Proceedings of the 28th Conference
on Decision and Control
Tampa, Florida • December 1989

The Stability Robustness of Generalized Eigenvalues*

L. Qiu and E.J. Davison
Department of Electrical Engineering
University of Toronto
Toronto, Ontario, Canada M5S 1A4

Abstract

This paper considers the stability robustness of the generalized eigenvalues of matrix pairs with real perturbations. The problem is to estimate the norm of the smallest destabilizing perturbation on a stable matrix pair. The method used is an extension of the one used in [12], which is based on the properties of some composite matrices. Sufficient conditions on the norm of the perturbations are given which guarantee the stability of the perturbed matrix pair. The results obtained can be applied to the stability robustness analysis of singularly perturbed systems and descriptor systems, and to a new kind of problem called the "the minimum phase robustness problem".

1 Introduction

In the stability robustness analysis of state space models, one of the important problems studied is to estimate the distance of a stable matrix to the set of all unstable matrices. Here a matrix is said to be stable if all its eigenvalues are contained in the open left half of the complex plane. Since this problem was first considered in [11] , it has been intensively studied (see [12] and the references therein). Different methods have been used and various results have been obtained. It appears, however, that no effort has been made to extend the available results to the generalized eigenvalue problem. For a pair of matrices (A, B), where A, B are square and have the same size, the generalized eigenvalues of (A, B) are the roots of the following polynomial in λ:

$$\det(A - \lambda B). \qquad (1)$$

The matrix pair (A, B) is said to be stable if all its generalized eigenvalues are located in the open left half of the complex plane. A pathological case occurs when $\det(A - \lambda B)$ vanishes identically. If this is the case, (A, B) is said to be degenerate and every point in the complex plane is a generalized eigenvalue of (A, B). This case will not be excluded from our discussion. Instead, it is treated as a special case of unstable matrix pairs. In this paper, we will consider how far a stable pair (A, B) is from unstable pairs. We will emphasize the case when B is singular. In this case, when B is singular, it is possible to find an arbitrarily small perturbation ΔB on B such that $B + \Delta B$ is nonsingular, and this implies that $(A, B + \Delta B)$ has more generalized eigenvalues than (A, B); furthermore, ΔB can be chosen in such a way that some of the extra generalized eigenvalues of $(A, B + \Delta B)$ are located in the closed right half of the complex plane. This means that the stability of (A, B) has zero tolerance to the perturbation of matrix B. Hence we assume in this paper that B is always fixed and is not subject to perturbation.

The problem considered in this paper is to estimate the smallest possible spectral norm of ΔA such that $(A + \Delta A, B)$ is unstable for a given stable matrix pair (A, B). This paper considers only real matrices. The method used is a generalization of the one given in [12], which is based on the properties of some composite matrices. Although the exact value of the norm of such ΔA is not in general obtained, the bounds obtained are tight. Some possible applications of the results obtained are in the stability robustness analysis of singularly perturbed systems [10] and descriptor systems [1] [2], and in a new kind of problem called the "minimum phase robustness problem", which deals with the robustness of the minimum phase property of systems [4]. Note that for the generalized eigenvalue problems involved in singularly perturbed systems and in the minimum phase robustness problem, the matrix B is always fixed. Due to the space limit, not all of the results are proved in this paper. Readers are referred to [13] for the missing proofs.

The following notation will be used throughout this paper. Let $\mathbf{R}$ and $\mathbf{C}$ be the fields of real numbers and complex numbers respectively. The real part of $\lambda \in \mathbf{C}$ is written as $\Re(\lambda)$ and the imaginary part $\Im(\lambda)$. Denote the sets $\{\lambda \in \mathbf{C} : \Re(\lambda) < 0\}$ and $\{\lambda \in \mathbf{C} : \Re(\lambda) \geq 0\}$ by $\mathbf{C}^-$ and $\mathbf{C}^+$ respectively. Let $\mathbf{F}$ be either $\mathbf{R}$ or $\mathbf{C}$. For $A \in \mathbf{F}^{m\times n}$, A' is the transpose of A and A^* is the conjugate transpose of A. The rank and the nullity of A are denoted by $\rho(A)$ and $\nu(A)$ respectively; a well-known relation between them is $\rho(A) + \nu(A) = n$. $\sigma_i(A)$, $i = 1, 2, \ldots, \min(m, n)$, denotes the i-th singular value of A with ordering $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min(m,n)}(A)$; in particular, $\sigma_1(A)$ and $\sigma_{\min(m,n)}(A)$ are denoted by $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ respectively. The spectral norm of A is denoted by $\|A\|_s$, which has the property that $\|A\|_s = \overline{\sigma}(A)$. For square matrices $A, B \in \mathbf{F}^{n\times n}$, the i-th generalized eigenvalue of (A, B) is denoted by $\lambda_i(A, B)$ with no specific ordering imposed: the set of all $\lambda_i(A, B)$ is denoted by $\Lambda(A, B)$ and the number of elements in $\Lambda(A, B)$ (including multiplicities) is denoted by $|\Lambda(A, B)|$. If (A, B) is degenerate, we write $\Lambda(A, B) = \mathbf{C}$ and $|\Lambda(A, B)| = \infty$.

2 Development

Given $A, B \in \mathbf{R}^{n\times n}$ with $\rho(B) = r$, define the distance of (A, B) from instability by

$$\mu(A, B) = \inf\{\|\Delta A\|_s : \Delta A \in \mathbf{R}^{n\times n} \text{ and } \Lambda(A + \Delta A, B) \not\subset \mathbf{C}^-\}. \qquad (2)$$

The purpose of this paper is to study the properties of $\mu(A, B)$ and to obtain bounds on $\mu(A, B)$. To keep the notation consistent with that used in [12], we write $\mu(A, I)$ as $\mu(A)$. Let a singular value decomposition of B be given as follows:

$$B = USV' = [U_1\, U_2] \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix} [V_1\, V_2]', \qquad (3)$$

where $U, V \in \mathbf{R}^{n\times n}$ are orthogonal matrices and $S_{11} \in \mathbf{R}^{r\times r}$ is a diagonal matrix with positive diagonal elements. This singular value decomposition will be used frequently, and for the sake of convenience, the notation used in (3) is assumed to hold throughout this paper.

