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The Stability Robustness of Generalized Eigenvalues

L. Qiu and E. J. Davison

***Abstract*—This note generalizes the concept of stability radius to matrix pairs. A matrix pair is said to be stable if its generalized eigenvalues are located in the open left half of the complex plane. The stability radius of a matrix pair (A, B) is defined to be the norm of the smallest perturbation ΔA such that $(A + \Delta A, B)$ is unstable. Our purpose is to estimate the stability radius of a given matrix pair. Depending on whether the matrices under consideration are complex or real, the problem can be classified into two cases. The complex case is easy and a complete solution is provided. The real case is more difficult and only a partial solution is given.**

I. Introduction

In the stability robustness analysis of state-space models, one of the important problems studied is to estimate the distance of a stable matrix to the set of all unstable matrices. Here a matrix is said to be stable if all of its eigenvalues are contained in the open left half of the complex plane. Since this problem was first considered in [14], it has been intensively studied; see [10], [11], [6], [13], [2], [16], [19]. Different methods have been used and various results have been obtained. It appears, however, that no effort has been made to extend the available results to the generalized eigenvalue problem.

Manuscript received November 7, 1989; revised November 27, 1990. Paper recommended by Past Associate Editor, M. Shayman. This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grant A4396.

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IEEE Log Number 9107963.

Denote the real and complex fields by $\mathbb{R}$ and $\mathbb{C}$, respectively, and let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. For a pair of matrices (A, B), where $A, B \in \mathbb{F}^{n \times n}$, the *generalized eigenvalues* of (A, B) are the roots of the following polynomial in λ:

$$\det(A - \lambda B).$$

The matrix pair (A, B) is said to be *stable* if all of its generalized eigenvalues are located in the open left half of the complex plane. A pathological case occurs when $\det(A - \lambda B)$ vanishes identically. If this is the case, (A, B) is said to be *singular* and every point in the complex plane is a generalized eigenvalue of (A, B). This case will not be excluded from our discussion. Instead, it is treated as a special case of unstable matrix pairs.

Let $A, B \in \mathbb{F}^{n \times n}$ be matrices. By carrying out a singular value decomposition of B, there exist unitary matrices $U, V \in \mathbb{F}^{n \times n}$ such that

$$A = URV^* = U \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} V^* \tag{1}$$
$$B = USV^* = U \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix} V^*$$

where S_{11} is a diagonal matrix with positive diagonal elements.

No assumption on matrix B has been made so far, but emphasis will be given to the case when B is singular. Assume that B is singular and the matrix pair (A, B) is stable. Let

$$\Delta B = U \begin{bmatrix} 0 & 0 \\ 0 & \Delta S_{22} \end{bmatrix} V^*.$$

Then

$$\det[A - \lambda(B + \Delta B)] = (-1)^n \det(UV^*) \det(S_{11}) \cdot \det(\Delta S_{22}) \lambda^n + \cdots + \det(A).$$

It is clear that a matrix ΔS_{22} with arbitrary small norm can be chosen such that the coefficients of the first term and the last term of the polynomial $\det[A - \lambda(B + \Delta B)]$ have opposite signs. If this is done, the polynomial has roots in the right half of the complex plane. This means that *the stability of (A, B) has zero tolerance to the unstructured uncertainty on matrix B*. Consequently, we assume in the following that B is always fixed and is not subject to uncertainty. It is noted that in many applications, the matrix B is a "structure" matrix rather than a "parameter" matrix, i.e., the elements of B contain only structural information regarding the problem considered, and hence are not subject to variation. On the other hand, in the case when the elements of B do contain uncertain parameters and when such uncertainty does not alter the rank of matrix B, then the perturbed matrix $B + \Delta B$ must admit a singular value decomposition

$$B + \Delta B = (U + \Delta U) \begin{bmatrix} S_{11} + \Delta S_{11} & 0 \\ 0 & 0 \end{bmatrix} (V + \Delta V)^*$$

where U, V, S_{11} are given as in (1). If ΔB is small, the matrices ΔU, ΔV, and ΔS_{11} can be chosen to be small. Since in this case the generalized eigenvalues of $(A + \Delta A, B + \Delta B)$ are the same as the generalized eigenvalues of

$$\left((U + \Delta U)^*(A + \Delta A)(V + \Delta V) \cdot \begin{bmatrix} (S_{11} + \Delta S_{11})^{-1} & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)$$

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the problem with both uncertain A and B can be transformed to a problem with uncertain A only. Therefore, the assumption that B is fixed imposes little limitation in applications.

