(A) \geq \mu_{\mathbb{C}}(A) = 0.5093$, respectively. The new robustness bounds are as follows:

$$\text{bound (4.7):} \quad \mu_{\mathbb{R}}(A) \geq \min\{\underline{\sigma}(A), \tfrac{1}{2}\sigma_{n^2-1}(A \oplus A)\}$$
$$= \min\{1.4704, \tfrac{1}{2} \times 1.3342\} = 0.6671,$$
$$\text{bound (8.3):} \quad \mu_{\mathbb{R}}(A) \geq \min\{\underline{\sigma}(A), \tfrac{1}{2}\underline{\sigma}(A \,\bar{\bar{\oplus}}\, A)\}$$
$$= \{1.4704, \tfrac{1}{2} \times 1.3342\} = 0.6671.$$

The new bounds (4.7) and (8.3) are tighter than the bounds given by Theorem 2.1(f) and (g). Bound (8.1) gives $\mu_{\mathbb{R}}(A) \geq \frac{1}{2}\underline{\sigma}(A \,\bar{\oplus}\, A) = 0.1894$ and the conjecture (8.6) is true in this case.

Example 2. In this example we examine how conservative the value $\mu_{\mathbb{C}}(A)$ can be when used as a lower bound for $\mu_{\mathbb{R}}(A)$ and how much improvement the new bounds possibly have over the previous bounds. The matrix to be considered is

$$A = \begin{bmatrix} -1 & k \\ -1 & -1 \end{bmatrix}, \qquad \text{where} \quad k \geq 1.$$

The eigenvalues of A are $-1 \pm j\sqrt{k}$ and

$$\sigma_3(A \oplus A) = \underline{\sigma}(A \,\bar{\bar{\oplus}}\, A) = \tfrac{1}{2}\mathrm{tr}(A) = 1,$$

$$\underline{\sigma}(A) = \left[\frac{3 + k^2 - \sqrt{(k^2+1)^2 - 8k + 4}}{2}\right]^{1/2} > 1,$$

and so from Theorem 5.2 or Theorem 8.3, we obtain $\mu_{\mathbb{R}}(A) = 1$. However, Theorem 2.1(g) gives

$$\mu_{\mathbb{C}}(A) = \inf_{\omega \in \mathbb{R}} \underline{\sigma}(j\omega I - A) \leq \underline{\sigma}(j\omega I - A)|_{\omega = \sqrt{k-1}}$$
$$= \left[\frac{(1+k)^2 - \sqrt{(1+k)^4 - 16k}}{2}\right]^{1/2},$$

which goes to zero as k goes to infinity. This implies that the conservatism of $\mu_{\mathbb{C}}(A)$ as a bound of $\mu_{\mathbb{R}}(A)$ can become arbitrarily large. Since bounds (4.7) and (8.3) give the exact value of $\mu_{\mathbb{R}}(A)$ when A is 2×2, there can be an arbitrary degree of improvement over the previous bounds (Theorem 2.1(f) and (g)) when they are applied to matrices with any size.

Example 3. This example considers a 46th-order state space model obtained from the design of a third-generation spacecraft which has three rigid body modes and twenty elastic body modes [S1]. The system has five inputs and five outputs. It is stabilized by a static output feedback controller so that the closed-loop poles closest to the imaginary axis are given by $-1.0026 \times 10^{-3} \pm j5.3242 \times 10^{-1}$. Denote the closed-loop state matrix by A_c. It is desired to obtain an estimate of $\mu_{\mathbb{R}}(A_c)$, the distance of A_c to the set of unstable 46×46 real matrices. It is clear that $\mu_{\mathbb{R}}(A_c) \leq 1.0026 \times 10^{-3}$. On using the algorithm of Section 9, it is determined after seven iterations that $\underline{\sigma}(A_c \,\bar{\bar{\oplus}}\, A_c) = 1.9903 \times 10^{-3}$. By using the usual QR method, it is determined that $\underline{\sigma}(A_c) = 2.0252 \times 10^{-1}$. Hence, on using bound (8.3), an estimate of $\mu_{\mathbb{R}}(A_c)$ is given by

$$0.9952 \times 10^{-3} \leq \mu_{\mathbb{R}}(A_c) \leq 1.0026 \times 10^{-3}.$$

This estimate is very tight.

Note that the dimensions of $A \oplus A$, $A \,\bar{\oplus}\, A$, and $A \,\bar{\bar{\oplus}}\, A$ are 2116×2116, 1081×1081, and 1035×1035, respectively.

11. Conclusions

A new method for the robust stability problem of linear time-invariant state space models with real perturbations is considered in this paper. The method is based on the properties of the Kronecker sum and two other composite matrices. Explicit bounds on the magnitude of unstructured real perturbations which do not destabilize a linear time-invariant stable system are obtained. These bounds are easy to compute, and although the dimensions of the composite matrices required are of the order n^2, it is shown that it is possible to compute the required singular values of the composite matrices in an efficient way without actually constructing them. The new method can also be adopted to analyze the stability robustness of (i) systems with structured perturbations [STA], (ii) generalized eigenvalue problems [QD4], and (iii) discrete-time systems [QD2].