If (A, B) itself is unstable, then obviously $\mu(A, B) = 0$. Another trivial case occurs when (A, B) is stable but $|\Lambda(A, B)| < r$. Let U and V be given by (3) and let

$$R = U'AV = [U_1\, U_2]'A[V_1\, V_2] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} \det(A - \lambda B) &= \det(U)\det(R - \lambda S)\det(V) \\ &= \pm[(-1)^r \det(R_{22})\lambda^r + \cdots + \det(R)]. \end{aligned}$$

*This work has been supported by the National Sciences and Engineering Research Council of Canada under grant no. A4396.

It is apparent that $|\Lambda(A,B)| < r$ if and only if $\det(R_{22}) = 0$. In this case, when $|\Lambda(A,B)| < r$, an arbitrarily small perturbation ΔR_{22} on submatrix R_{22} can be found such that $\det(R_{22}+\Delta R_{22})$ is nonzero and $\text{sign}[(-1)^r \det(R_{22}+\Delta R_{22})] = -\text{sign}[\det(R)]$. Let

$$\Delta A = U \begin{bmatrix} 0 & 0 \\ 0 & \Delta R_{22} \end{bmatrix} V'.$$

Then $(A+\Delta A, B)$ has r generalized eigenvalues and at least one of them is in $\mathbf{C}^+$. This shows that $\mu(A,B) = 0$ if $|\Lambda(A,B)| < r$. As a consequence, we always assume in the following development that (A,B) is stable and $|\Lambda(A,B)| = r$.

The following theorem gives some quick facts about $\mu(A,B)$.

Theorem 1 *Given $A, B \in \mathbf{R}^{n\times n}$ such that $\rho(B) = r$, (A,B) is stable and $|\Lambda(A,B)| = r$, then*

(a) $\mu(A,B) > 0$;

(b) $\mu(\alpha A, \beta B) = \alpha\mu(A,B)$ *for any* $\alpha > 0$, $\beta > 0$;

(c) $\mu(W_1AW_2, W_1BW_2) = \mu(A,B)$ *for any orthogonal matrices* W_1, $W_2 \in \mathbf{R}^{n\times n}$;

(d) $\mu(A,B) \le \underline{\sigma}(A)$;

(e) $\mu(A,B) \le \underline{\sigma}(U_2'AV_2)$.

Proof

(a) If ΔA is sufficiently small, $\det(A+\Delta A-\lambda B)$ will have the same degree as $\det(A-\lambda B)$; continuity arguments then show $\mu(A,B) > 0$.

(b) $(\alpha A+\alpha\Delta A, \beta B)$ has a generalized eigenvalue λ_i if and only if $(A+\Delta A, B)$ has a generalized eigenvalue $\frac{\beta}{\alpha}\lambda_i$. So the stability of $(\alpha A+\alpha\Delta A, \beta B)$ is equivalent to the stability of $(A+\Delta A, B)$. The result follows if we note that $\|\alpha\Delta A\|_s = \alpha\|\Delta A\|_s$.

(c) The result becomes obvious if we notice that $\det(A+\Delta A-\lambda B) = \det(W_1AW_2+W_1\Delta AW_2-\lambda W_1BW_2)$ and $\|\Delta A\|_s = \|W_1\Delta AW_2\|_s$.

(d) Let ΔA be chosen such that $\|\Delta A\|_s = \underline{\sigma}(A)$ and $\det(A+\Delta A) = 0$; then $(A+\Delta A, B)$ is unstable. This shows $\mu(A,B) \le \|\Delta A\|_s = \underline{\sigma}(A)$.

(e) This inequality will become clear at the end of this section. The proof is omitted here. □

Let (A,B) be stable and $(A+\Delta A, B)$ be unstable for some ΔA. Intuitively, the behavior of the generalized eigenvalues of $(A+\alpha\Delta A, B)$ when α sweeps continuously from 0 to 1 will, in the ideal case, have three possibilities: (i) one generalized eigenvalue of $(A+\alpha\Delta A, B)$ shifts from $\mathbf{C}^-$ to $\mathbf{C}^+$ across the origin; (ii) a pair of generalized eigenvalues of $(A+\alpha\Delta A, B)$ shift from $\mathbf{C}^-$ to $\mathbf{C}^+$ across the imaginary axis; (iii) one of the generalized eigenvalues of $(A+\alpha\Delta A, B)$ disappears at infinity and then appears in $\mathbf{C}^+$. However, the actual situation may be more complicated since possibly for some $\alpha \in (0,1]$, $(A+\alpha\Delta A, B)$ becomes degenerate. Nevertheless let us define the following three quantities:

$$\mu_1(A,B) = \inf\{\|\Delta A\|_s : \Delta A \in \mathbf{R}^{n\times n} \text{ and } \det(A+\Delta A) = 0\} \quad (4)$$

$$\mu_2(A,B) = \inf\{\|\Delta A\|_s : \Delta A \in \mathbf{R}^{n\times n},\ |\Lambda(A+\Delta A,B)| = r, \text{ and } \Lambda(A+\Delta A,B)\cap(\partial\mathbf{C}^-\setminus\{0\}) \neq \varnothing\} \quad (5)$$

$$\mu_3(A,B) = \inf\{\|\Delta A\|_s : \Delta A \in \mathbf{R}^{n\times n} \text{ and } |\Lambda(A+\Delta A,B)| < r\} \quad (6)$$

where $\partial\mathbf{C}^- = \{j\omega : \omega \in \mathbf{R}\}$.

The following theorem simplifies the analysis of $\mu(A,B)$:

Theorem 2 *Given $A, B \in \mathbf{R}^{n\times n}$ such that $\rho(B) = r$, (A,B) is stable and $|\Lambda(A,B)| = r$, then*

$$\mu(A,B) = \min\{\mu_1(A,B), \mu_2(A,B), \mu_3(A,B)\}. \quad (7)$$

Proof It is clear that $\mu(A,B) \le \min\{\mu_1(A,B), \mu_2(A,B), \mu_3(A,B)\}$. Now assume that for some $\Delta A \in \mathbf{R}^{n\times n}$, $(A+\Delta A, B)$ is unstable. Fix this ΔA and consider the matrix pair $(A+\alpha\Delta A, B)$ for $\alpha \in (0,1]$ in the following three possible cases:

(i) $(A+\alpha\Delta A, B)$ is degenerate for some $\alpha \in (0,1]$.