The problem considered in this note is to estimate the smallest possible spectral norm of $\Delta A \in \mathbb{F}^{n\times n}$ such that $(A + \Delta A, B)$ is unstable for given $A, B \in \mathbb{F}^{n\times n}$. The problem can be classified into two cases depending on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. The complex case is relatively simple and admits a complete solution. The method used is a generalization of the one used in [11], [6], [13], [2]. The real case is more difficult and only partial solutions are given. The method used is a generalization of the one given in [16], which is based on the properties of tensor products (or, in other words, composite matrices). Possible applications of the results obtained include the stability robustness analysis of singularly perturbed systems [8], descriptor systems [3], and the robustness of the minimum phase property of a system [4], [9]. Note that for the generalized eigenvalue problems involved in singularly perturbed systems and in the minimum phase robustness problem, the matrix B is always fixed.

The following notation will be used throughout this note. The real part of $\lambda \in \mathbb{C}$ is written as $\mathscr{R}(\lambda)$ and the imaginary part $\mathscr{I}(\lambda)$. Denote the sets $\{\lambda \in \mathbb{C}: \mathscr{R}(\lambda) < 0\}$, $\{\lambda \in \mathbb{C}: \mathscr{R}(\lambda) = 0\}$, and $\{\lambda \in \mathbb{C}: \mathscr{R}(\lambda) > 0\}$ by $\mathbb{C}^-$, $\mathbb{C}^0$, and $\mathbb{C}^+$, respectively. Let $A \in \mathbb{F}^{m\times n}$. The rank and the nullity of A are denoted by rank(A) and null(A), respectively; a well-known relation between them is $\operatorname{rank}(A) + \operatorname{null}(A) = n$. $\sigma_i(A)$, $i = 1, 2, \cdots, \min(m, n)$, denotes the ith singular value of A with order $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_{\min(m,n)}(A)$; in particular, $\sigma_1(A)$ and $\sigma_{\min(m,n)}(A)$ are denoted by $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively. The norm of A, denoted by $\|A\|$, is assumed to be the spectral norm, i.e., $\|A\| = \bar{\sigma}(A)$. For square matrices $A, B \in \mathbb{F}^{n\times n}$, the set of all generalized eigenvalues of (A, B) is denoted by $\Lambda(A, B)$ and the number of elements in $\Lambda(A, B)$ (including multiplicities) is denoted by $|\Lambda(A, B)|$. If (A, B) is nonsingular, we have $|\Lambda(A, B)| \leq \operatorname{rank}(B)$. If (A, B) is singular, we write $\Lambda(A, B) = \mathbb{C}$ and $|\Lambda(A, B)| = \infty$.

II. Development

For $A, B \in \mathbb{F}^{n\times n}$ with (A, B) stable, we define the *stability radius* of (A, B) by

$$r_{\mathbb{F}}(A, B) = \inf\{\|\Delta A\|: \Delta A \in \mathbb{F}^{n\times n} \text{ and } \Lambda(A + \Delta A, B) \not\subset \mathbb{C}^-\}. \quad (2)$$

A trivial case occurs when (A, B) is stable but $|\Lambda(A, B)| < \operatorname{rank}(B) = l$. Using the same notation as in (1), we have

$$\det(A - \lambda B) = (-1)^l \det(UV^*) \det(S_{11}) \det(R_{22}) \lambda^l + \cdots + \det(A).$$

It is apparent that $|\Lambda(A, B)| < l$ if and only if $\det(R_{22}) = 0$. In the case when $|\Lambda(A, B)| < l$, an arbitrarily small matrix ΔA of the form

$$\Delta A = U \begin{bmatrix} 0 & 0 \\ 0 & \Delta R_{22} \end{bmatrix} V^*$$

can be found such that $\det(R_{22} + \Delta R_{22})$ is nonzero and the signs of $\det(A + \Delta A)$ and $(-1)^l \det(UV^*) \det(S_{11}) \det(R_{22} + \Delta R_{22})$ are opposite. In this case the polynomial

$$\det(A + \Delta A - \lambda B) = (-1)^l \det(UV^*) \det(S_{11}) \det(R_{22} + \Delta R_{22}) \lambda^l + \cdots + \det(A + \Delta A)$$

has l roots and at least one of them is in $\mathbb{C}^+$. This shows that if $|\Lambda(A, B)| < \operatorname{rank}(B)$ then $r_{\mathbb{F}}(A, B) = 0$. As a consequence, we always assume in the following development that $|\Lambda(A, B)| = \operatorname{rank}(B)$.