Appendix 1

Proof of Proposition 7.1. In order to simplify the proof, we derive an equality first. Let $A, B \in \mathbb{F}^{n \times n}$ and E_{ij} be defined as in Section 7. Then

$$\begin{aligned}
[\mathrm{Vec}(E_{ij})]'(A \otimes B)\,\mathrm{Vec}(E_{kl}) &= [(A' \otimes I)\,\mathrm{Vec}(E_{ij})]'[(I \otimes B)\,\mathrm{Vec}(E_{kl})] \\
&= [\mathrm{Vec}(E_{ij}A)]'[\mathrm{Vec}(BE_{kl})] = \mathrm{tr}[(E_{ij}A)'BE_{kl}] \\
&= \mathrm{tr}(A'E_{ji}BE_{kl}) = \mathrm{tr}(A'b_{ik}E_{jl}) = b_{ik}\,\mathrm{tr}(E_{ij}A) = a_{jl}b_{ik}.
\end{aligned}$$

Thus $[\mathrm{Vec}(E_{ij})]'(A \otimes B)\,\mathrm{Vec}(E_{kl}) = a_{jl}b_{ik}$.

Let the ith and jth pairs in the sequence (7.1) be (i_1, i_2) and (j_1, j_2). If $i_1 = i_2$ and $j_1 = j_2$,

$$\begin{aligned}
[T_1'(A \otimes B)T_1]_{ij} &= u_i'(A \otimes B)u_j \\
&= [\mathrm{Vec}(E_{i_1 i_1})]'(A \otimes B)\,\mathrm{Vec}(E_{j_1 j_1}) \\
&= a_{i_1 j_1} b_{i_1 j_1}.
\end{aligned}$$

If $i_1 \neq i_2$ and $j_1 \neq j_2$,

$$[T_1'(A\otimes B)T_1]_{ij} = u_i'(A \otimes B)u_j$$

$$= \left[\mathrm{Vec}\left(\frac{\sqrt{2}}{2}E_{i_1i_2}+\frac{\sqrt{2}}{2}E_{i_2i_1}\right)\right]'(A\otimes B)\left[\mathrm{Vec}\left(\frac{\sqrt{2}}{2}E_{j_1j_2}+\frac{\sqrt{2}}{2}E_{j_2j_1}\right)\right]$$

$$= \tfrac{1}{2}(a_{i_1j_1}b_{i_2j_2} + a_{i_1j_2}b_{i_2j_1} + a_{i_2j_1}b_{i_1j_2} + a_{i_2j_2}b_{i_1j_1}).$$

If $i_1 = i_2$ and $j_1 \neq j_2$,

$$[T_1'(A \otimes B)T_1]_{ij} = u_i'(A \otimes B)u_j$$

$$= [\mathrm{Vec}(E_{i_1i_1})]'(A \otimes B)\left[\mathrm{Vec}\left(\frac{\sqrt{2}}{2}E_{j_1j_2} + \frac{\sqrt{2}}{2}E_{j_2j_1}\right)\right]$$

$$= \frac{\sqrt{2}}{2}(a_{i_1j_2}b_{i_1j_1} + a_{i_1j_1}b_{i_1j_2})$$

$$= \frac{\sqrt{2}}{2}(a_{i_1j_1}b_{i_2j_2} + a_{i_2j_2}b_{i_1j_1}).$$

If $i_1 \neq i_2$ and $j_1 = j_2$,

$$[T_1'(A \otimes B)T_1]_{ij} = u_i'(A \otimes B)u_j$$

$$= \left[\mathrm{Vec}\left(\frac{\sqrt{2}}{2}E_{i_1i_2} + \frac{\sqrt{2}}{2}E_{i_2i_1}\right)\right]'(A \otimes B)[\mathrm{Vec}(E_{j_1j_1})]$$

$$= \frac{\sqrt{2}}{2}(a_{i_2j_1}b_{i_1j_1} + a_{i_1j_1}b_{i_2j_1})$$

$$= \frac{\sqrt{2}}{2}(a_{i_1j_1}b_{i_2j_2} + a_{i_2j_2}b_{i_1j_1}).$$

This proves (7.11).