For such an α, $A+\alpha\Delta A$ is singular. This implies $\|\Delta A\|_s \ge \|\alpha\Delta A\|_s \ge \mu_1(A,B)$.

(ii) $(A+\alpha\Delta A, B)$ is non-degenerate for all $\alpha \in (0,1]$, but $|\Lambda(A+\alpha\Delta A, B)| < r$ for some $\alpha \in (0,1]$.

Choose an $\alpha \in (0,1]$ such that $|\Lambda(A+\alpha\Delta A, B)| < r$; then by the definition of $\mu_3(A,B)$, we have $\|\alpha\Delta A\|_s \ge \mu_3(A,B)$. This implies $\|\Delta A\|_s \ge \mu_3(A,B)$.

(iii) $(A+\alpha\Delta A, B)$ is non-degenerate and $|\Lambda(A+\alpha\Delta A, B)| = |\Lambda(A,B)|$ for all $\alpha \in (0,1]$.

In this case, $\det(A+\alpha\Delta A-\lambda B)$ has the same degree as $\det(A-\lambda B)$ for all $\alpha \in (0,1]$, and its coefficients are continuous functions of α. Since all the roots of $\det(A-\lambda B)$ are contained in $\mathbf{C}^-$ and at least one of the roots of $\det(A+\Delta A-\lambda B)$ is in $\mathbf{C}^+$, there must be an $\alpha \in (0,1]$ such that at least one of the roots of $\det(A+\alpha\Delta A-\lambda B)$ is in $\partial\mathbf{C}^-$. If this root is at the origin, then $\|\Delta A\|_s \ge \|\alpha\Delta A\|_s \ge \mu_1(A,B)$; otherwise $\|\Delta A\|_s \ge \|\alpha\Delta A\|_s \ge \mu_2(A,B)$.

For any of the three possible cases, we have

$$\mu(A,B) \ge \min\{\mu_1(A,B), \mu_2(A,B), \mu_3(A,B)\}.$$

This completes the proof. □

$\mu_1(A,B)$ can be easily obtained as

$$\mu_1(A,B) = \underline{\sigma}(A). \quad (8)$$

It has already been shown that $|\Lambda(A+\Delta A, B)| < r$ if and only if $U_2'(A+\Delta A)V_2$ is singular, where U_2 and V_2 are given by (3). Let ΔA be such a matrix that $|\Lambda(A+\Delta A, B)| < r$; then $\|\Delta A\|_s \ge \|U_2'\Delta AV_2\|_s \ge \underline{\sigma}(U_2'AV_2)$. This shows $\mu_3(A,B) \ge \underline{\sigma}(U_2'AV_2)$. On the other hand, it is known that there exists a matrix $\overline{\Delta A}$ such that $\|\overline{\Delta A}\|_s = \underline{\sigma}(U_2'AV_2)$ and $U_2'AV_2 + \overline{\Delta A}$ is singular. Let $\Delta A = U \begin{bmatrix} 0 & 0 \\ 0 & \overline{\Delta A} \end{bmatrix} V'$. Then $\|\Delta A\|_s = \|\overline{\Delta A}\|_s = \underline{\sigma}(U_2'AV_2)$ and $U_2'(A+\Delta A)V_2 = U_2'AV_2 + \overline{\Delta A}$ which implies $|\Lambda(A+\Delta A, B)| < r$. This shows $\mu_3(A,B) \le \underline{\sigma}(U_2'AV_2)$. As a consequence, we conclude that

$$\mu_3(A,B) = \underline{\sigma}(U_2'AV_2). \quad (9)$$

The inequality given by Theorem 1(e) now directly follows Theorem 2 and equation (9).

It will now be shown that $\mu(A,B)$ can be obtained for a special class of matrix pairs. Consider a matrix pair (A,B) with $\rho(B) = 1$. Then $(A+\Delta A, B)$ has at most one generalized eigenvalue for any perturbation ΔA, so it is impossible to have $\Lambda(A+\Delta A, B)\cap(\partial\mathbf{C}^-\setminus\{0\}) \neq \varnothing$ for real ΔA. This means $\mu_2(A,B) = \infty$. Consequently, for $A, B \in \mathbf{R}^{n\times n}$ with $\rho(B) = |\Lambda(A,B)| = 1$ and (A,B) is stable, we have

$$\mu(A,B) = \min\{\underline{\sigma}(A), \underline{\sigma}(U_2'AV_2)\}. \quad (10)$$

The following analysis will be focused on $\mu_2(A,B)$ for general matrix pairs with $\rho(B) = |\Lambda(A,B)| > 1$. The exact value of $\mu_2(A,B)$ is not obtained in this paper. Instead, some lower bounds on $\mu_2(A,B)$ is obtained. These lower bounds are based on the properties of some composite matrices and are easy to compute.

3 Composite Matrices

Let $\mathbf{F}$ be the field $\mathbf{R}$ or $\mathbf{C}$. Let $A \in \mathbf{F}^{m\times n}$, $B \in \mathbf{F}^{p\times q}$. Denote the Kronecker product of A and B by $A\otimes B$. Among the many nice properties of the Kronecker product, those which will be used in our development are listed in the following lemma:

Lemma 1

(a) *If $\alpha, \beta \in \mathbf{F}$, then*

$$A\otimes(\alpha B+\beta C) = \alpha(A\otimes B)+\beta(A\otimes C)$$

$$(\alpha A+\beta B)\otimes C = \alpha(A\otimes C)+\beta(B\otimes C)$$

(b) $(A \otimes B)(C \otimes D) = AC \otimes BD$

(c) *If* $x_1, x_2, \ldots, x_p \in \mathsf{F}^m$ *are linearly independent and* $y_1, y_2, \ldots, y_q \in \mathsf{F}^n$ *are linearly independent, then* $\{x_i \otimes y_j : i = 1, 2, \ldots, p, j = 1, 2, \ldots, q\}$ *are linearly independent.*

(d) $\|A \otimes B\|_s = \|A\|_s \|B\|_s$.