Let (A, B) be a stable matrix pair with $|\Lambda(A, B)| = \operatorname{rank}(B)$ and assume that $(A + \Delta A, B)$ is unstable for some ΔA. Intuitively, the behavior of the generalized eigenvalues of $(A + \alpha\Delta A, B)$ when α sweeps continuously from 0 to 1 will, in the ideal case, have three possibilities: 1) a generalized eigenvalue of $(A + \alpha\Delta A, B)$ shifts from $\mathbb{C}^-$ to $\mathbb{C}^+ \cup \mathbb{C}^0$ across the origin; 2) a generalized eigenvalue of $(A + \alpha\Delta A, B)$ shifts from $\mathbb{C}^-$ to $\mathbb{C}^+ \cup \mathbb{C}^0$ across the imaginary axis; 3) one of the generalized eigenvalues of $(A + \alpha\Delta A, B)$ disappears at infinity and then appears in $\mathbb{C}^+ \cup \mathbb{C}^0$. However, the actual situation may be more complicated since it is possible that for some $\alpha \in (0, 1]$, $(A + \alpha\Delta A, B)$ becomes singular. Nevertheless, let us define the following three quantities:

$$r_{\mathbb{F}_0}(A, B) = \inf\{\|\Delta A\|: \Delta A \in \mathbb{F}^{n\times n} \text{ and } \det(A + \Delta A) = 0\} \quad (3)$$

$$r_{\mathbb{F}_\omega}(A, B) = \inf\{\|\Delta A\|: \Delta A \in \mathbb{F}^{n\times n}, |\Lambda(A + \Delta A, B)| = \operatorname{rank}(B) \text{ and } \Lambda(A + \Delta A, B) \cap (\mathbb{C}^0 \setminus \{0\}) \neq \varnothing\} \quad (4)$$

$$r_{\mathbb{F}_\infty}(A, B) = \inf\{\|\Delta A\|: \Delta A \in \mathbb{F}^{n\times n} \text{ and } |\Lambda(A + \Delta A, B)| < \operatorname{rank}(B)\}. \quad (5)$$

The following proposition simplifies the analysis of $r_{\mathbb{F}}(A, B)$.

Proposition 1: Let $A, B \in \mathbb{F}^{n\times n}$ be matrices with (A, B) stable and $|\Lambda(A, B)| = \operatorname{rank}(B)$. Then

$$r_{\mathbb{F}}(A, B) = \min\{r_{\mathbb{F}_0}(A, B), r_{\mathbb{F}_\omega}(A, B), r_{\mathbb{F}_\infty}(A, B)\}. \quad (6)$$

Proof: It is clear that $r_{\mathbb{F}}(A, B) \leq \min\{r_{\mathbb{F}_0}(A, B), r_{\mathbb{F}_\omega}(A, B), r_{\mathbb{F}_\infty}(A, B)\}$. Now assume that for some $\Delta A \in \mathbb{F}^{n\times n}$, $(A + \Delta A, B)$ is unstable. Fix this ΔA, and consider the matrix pair $(A + \alpha\Delta A, B)$ for $\alpha \in (0, 1]$ in the following three possible cases.

1) $(A + \alpha\Delta A, B)$ is singular for some $\alpha \in (0, 1]$.

For such an α, $A + \alpha\Delta A$ is singular. This implies that $\|\Delta A\| \geq \|\alpha\Delta A\| \geq r_{\mathbb{F}_0}(A, B)$.

2) $(A + \alpha\Delta A, B)$ is nonsingular for all $\alpha \in (0, 1]$, but $|\Lambda(A + \alpha\Delta A, B)| < \operatorname{rank}(B)$ for some $\alpha \in (0, 1]$.