Let the rth and sth pairs in the sequence (7.3) be (r_1, r_2) and (s_1, s_2). Then

$$[T_2'(A\otimes B)T_2]_{rs} = v_r'(A \otimes B)v_s$$

$$= \left[\mathrm{Vec}\left(\frac{\sqrt{2}}{2}E_{r_1r_2}-\frac{\sqrt{2}}{2}E_{r_2r_1}\right)\right]'(A\otimes B)\left[\mathrm{Vec}\left(\frac{\sqrt{2}}{2}E_{s_1s_2}-\frac{\sqrt{2}}{2}E_{s_2s_1}\right)\right]$$

$$= \tfrac{1}{2}(a_{r_1s_1}a_{r_2s_2} - a_{r_1s_2}b_{r_2s_1} - a_{r_2s_1}b_{r_1s_2} + a_{r_2s_2}b_{r_1s_1}).$$

This proves equality (7.12). ■

Appendix 2

Proof of Proposition 7.3. (a) This property directly follows Proposition 3.1(a) and Proposition 7.1.

(b) Since $\|T_1\|_s = 1$ and $\|A \otimes B\|_s = \|A\|_s\|B\|_s$,

$$\|A \bar{\otimes} B\|_s = \|T_1'(A \otimes B)T_1\|_s \leq \|T_1'\|_s\|A \otimes B\|_s\|T_1\|_s = \|A\|_s\|B\|_s.$$

Similarly, $\|A \bar{\bar{\otimes}} B\|_s \leq \|A\|_s\|B\|_s$ due to $\|T_2\|_s = 1$.

(c) Let $P \in \mathbb{C}^{n \times n}$ be a nonsingular matrix such that $B := P^{-1}AP$ is an upper triangular matrix. Then $\lambda_i(A) = b_{ii}$. By (7.13) and (7.14), we obtain

$$\begin{bmatrix} A \bar{\oplus} A & 0 \\ 0 & A \bar{\bar{\oplus}} A \end{bmatrix} = T'(A \oplus A)T,$$

$$\begin{bmatrix} P \bar{\otimes} P & 0 \\ 0 & P \bar{\bar{\otimes}} P \end{bmatrix} = T'(P \otimes P)T,$$

and

$$\begin{aligned} \begin{bmatrix} (P \bar{\otimes} P)^{-1} & 0 \\ 0 & (P \bar{\bar{\otimes}} P)^{-1} \end{bmatrix} &= \begin{bmatrix} P \bar{\otimes} P & 0 \\ 0 & P \bar{\bar{\otimes}} P \end{bmatrix}^{-1} \\ &= [T'(P \otimes P)T]^{-1} \\ &= T'(P \otimes P)^{-1}T \qquad \text{(since } T \text{ is real orthogonal)} \\ &= T'(P^{-1} \otimes P^{-1})T \qquad \text{(by Proposition 3.1(d)).} \end{aligned}$$

Then

$$\begin{aligned} &\begin{bmatrix} (P \bar{\otimes} P)^{-1}(A \bar{\oplus} A)(P \bar{\otimes} P) & 0 \\ 0 & (P \bar{\bar{\otimes}} P)^{-1}(A \bar{\bar{\oplus}} A)(P \bar{\bar{\otimes}} P) \end{bmatrix} \\ &= T'(P^{-1} \otimes P^{-1})(A \oplus A)(P \otimes P)T \\ &= T'(P^{-1}AP \oplus P^{-1}AP)T \\ &= \begin{bmatrix} P^{-1}AP \bar{\oplus} P^{-1}AP & 0 \\ 0 & P^{-1}AP \bar{\bar{\oplus}} P^{-1}AP \end{bmatrix}. \end{aligned}$$

Therefore,

$$B \bar{\oplus} B = P^{-1}AP \bar{\oplus} P^{-1}AP = (P \bar{\otimes} P)^{-1}(A \bar{\oplus} A)(P \bar{\otimes} P),$$

and

$$B \bar{\bar{\oplus}} B = P^{-1}AP \bar{\bar{\oplus}} P^{-1}AP = (P \bar{\bar{\otimes}} P)^{-1}(A \bar{\bar{\oplus}} A)(P \bar{\bar{\otimes}} P).$$

This implies that the spectrum of $A \bar{\oplus} A$ is the same as that of $B \bar{\oplus} B$. From Definition 7.1, we can see that $B \bar{\oplus} B$ is also an upper triangular matrix and the (i, i)th element of $B \bar{\oplus} B$ is just $b_{i_1 i_1} + b_{i_2 i_2} = \lambda_{i_1}(A) + \lambda_{i_2}(A)$, where (i_1, i_2) is the ith pair in the sequence (7.1). This proves $\mathrm{sp}(A \bar{\oplus} A) = \{\lambda_i(A) + \lambda_j(A), i = 1, 2, \ldots, n, j \geq i\}$. The proof for $\mathrm{sp}(A \bar{\bar{\oplus}} A) = \{\lambda_i(A) + \lambda_j(A), i = 1, 2, \ldots, n - 1, j > i\}$ is similar to the above by using the fact that the spectrum of $A \bar{\bar{\oplus}} A$ is the same as that of $B \bar{\bar{\oplus}} B$, and $B \bar{\bar{\oplus}} B$ is upper triangular. ■

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