The Kronecker product can be considered as a matrix composition; two additional matrix compositions are also important in our context. Let $A = [a_{ij}] \in \mathsf{F}^{n \times n}$, $B = [b_{ij}] \in \mathsf{F}^{n \times n}$. Let (i_1, i_2) and (j_1, j_2) be the i-th and j-th pairs of integers respectively in the sequence

$$(1,1), \ldots, (1,n), (2,2), \ldots, (2,n), (3,3), \ldots, (n-1,n), (n,n). \quad (11)$$

Definition 1 [12]

$$A \overline{\otimes} B = [c_{ij}] \in \mathsf{F}^{\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)},$$

where

$$c_{ij} = \begin{cases} a_{i_1 j_1} b_{i_1 j_1} & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ \frac{1}{2}(a_{i_1 j_1} b_{i_2 j_2} + a_{i_1 j_2} b_{i_2 j_1} & \\ \quad + a_{i_2 j_1} b_{i_1 j_2} + a_{i_2 j_2} b_{i_1 j_1}) & \text{if } i_1 \neq i_2 \text{ and } j_1 \neq j_2 \\ \frac{\sqrt{2}}{2}(a_{i_1 j_1} b_{i_2 j_2} + a_{i_2 j_2} b_{i_1 j_1}) & \text{otherwise.} \end{cases} \quad (12)$$

Let (r_1, r_2) and (s_1, s_2) be the r-th and s-th pairs of integers respectively in the sequence

$$(1,2), (1,3), \ldots, (1,n), (2,3), \ldots, (2,n), (3,4), \ldots, (n-1,n). \quad (13)$$

Definition 2 [12]

$$A \overline{\overline{\otimes}} B = [d_{rs}] \in \mathsf{F}^{\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)},$$

where

$$d_{rs} = \frac{1}{2}(a_{r_1 s_1} b_{r_2 s_2} - a_{r_1 s_2} b_{r_2 s_1} - a_{r_2 s_1} b_{r_1 s_2} + a_{r_2 s_2} b_{r_1 s_1}). \quad (14)$$

The operations $\overline{\otimes}$ and $\overline{\overline{\otimes}}$ are closely related to the Kronecker product operation $\otimes$. Given $A, B \in \mathsf{F}^{n \times n}$, $A \otimes B$ can be considered as a linear operator on the space F^{n^2}. The space F^{n^2} is an n^2- dimensional Hilbert space if it is equipped with the inner product $\langle x, y \rangle = x^* y$, $\forall x, y \in \mathsf{F}^{n^2}$. The space $\mathsf{F}^{n \times n}$ with inner product $\langle X, Y \rangle = \mathrm{tr}(X^* Y)$, $\forall X, Y \in \mathsf{F}^{n \times n}$ is also an n^2-dimensional Hilbert space. Now define a linear operator Vec: $\mathsf{F}^{n \times n} \to \mathsf{F}^{n^2}$ by

$$\mathrm{Vec} \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} = [x_{11} \cdots x_{n1} x_{12} \cdots x_{n2} \cdots x_{nn}]'. \quad (15)$$

We need two properties of Vec to proceed.

Lemma 2 [7]

(a) $\mathrm{tr}(X^* Y) = [\mathrm{Vec}(X)]^* \mathrm{Vec}(Y)$

(b) $(A \otimes B)\mathrm{Vec}(X) = \mathrm{Vec}(BXA')$.

Lemma 2 implies that Vec is an isomorphism. Under this isomorphism, operator $A \otimes B$ on F^{n^2} becomes an operator F mapping $X \in \mathsf{F}^{n \times n}$ to $F(X) = BXA'$.

Let $\mathbf{S_1} \subset \mathsf{F}^{n \times n}$ be the subspace of all symmetric matrices, and let $\mathbf{S_2} \subset \mathsf{F}^{n \times n}$ be the subspace of all skew-symmetric matrices. Formally $\mathbf{S_1} = \{X \in \mathsf{F}^{n \times n} : X' = X\}$ and $\mathbf{S_2} = \{X \in \mathsf{F}^{n \times n} : X' = -X\}$. It is shown in [12] that $\mathbf{S_1}$ and $\mathbf{S_2}$ are orthogonal complements to each other. Since Vec is an isomorphism, $\mathrm{Vec}(\mathbf{S_1})$ and $\mathrm{Vec}(\mathbf{S_2})$ are also orthogonal compliments to each other on F^{n^2}.

Define $E_{ij} \in \mathsf{F}^{n \times n}$ to be the matrix with 1 in the (i,j)-th entry and 0 elsewhere. Let (i_1, i_2) be the i-th pair of integers in the sequence (11) and let

$$U_i = \begin{cases} E_{i_1 i_1} & \text{if } i_1 = i_2 \\ \frac{\sqrt{2}}{2}(E_{i_1 i_2} + E_{i_2 i_1}) & \text{otherwise.} \end{cases}$$

Then $\{U_1, U_2, \ldots, U_{\frac{1}{2}n(n+1)}\}$ is an orthonormal basis of $\mathbf{S_1}$.

Let (r_1, r_2) be the r-the pair of integers in the sequence (13) and let

$$V_r = \frac{\sqrt{2}}{2}(E_{r_1 r_2} - E_{r_2 r_1}).$$

Then $\{V_1, V_2, \ldots, V_{\frac{1}{2}n(n-1)}\}$ is an orthonormal basis of $\mathbf{S_2}$.

Let $u_i = \mathrm{Vec}(U_i)$, $i = 1, 2, \ldots, \frac{1}{2}n(n+1)$ and $v_i = \mathrm{Vec}(V_i)$, $i = 1, 2, \ldots, \frac{1}{2}n(n-1)$. Then $\{u_1, u_2, \ldots, u_{\frac{1}{2}n(n+1)}\}$ is an orthonormal basis of $\mathrm{Vec}(\mathbf{S_1})$, and $\{v_1, v_2, \ldots, v_{\frac{1}{2}n(n-1)}\}$ is an orthonormal basis of $\mathrm{Vec}(\mathbf{S_2})$. Define

$$T_1 = [u_1 \, u_2 \ldots u_{\frac{1}{2}n(n+1)}] \in \mathsf{F}^{n \times \frac{1}{2}n(n+1)} \quad (16)$$

$$T_2 = [v_1 \, v_2 \ldots v_{\frac{1}{2}n(n-1)}] \in \mathsf{F}^{n \times \frac{1}{2}n(n-1)}. \quad (17)$$

The above construction implies that $[T_1 \, T_2]$ is a real orthogonal matrix.