Choose an $\alpha \in (0, 1]$ such that $|\Lambda(A + \alpha\Delta A, B)| < \operatorname{rank}(B)$; then by the definition of $r_{\mathbb{F}_\infty}(A, B)$, we have $\|\alpha\Delta A\| \geq r_{\mathbb{F}_\infty}(A, B)$. This implies $\|\Delta A\| \geq r_{\mathbb{F}_\infty}(A, B)$.

3) $(A + \alpha\Delta A, B)$ is nonsingular and $|\Lambda(A + \alpha\Delta A, B)| = |\Lambda(A, B)|$ for all $\alpha \in (0, 1]$.

In this case, $\det(A + \alpha\Delta A - \lambda B)$ has the same degree as $\det(A - \lambda B)$ for all $\alpha \in (0, 1]$, and its coefficients are continuous functions of α. Since all the roots of $\det(A - \lambda B)$ are contained in $\mathbb{C}^-$ and at least one of the roots of $\det(A + \Delta A - \lambda B)$ is in $\mathbb{C}^+ \cup \mathbb{C}^0$, there must be an $\alpha \in (0, 1]$ such that at least one of the roots of $\det(A + \alpha\Delta A - \lambda B)$ is in $\mathbb{C}^0$. If this root is at the origin, then $\|\Delta A\| \geq \|\alpha\Delta A\| \geq r_{\mathbb{F}_0}(A, B)$; otherwise $\|\Delta A\| \geq \|\alpha\Delta A\| \geq r_{\mathbb{F}_\omega}(A, B)$.

For each of the three cases, we have $\|\Delta A\| \geq \min\{r_{\mathbb{F}_0}(A, B), r_{\mathbb{F}_\omega}(A, B), r_{\mathbb{F}_\infty}(A, B)\}$. Therefore, $r_{\mathbb{F}}(A, B) \geq \min\{r_{\mathbb{F}_0}(A, B), r_{\mathbb{F}_\omega}(A, B), r_{\mathbb{F}_\infty}(A, B)\}$.

This completes the proof. □

$r_{\mathbb{F}_0}(A, B)$ can be easily obtained as

$$r_{\mathbb{F}_0}(A; B) = \underline{\sigma}(A). \quad (7)$$

Introduce a decomposition of matrices A, B of the form of (1) and let

$$\Delta A = U\begin{bmatrix} \Delta R_{11} & \Delta R_{12} \\ \Delta R_{21} & \Delta R_{22} \end{bmatrix} V^*.$$

Then $|\Lambda(A + \Delta A, B)| < \operatorname{rank}(B)$ if and only if $\det(R_{22} + \Delta R_{22}) = 0$. If ΔA satisfies $|\Lambda(A + \Delta A, B)| < \operatorname{rank}(B)$, then $\|\Delta A\| \geq \|\Delta R_{22}\| \geq \underline{\sigma}(R_{22})$. This shows that $r_{\mathbb{F}_\infty}(A, B) \geq \underline{\sigma}(R_{22})$. On the other hand, let

$$\Delta A = U\begin{bmatrix} 0 & 0 \\ 0 & \Delta R_{22} \end{bmatrix} V^*$$

and ΔR_{22} be a matrix with $\|\Delta R_{22}\| = \underline{\sigma}(R_{22})$ and $\det(R_{22} + \Delta R_{22}) = 0$. Then $\|\Delta A\| = \underline{\sigma}(R_{22})$ and $|\Lambda(A + \Delta A, B)| < \operatorname{rank}(B)$. This shows that $r_{\mathbb{F}_\infty}(A, B) \leq \underline{\sigma}(R_{22})$. As a result, we conclude that

$$r_{\mathbb{F}_\infty}(A, B) = \underline{\sigma}(R_{22}). \tag{8}$$

However, $r_{\mathbb{F}_\omega}(A, B)$ cannot be so easily obtained for general matrices.

Before ending this section, we give some quick facts about $r_{\mathbb{F}}(A, B)$.

Proposition 2: Let A, $B \in \mathbb{F}^{n \times n}$ be matrices with (A, B) stable and $|\Lambda(A, B)| = \operatorname{rank}(B)$. Then

a) $r_{\mathbb{F}}(A, B) > 0$,

b) $r_{\mathbb{F}}(\alpha A, \beta B) = \alpha r_{\mathbb{F}}(A, B)$ for all $\alpha > 0$, $\beta > 0$,

c) $r_{\mathbb{F}}(W_1 A W_2, W_1 B W_2) = r_{\mathbb{F}}(A, B)$ for all unitary matrices $W_1, W_2 \in \mathbb{F}^{n \times n}$,

d) $r_{\mathbb{F}}(A, B) \leq \underline{\sigma}(A)$,

e) $r_{\mathbb{F}}(A, B) \leq \underline{\sigma}(R_{22})$, where R_{22} is given by (1).

III. Complex Stability Radius

Theorem 1: Let A, $B \in \mathbb{C}^{n \times n}$ be matrices with (A, B) stable and $|\Lambda(A, B)| = \operatorname{rank}(B)$. Then

$$r_{\mathbb{C}}(A, B) = \inf_{\omega \in \mathbb{R}} \underline{\sigma}(A - j\omega B). \tag{9}$$

Proof: For each $\omega \in \mathbb{R} \setminus \{0\}$, there exists $\Delta A \in \mathbb{C}^{n \times n}$ such that $\|\Delta A\| = \underline{\sigma}(A - j\omega B)$ and $A + \Delta A - j\omega B$ is singular, i.e., $(A + \Delta A, B)$ has a generalized eigenvalue at $j\omega$. Therefore, $r_{\mathbb{C}_\omega}(A, B) \leq \underline{\sigma}(A - j\omega B)$ for each $\omega \in \mathbb{R} \setminus \{0\}$ or, equivalently, $r_{\mathbb{C}_\omega}(A, B) \leq \inf_{\omega \in \mathbb{R} \setminus \{0\}} \underline{\sigma}(A - j\omega B)$.

Now assume that for some $\Delta A \in \mathbb{C}^{n \times n}$, there exists $\bar{\omega} \in \mathbb{R} \setminus \{0\}$ such that $\det(A + \Delta A - j\bar{\omega} B) = 0$. Then $\|\Delta A\| \geq \underline{\sigma}(A - j\bar{\omega}B)$. This shows $r_{\mathbb{C}_\omega}(A, B) \geq \inf_{\omega \in \mathbb{R} \setminus \{0\}}(A - j\omega B)$. Since $\omega \to \underline{\sigma}(A - j\omega B)$ is a continuous function, it follows that:

$$r_{\mathbb{C}_\omega}(A, B) = \inf_{\omega \in \mathbb{R}} (A - j\omega B).$$

Assume that A, B admit a decomposition as in (1). Then by Proposition 1,

$$r_{\mathbb{C}}(A, B) = \min\left\{\underline{\sigma}(A), \inf_{\omega \in \mathbb{R}} (A - j\omega B), \underline{\sigma}(R_{22})\right\}.$$

It is trivial to see

$$\inf_{\omega \in \mathbb{R}} (A - j\omega B) \leq \underline{\sigma}(A).$$

The proof will be completed if we can show

$$\inf_{\omega \in \mathbb{R}} (A - j\omega B) \leq \underline{\sigma}(R_{22}).$$

We do this by showing

$$\lim_{\omega \to \infty} \underline{\sigma}(A - j\omega B) = \lim_{\omega \to -\infty} \underline{\sigma}(A - j\omega B) = \underline{\sigma}(R_{22}).$$

This can be done in many ways; we proceed by using the concept of the transfer matrix of a generalized system. Assume that (A, B) admits a decomposition in the form of (1). Then

$$\underline{\sigma}(A - j\omega B) = \underline{\sigma}(R - j\omega S) = \bar{\sigma}^{-1}\left[(R - j\omega S)^{-1}\right] = \bar{\sigma}^{-1}\left(\begin{bmatrix} R_{11} - j\omega S_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}^{-1}\right).$$