The following lemma gives a relation among operators $\otimes$, $\overline{\otimes}$, $\overline{\overline{\otimes}}$.

Lemma 3 [12] *Let* $A, B \in \mathsf{F}^{n \times n}$. *Then*

$$A \overline{\otimes} B = T_1'(A \otimes B)T_1$$
$$A \overline{\overline{\otimes}} B = T_2'(A \otimes B)T_2.$$

From Lemma 3, many interesting properties of $\otimes$ and $\overline{\overline{\otimes}}$ can be established [12]. Lemma 4 states some properties which are useful for the development of the main result in the next section.

Lemma 4 [12]

(a) *If* $\alpha, \beta \in \mathsf{F}$, *then*

$$\begin{aligned} A \overline{\otimes}(\alpha B + \beta C) &= \alpha(A \overline{\otimes} B) + \beta(A \overline{\otimes} C) \\ (\alpha A + \beta B) \overline{\otimes} C &= \alpha(A \overline{\otimes} C) + \beta(B \overline{\otimes} C) \\ A \overline{\overline{\otimes}}(\alpha B + \beta C) &= \alpha(A \overline{\overline{\otimes}} B) + \beta(A \overline{\overline{\otimes}} C) \\ (\alpha A + \beta B) \overline{\overline{\otimes}} C &= \alpha(A \overline{\overline{\otimes}} C) + \beta(B \overline{\overline{\otimes}} C). \end{aligned}$$

(b) $\|A \overline{\otimes} B\|_s \leq \|A\|_s \|B\|_s$, $\|A \overline{\overline{\otimes}} B\|_s \leq \|A\|_s \|B\|_s$.

In the derivation of the lower bounds of $\mu(A, B)$, the matrices $A \otimes B + B \otimes A$, $A \overline{\otimes} B + B \overline{\otimes} A$, $A \overline{\overline{\otimes}} B + B \overline{\otimes} A$ play important roles. In the rest of this section, we will investigate some properties of these matrices.

Consider $A \otimes B + B \otimes A$ as a linear operator on F^{n^2}. Then under isomorphism Vec, it is equivalent to a linear operator M on $\mathsf{F}^{n \times n}$ mapping $X \in \mathsf{F}^{n \times n}$ to $M(X) = AXB' + BXA'$. This operator has the following nice property.

Lemma 5

$$M(\mathbf{S_1}) \subset \mathbf{S_1} \quad \text{and} \quad M(\mathbf{S_2}) \subset \mathbf{S_2}.$$

PROOF For all $X_1 \in \mathbf{S_1}$, we have $X_1' = X_1$. Thus

$$\begin{aligned} [M(X_1)]' &= (AX_1B' + BX_1A')' = AX_1'B' + BX_1A' \\ &= AX_1B' + BX_1A' = M(X_1), \end{aligned}$$

so that $M(\mathbf{S_1}) \subset \mathbf{S_1}$.

For all $X_2 \in \mathbf{S_2}$, we have $X_2' = -X_2$. Thus

$$\begin{aligned} [M(X_2)]' &= (AX_2B' + BX_2A')' = AX_2'B' + BX_2A' \\ &= -AX_1B' - BX_1A' = -M(X_1), \end{aligned}$$

so that $M(\mathbf{S_2}) \subset \mathbf{S_2}$. □

This lemma states that $\mathbf{S_1}$ and $\mathbf{S_2}$ are reducing subspaces of $\mathsf{F}^{n \times n}$ for operator M. Since Vec is an isomorphism between $\mathsf{F}^{n \times n}$ and F^{n^2}, $\mathrm{Vec}(\mathbf{S_1})$ and $\mathrm{Vec}(\mathbf{S_2})$ are reducing subspaces of F^{n^2} for operator $A \otimes B + B \otimes A$. This fact and the orthogonality of $\mathrm{Vec}(\mathbf{S_1})$ and $\mathrm{Vec}(\mathbf{S_2})$, along with Lemma 3 imply the following lemma:

Lemma 6

$$T'(A \otimes B + B \otimes A)T = \begin{bmatrix} A \overline{\otimes} B + B \overline{\otimes} A & 0 \\ 0 & A \overline{\overline{\otimes}} B + B \overline{\overline{\otimes}} A \end{bmatrix} \quad (18)$$

This mean that $A\overline{\otimes}B + B\overline{\otimes}A$ and $A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A$ are the matrix representations of the restrictions of $A\otimes B + B\otimes A$ to $\mathrm{Vec}(\mathbf{S_1})$ and $\mathrm{Vec}(\mathbf{S_2})$ respectively, when the basis $\{u_1, u_2, \ldots, u_{\frac{1}{2}n(n+1)}\}$ is used for $\mathrm{Vec}(\mathbf{S_1})$ and the basis $\{v_1, v_2, \ldots, v_{\frac{1}{2}n(n-1)}\}$ is used for $\mathrm{Vec}(\mathbf{S_2})$.

4 Main Result

In this section, the lower bounds of $\mu(A, B)$ will be derived. These bounds are based on the composite matrices developed in the last section. Throughout this section, it is assumed that $A, B \in \mathbf{R}^{n\times n}$, and $|\Lambda(A, B)| = \rho(B) = r$. To rule out trivial cases, we assume $r > 1$. Denote $s = \nu(B) = n - r$. The following lemmas are required in the proof of the main result.

Lemma 7 [13] *Assume* $0 \notin \{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in \Lambda(A, B)\}$. *Then*

(a) $\nu(A\otimes B + B\otimes A) = s^2$,

(b) $\nu(A\overline{\otimes}B + B\overline{\otimes}A) = \frac{1}{2}s(s+1)$,

(c) $\nu(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A) = \frac{1}{2}s(s-1)$.

Lemma 8 [13] *Assume that* $\Lambda(A, B) \cap \{\partial\mathbf{C}^- \setminus \{0\}\} \neq \varnothing$. *Then*

(a) $\nu(A\otimes B + B\otimes A) \geq s^2 + 2$,

(b) $\nu(A\overline{\otimes}B + B\overline{\otimes}A) \geq \frac{1}{2}s(s+1) + 1$,

(c) $\nu(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A) \geq \frac{1}{2}s(s-1) + 1$.

Lower bounds of $\mu_2(A, B)$ can then be obtained as a consequence of Lemma 7-8.

Theorem 3 *Given* $A, B \in \mathbf{R}^{n\times n}$ *such that* (A, B) *is stable and* $s = \nu(B) = n - |\Lambda(A, B)|$, *then*

$$\mu_2(A, B) \geq \frac{1}{2\|B\|_s}\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A) \quad (19)$$

$$\mu_2(A, B) \geq \frac{1}{2\|B\|_s}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A) \quad (20)$$

$$\mu_2(A, B) \geq \frac{1}{2\|B\|_s}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A). \quad (21)$$

PROOF If $\|\Delta A\|_s < \frac{1}{2\|B\|_s}\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A)$, then

$$\|\Delta A\otimes B + B\otimes\Delta A\|_s \leq 2\|\Delta A\|_s\|B\|_s < \sigma_{n^2-s^2-1}(A\otimes B + B\otimes A).$$

This implies that the nullity of

$$(A + \Delta A)\otimes B + B\otimes(A + \Delta A) = A\otimes B + B\otimes A + \Delta A\otimes B + B\otimes\Delta A$$

is less than $s^2 + 2$. By Lemma 8, $\Lambda(A + \Delta A, B)$ has no element in $\{\partial\mathbf{C}^- \setminus \{0\}\}$. Therefore, if $\Lambda(A + \Delta A, B) \cap \{\partial\mathbf{C}^- \setminus \{0\}\} \neq \varnothing$, $\|\Delta A\|_s$ must be greater than or equal to $\frac{1}{2\|B\|_s}\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A)$. This proves (19). The proofs of (20) and (21) are similar. □

Let a singular value decomposition of B be given by (3); then the following lemma is obtained.

Lemma 9 [13]

$$\frac{1}{2\|B\|_s}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A) \leq \underline{\sigma}(A), \quad (22)$$

$$\frac{1}{2\|B\|_s}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A) \leq \underline{\sigma}(U_2'AV_2). \quad (23)$$

The main result of this paper which gives the lower bounds of $\mu(A, B)$ is then obtained as an immediate consequence of Theorem 3, Lemma 9 and equation (7)-(9).

Theorem 4 *Given* $A, B \in \mathbf{R}^{n\times n}$ *such that* (A, B) *is stable and* $s = \nu(B) = n - |\Lambda(A, B)|$, *then*

$$\mu(A, B) \geq \min\{\underline{\sigma}(A), \frac{1}{2\|B\|_s}\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A), \underline{\sigma}(U_2'AV_2)\} \quad (24)$$

$$\mu(A, B) \geq \min\{\frac{1}{2\|B\|_s}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A), \underline{\sigma}(U_2'AV_2)\} \quad (25)$$

$$\mu(A, B) \geq \min\{\underline{\sigma}(A), \frac{1}{2\|B\|_s}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A)\}. \quad (26)$$

Theorem 4 actually provides a sufficient condition for the robust stability of a matrix pair, i.e. if $\|\Delta A\|_s$ is less than any one of the quantities on the left hand sides of (24)-(26), $(A + \Delta A, B)$ is always stable.

It is of interest to have certain knowledge on how tight the bounds given in Theorem 4 are. For general matrix pairs, it is very hard to estimate the conservatism of the lower bounds obtained. However, this can be done for some special classes of matrix pairs. A matrix is called a partial isometry if all its singular values are either one or zero. The spectral norm of a nonzero partial isometry is one. Two matrices $C, D \in \mathbf{C}^{n\times n}$ are said to be simultaneously diagonalizable by unitary transformations if there exist unitary matrices $P, Q \in \mathbf{C}^{n\times n}$ such that P^*CQ and P^*DQ are diagonal matrices.

Theorem 5 *Given* $A, B \in \mathbf{R}^{n\times n}$ *such that* (A, B) *is stable and* $s = \nu(B) = n - |\Lambda(A, B)|$, *assume that* B *is a nonzero partial isometry and that* A, B *are simultaneously diagonalizable by unitary transformations; then*

$$\mu(A, B) \leq \min\{\underline{\sigma}(U_2'AV_2), -\Re(\lambda_i) : \lambda_i \in \Lambda(A, B)\} \quad (27)$$

and

$$\begin{aligned}\mu(A, B) &\geq \min\{\underline{\sigma}(A), \frac{1}{2}\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A), \underline{\sigma}(U_2'AV_2)\}\\ &= \min\{\frac{1}{2}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A), \underline{\sigma}(U_2'AV_2)\}\\ &= \min\{\underline{\sigma}(A), \frac{1}{2}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A)\}\\ &= \min\{\frac{1}{2}\underline{\sigma}(U_2'AV_2), -\Re(\lambda_i) : \lambda_i \in \Lambda(A, B)\}. \quad (28)\end{aligned}$$

The proof of Theorem 5 is given in [13]. This theorem implies that in the case when B is a partial isometry and A, B are simultaneously diagonalizable by unitary transformations, the conservatism of the lower bounds (24)-(26) are at worst 50%, i.e. the differences of $\mu(A, B)$ and its lower bounds are at worst 50% of $\mu(A, B)$. If $\frac{1}{2}\underline{\sigma}(U_2'AV_2) \geq \min\{-\Re(\lambda_i) : \lambda_i \in \Lambda(A, B)\}$, the exact value of $\mu(A, B)$ is obtained as $\min\{-\Re(\lambda_i) : \lambda_i \in \Lambda(A, B)\}$.

In many applications, such as the singularly perturbed system problem and the minimum phase problem, the matrix B is always a partial isometry. Let a singular value decomposition of B be given by (3). Then since a necessary and sufficient condition for $A, B \in \mathbf{C}^{n\times n}$ to be simultaneously diagonalizable by unitary transformations is that AB^* and B^*A are both normal [8], it can be shown that when B is a nonzero partial isometry, the necessary and sufficient condition for A, B to be simultaneously diagonalizable by unitary transformations is that $U_1'AV_1$ is normal, $U_1'AV_2 = 0$ and $U_2'AV_1 = 0$.