The matrix

$$\begin{bmatrix} R_{11} - j\omega S_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}^{-1}$$

is the frequency response of the following generalized system:

$$\begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Its equivalent ordinary system has a state-space realization

$$\dot{x}_1 = S_{11}^{-1}(R_{11} - R_{12}R_{22}^{-1}R_{21})x_1 + \begin{bmatrix} -I & R_{12}R_{22}^{-1} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I \\ -R_{22}^{-1}R_{21} \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ 0 & R_{22}^{-1} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The two systems must have the same frequency response, therefore,

$$\begin{bmatrix} R_{11} - j\omega S_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}^{-1} = \begin{bmatrix} I \\ -R_{22}^{-1}R_{21} \end{bmatrix}\left[j\omega I - S_{11}^{-1}(R_{11} - R_{12}R_{22}^{-1}R_{21})\right]^{-1} \cdot \begin{bmatrix} -I & R_{12}R_{22}^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & R_{22}^{-1} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

It now becomes obvious that

$$\lim_{\omega \to \infty} \underline{\sigma}(A - j\omega B) = \lim_{\omega \to -\infty} \underline{\sigma}(A - j\omega B) = \bar{\sigma}^{-1}(R_{22}^{-1}) = \underline{\sigma}(R_{22}).$$

□

A straightforward method to compute $r_{\mathbb{C}}(A, B)$ is to use a "brute force" search to find the infimum on the right-hand side of (9). Potential difficulties exist, however, due to the facts that $\omega \to \underline{\sigma}(A - j\omega B)$ is nonconvex and that the infimum is taken over an infinite interval. On noticing that

$$\inf_{\omega \in \mathbb{R}} \underline{\sigma}(A - j\omega B) = \left[\sup_{\omega \in \mathbb{R}} \|(A - j\omega B)^{-1}\|\right]^{-1}$$

and $(A - sB)^{-1}$ is a proper stable real rational matrix, the method given in [1] for the computation of the $\mathcal{H}_\infty$ norm of a stable real rational matrix can be used to compute the left-hand side of (9).

IV. Real Stability Radius of Matrix Pairs

A lower bound of the real stability radius of a matrix pair is given by its complex stability radius.

Corollary 1: Let A, $B \in \mathbb{R}^{n \times n}$ be matrices with (A, B) stable and $|\Lambda(A, B)| = \operatorname{rank}(B)$. Then

$$r_{\mathbb{R}}(A, B) \geq \inf_{\omega \in \mathbb{R}} \underline{\sigma}(A - j\omega B). \tag{10}$$

The purpose of this section is to develop lower bounds of the real stability radius which, at least in some cases, improve the lower bound given in Corollary 1.

By Proposition 1

$$r_{\mathbb{R}}(A,B) = \min\{r_{\mathbb{R}_0}(A,B), r_{\mathbb{R}_\omega}(A,B), r_{\mathbb{R}_\infty}(A,B)\}$$

where $r_{\mathbb{R}_0}(A,B)$ and $r_{\mathbb{R}_\infty}(A,B)$ are given by (7) and (8). The difficulty of analysis lies in determining $r_{\mathbb{R}_\omega}(A,B)$.

For a real matrix pair (A,B) with $\text{rank}(B)=1$, it is impossible to have a real ΔA such that $(A+\Delta A, B)$ is nonsingular and has imaginary eigenvalues. So in this case $r_{\mathbb{R}_\omega}(A,B)=\infty$ and

$$r_{\mathbb{R}}(A,B) = \min\{r_{\mathbb{R}_0}(A,B), r_{\mathbb{R}_\infty}(A,B)\} = \min\{\underline{\sigma}(A), \underline{\sigma}(R_{22})\}. \quad (11)$$

It is of interest to note that if $A, B\in\mathbb{R}^{n\times n}$ and $\text{rank}(B)=1$, then

$$\inf_{\omega\in\mathbb{R}} \underline{\sigma}(A-j\omega B) = \min\{\underline{\sigma}(A), \underline{\sigma}(R_{22})\}.$$

This means that the inequality (10) becomes an equality when $\text{rank}(B)=1$.

In this note we are unable to obtain a formula which gives the exact value of $r_{\mathbb{R}}(A,B)$ for general real (A,B) pairs. Instead, some lower bounds on $r_{\mathbb{R}}(A,B)$ are obtained by using tensor products. In the following, we use $\otimes$, $\vee$, and $\wedge$ to denote the tensor product, the symmetric tensor product, and the skew-symmetric tensor product, respectively. A good reference in the basic concepts of tensor product is [12]. Readers who are not familiar with tensor products can follow the development in [16] which is based on pure matrix arguments. (In [16], the symmetric tensor product and the skew-symmetric tensor product are denoted by $\overline{\otimes}$ and $\overline{\overline{\otimes}}$ instead of $\vee$ and $\wedge$.)