Theorem 4 gives three lower bounds on $\mu(A, B)$. It is of interest to give a comparison between them, i.e. it is desired to know which bound is the best and which is the worst. From Lemma 5, it can be seen that the singular values of $A\otimes B + B\otimes A$ are just the singular values of $A\overline{\otimes}B + B\overline{\otimes}A$ together with the singular values of $A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A$. It is known that $\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A)$ and $\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A)$ are the smallest nonzero singular values of $(A\overline{\otimes}B + B\overline{\otimes}A)$ and $(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A)$ respectively, and that $\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A)$ are the second smallest nonzero singular value of $A\otimes B + B\otimes A$. Thus $\sigma_{n^2-s^2-1}(A\otimes B + B\otimes A)$ must lie between $\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A)$ and $\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A)$. Examples show that both $\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}s(s+1)}(A\overline{\otimes}B + B\overline{\otimes}A)$ and $\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}s(s-1)}(A\overline{\overline{\otimes}}B + B\overline{\overline{\otimes}}A)$ can in fact be equal to the smallest singular value of $A\otimes B + B\otimes A$. The conclusion is that amongst the three bounds (24)-(26), either bound (25) or bound (26) can be the best or the worst, and bound (24) always lies between bound (25) and (26).

5 Applications and Examples

There are a number of problems in control theory which involves the stability of the generalized eigenvalues. We will show how the results developed in the previous sections can be used to analyze the stability robustness of such problems.

I Singularly perturbed systems

A singularly perturbed system (with zero inputs) is described by a state space equation in the following form [10] .

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \qquad (29)$$

where $x \in \mathbf{R}^{n_1}$, $z \in \mathbf{R}^{n_2}$, and A_{ij}, $i, j = 1, 2$, are matrices of compatible sizes. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}.$$

It is known from [10] that if A_{22} is stable and (A, E) is stable, then there exists $\epsilon^* > 0$ such that $\forall \epsilon \in [0, \epsilon^*]$, system (29) is asymptotically stable. Now suppose that matrix A is subject to an unstructured real parameter perturbation ΔA. Then the perturbed system is given by

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = (A + \Delta A) \begin{bmatrix} x \\ z \end{bmatrix}.$$

A question arises: under what condition on the norm of ΔA is the stability property of the perturbed system still maintained? It is clear that a sufficient condition is that $\|\Delta A\|_s < \min\{\mu(A_{22}), \mu(A, E)\}$. Further investigation shows that this condition is almost necessary in the sense that for any $\delta > 0$, there exists a perturbation ΔA with $\|\Delta A\|_s \leq \delta + \min\{\mu(A_{22}), \mu(A, E)\}$ such that the system does not have the stability property.

Example 1

The following singularly perturbed system represents a voltage regulator controlled by a so-called corrected near-optimal state feedback law [10].

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1.6 & 0 & 0 \\ 0 & 0 & -1.429 & 8.571 & 0 \\ 0 & 0 & 0 & -2.5 & 7.5 \\ -0.918 & -0.19 & -0.011 & -0.038 & -1.287 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix},$$

where $x \in \mathbf{R}^2$, $z \in \mathbf{R}^3$. From the bounds given in [12], we obtain

$$\begin{aligned} \mu(A_{22}) &\geq \min\{\underline{\sigma}(A_{22}), \frac{1}{2}\underline{\sigma}(A_{22}\overline{\overline{\oplus}}A_{22})\} \\ &= \min\{0.1094, 0.2282\} = 0.1094. \end{aligned}$$

Since $\underline{\sigma}(A_{22}) < \frac{1}{2}\underline{\sigma}(A_{22}\overline{\overline{\oplus}}A_{22})$ in this case, we actually have $\mu(A_{22}) = 0.1094$.

From the bounds given in Theorem 3, we obtain

$$\begin{aligned} \mu(A, E) &\geq \min\{\underline{\sigma}(A), \frac{1}{2}\sigma_{15}(A\otimes E + E\otimes A), \underline{\sigma}(A_{22})\} \\ &= \min\{0.5047, 0.0919, 0.1094\} = 0.0919, \\ \mu(A, E) &\geq \min\{\frac{1}{2}\sigma_9(A\overline{\otimes}E + E\overline{\otimes}A), \underline{\sigma}(A_{22})\} \\ &= \min\{0.0919, 0.1094\} = 0.0919, \\ \mu(A, E) &\geq \min\{\underline{\sigma}(A), \frac{1}{2}\sigma_7(A\overline{\overline{\otimes}}E + E\overline{\overline{\otimes}}A)\} \\ &= \min\{0.5047, 0.0120\} = 0.0120. \end{aligned}$$

From this computed data, we claim that for any perturbation ΔA with $\|\Delta A\|_s < 0.0919$, the singularly perturbed system maintains the desired stability property. Note that, since $\mu(A_{22}) = 0.1094$, there exists a ΔA with $\|\Delta A\|_s = 0.1094$ such that the system (29) does not have the stability property. Hence the condition $\|\Delta A\|_s < 0.0919$ is not over conservative.

II Descriptor systems

Descriptor systems are systems described by the state space equations in the following form

$$E\dot{x} = Ax + Bu, \qquad (30)$$

where E is usually a singular matrix. For any initial condition, the homogeneous solution of equation (30) goes to zero asymptotically if and only if the matrix pair (A, E) is stable [1]. In this case, we say system (30) is stable. Hence $\mu(A, E)$ provides a reasonable measure to the stability robustness of the system. In general, there may also be perturbations on the elements of the matrix E; the robustness property of system (30) with respect to these perturbations can not be treated in the frame work of this paper.

Example 2

The following descriptor system is obtained in an optimal state feedback design [2].

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.3536 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since $\rho(B) = 1$, $\mu(A, E)$ can be obtained exactly by (10). It is easy to compute $\underline{\sigma}(A) = 0.4644$. A singular value decomposition of E is given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which then implies $\underline{\sigma}(U_2' A V_2) = 0.3536$. From (10), we then obtain the exact bound $\mu(A, E) = 0.3536$.

III The minimun phase robustness problem

Assume a system with equal number of inputs and outputs is described by the following state space model

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du, \end{aligned} \qquad (31)$$

where $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^m$ and A, B, C, D are matrices of appropriate dimensions. For simplicity system (31) is called system (A, B, C, D) in the following. Denote $F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $G = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$; then the transmission zeros [4] of system (A, B, C, D) are defined to be the generalized eigenvalues of the matrix pair (F, G). If (F, G) is stable, the system (A, B, C, D) is said to be minimum phase; otherwise it is said to be non-minimum phase. The requirement that a system be minimum phase occurs often in a large number of problems, e.g. in the "perfect control problem"[3]. A question which immediately arises in this case is as follows: assume a system is minimum phase; then what is the largest class of plant perturbations which have the property that the perturbed system remains minimum phase? This is called the minimum phase robustness problem.

Assume now that (A, B, C, D) is minimum phase and matrices A, B, C, D are subject to real unstructured parameter perturbations ΔA, ΔB, ΔC, ΔD. This leads to a perturbation $\Delta F = \begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix}$ on matrix F. If we know nothing about ΔF except its norm, then the system $(A + \Delta A, B + \Delta B, C + \Delta C, D + \Delta D)$ remains minimum phase if $\|\Delta F\|_s < \mu(F, G)$. On the other hand, there exists ΔF with $\|\Delta F\|_s > \mu(F, G)$, but arbitrarily close to $\mu(F, G)$, such that the system $(A+\Delta A, B+\Delta B, C+\Delta C, D+\Delta D)$ is non-minimum phase. Therefore $\mu(F, G)$ gives a measure on the robustness of the minimum phase property against unstructured perturbations on system (A, B, C, D).

Example 3

The following minimum phase system is a balanced realization of an outer function obtained in an H_∞ design [5].

$$A = \begin{bmatrix} -0.2310 & -0.2834 & -0.2234 \\ -0.2834 & -0.4936 & -0.8628 \\ 0.2234 & 0.8628 & -0.3754 \end{bmatrix} \qquad B = \begin{bmatrix} 0.4193 \\ 0.3333 \\ -0.1798 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.4193 & 0.3333 & 0.1798 \end{bmatrix} \qquad D = 0.1.$$

From Theorem 4, the lower bounds of $\mu(F,G)$ are obtained as

$$\begin{aligned}
\mu(F,G) &\geq \min\{\underline{\sigma}(F), \frac{1}{2}\sigma_{14}(F\otimes G + G\otimes F), \underline{\sigma}(D)\} \\
&= \min\{0.2078, \frac{1}{2}\times 0.2271, 0.1\} = 0.1, \\
\mu(F,G) &\geq \min\{\frac{1}{2}\sigma_9(F\overline{\otimes} G + G\overline{\otimes} F), \underline{\sigma}(D)\} \\
&= \min\{\frac{1}{2}\times 0.2271, 0.1\} = 0.1, \\
\mu(F,G) &\geq \min\{\underline{\sigma}(F), \frac{1}{2}\sigma_6(F\overline{\overline{\otimes}} G + G\overline{\overline{\otimes}} F)\} \\
&= \min\{0.2078, \frac{1}{2}\times 0.0868\} = 0.0434.
\end{aligned}$$

From Theorem 1(d)-(e), two upper bound of $\mu(F,G)$ are obtained as

$$\mu(F,G) \leq \underline{\sigma}(F) = 0.2078,$$

$$\mu(F,G) \leq \underline{\sigma}(D) = 0.1.$$

On summarizing these inequalities, we then obtain that $\mu(F,G) = 0.1$. $\mu(F,G)$ is exactly obtained in this case.

6 Conclusion

This paper extends some recent results on the stability robustness of ordinary state space models to the stability robustness of the generalized eigenvalue problem. The generalized eigenvalue problem is much more complicated than the ordinary eigenvalue problem. This makes the extension a nontrivial task. The main achievement of this paper is in obtaining some sufficient conditions on the norm of the perturbation ΔA to ensure that $(A + \Delta A, B)$ is stable for a given stable matrix pair (A, B) . The results obtained can be applied to a number of important problems in control theory which involve the generalized eigenvalue problem. Application of the results obtained are given for the stability robustness of singularly perturbed systems, descriptor systems and for a new problem called the minimum phase robustness problem.

The stability robustness bounds obtained in this paper are given in terms of the singular values of some composite matrices which have much larger size than the original matrices. This brings a major concern in the computation of these bounds when large matrix pairs are considered. An alternative method is given in [13] to determine the required singular values without constructing the composite matrices explicitly.

References

[1] S.L. Campbell, *Singular Systems of Differential Equations,* Pitman Publishing Ltd., London, 1980.

[2] D. Cobb, "Descriptor variable systems and optimal state regulation", *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 601-611, 1983.

[3] E.J. Davison, B. Scherzinger, "Perfect control of the robust servomechanism problem", *IEEE Trans. Automat. Contr.,* vol. AC-32, pp. 689-702, 1987.

[4] E.J. Davison, S.H. Wong, "Properties and calculation of transmission zeros of linear multivariable systems", *Automatica*, vol. 10, pp. 643-658, 1974.

[5] B.A. Francis, *A Course in H_∞ Control Theory*, Springer-Verlag Berlin, Heidelberg, 1987.

[6] F.R. Gantmacher, *Matrix Theory,* Vol. II, Chelsea, New York, 1960.

[7] A. Graham, *Kronecker Products and Matrix Calculus with Applications,* Ellis Horwood Ltd., England, 1981.

[8] R.A. Horn, C.A. Johnson, *Matrix Analysis,* Cambridge University Press, Cambridge, 1985.

[9] T. Kailath, *Linear Systems,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1980.

[10] P. Kokotovic, H.K. Khalil, J. O'Reilly, *Singular Perturbation Method in Control: Analysis and Design,* Academic Press, Orlando, 1986.

[11] R.V. Patel, M. Toda, "Quantitative measures of robustness for multivariable systems", *Proc. Joint Automatic Control Conference,* Paper TD8-A, 1980.

[12] L. Qiu, E.J. Davison, "A new method for the stability robustness determination of state space models wiyh real perturbations", *Proc. IEEE Conference on Decision and Control,* pp. 538-543, 1988.

[13] L. Qiu, E.J. Davison, "The stability robustness of generalized eigenvalues", Systems Control Group Report, No. 8911, Dept. of Electrical Eng., Univ. of Toronto, 1989.

[14] G.W. Stewart, "On the sensitivity of the eigenvalue problem $Ax = \lambda Bx$", *SIAM J. Numer. Anal.,* vol. 9, no. 4, pp. 669-686, 1972.

[15] G.W. Stewart, "Gershgorin theory for the generalized eigenvalue problem $Ax = \lambda Bx$", *Math. of Comp.,* vol. 29, no. 130, pp. 600-606, 1975.