In the remaining part of this section, we use m to denote $\text{null}(B)$. The following three lemmas are stated without proof. Their proofs can be found in [15].

Lemma 1: Let $A, B\in\mathbb{R}^{n\times n}$ be matrices with $|\Lambda(A,B)| = \text{rank}(B)$ and $0\notin\{\lambda_i+\lambda_j: \lambda_i, \lambda_j\in\Lambda(A,B)\}$. Then

a) $\text{null}(A\otimes B + B\otimes A) = m^2$,

b) $\text{null}(A\vee B + B\vee A) = \frac{1}{2}m(m+1)$,

c) $\text{null}(A\wedge B + B\wedge A) = \frac{1}{2}m(m-1)$.

Lemma 2: Let $A, B\in\mathbb{R}^{n\times n}$ be matrices with (A,B) nonsingular and $\Lambda(A,B)\cap(\mathbb{C}^0\setminus\{0\})\neq\varnothing$. Then

a) $\text{null}(A\otimes B + B\otimes A)\geq m^2+2$,

b) $\text{null}(A\vee B + B\vee A)\geq \frac{1}{2}m(m+1)+1$,

c) $\text{null}(A\wedge B + B\wedge A)\geq \frac{1}{2}m(m-1)+1$.

Lemma 3: For each $A, B\in\mathbb{C}^{n\times n}$,

$$\frac{1}{2\|B\|}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)}(A\vee B + B\vee A)\leq \underline{\sigma}(A) \quad (12)$$

$$\frac{1}{2\|B\|}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}m(m-1)}(A\wedge B + B\wedge A)\leq \underline{\sigma}(R_{22}) \quad (13)$$

where R_{22} is given by (1).

The main result of this section which gives the lower bounds of $r_{\mathbb{R}}(A,B)$ is as follows.

Theorem 2: Let $A, B\in\mathbb{R}^{n\times n}$ be matrices with (A,B) stable and $|\Lambda(A,B)| = \text{rank}(B)$. Then

$$r_{\mathbb{R}}(A,B)\geq\min\left\{\underline{\sigma}(A), \frac{1}{2\|B\|}\sigma_{n^2-m^2-1}(A\otimes B + B\otimes A), \underline{\sigma}(R_{22})\right\} \quad (14)$$

$$r_{\mathbb{R}}(A,B)\geq\min\left\{\frac{1}{2\|B\|}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)}(A\vee B + B\vee A), \underline{\sigma}(R_{22})\right\} \quad (15)$$

$$r_{\mathbb{R}}(A,B)\geq\min\left\{\underline{\sigma}(A), \frac{1}{2\|B\|}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}m(m-1)}(A\wedge B + B\wedge A)\right\} \quad (16)$$

where R_{22} is given by (1).

Proof: If we can prove the following inequalities:

$$r_{\mathbb{R}_\omega}(A,B)\geq\frac{1}{2\|B\|}\sigma_{n^2-m^2-1}(A\otimes B + B\otimes A) \quad (17)$$

$$r_{\mathbb{R}_\omega}(A,B)\geq\frac{1}{2\|B\|}\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)}(A\vee B + B\vee A) \quad (18)$$

$$r_{\mathbb{R}_\omega}(A,B)\geq\frac{1}{2\|B\|}\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}m(m-1)}(A\wedge B + B\wedge A) \quad (19)$$

then inequalities (14)–(16) follow immediately from Proposition 1, Lemma 3, and identities (7) and (8).

Assume $\|\Delta A\| < 1/2\|B\|\sigma_{n^2-m^2-1}(A\otimes B + B\otimes A)$. Then

$$\|\Delta A\otimes B + B\otimes\Delta A\|\leq 2\|\Delta A\|\,\|B\| < \sigma_{n^2-m^2-1}(A\otimes B + B\otimes A).$$

This implies that the nullity of

$$(A+\Delta A)\otimes B + B\otimes(A+\Delta A) = A\otimes B + B\otimes A + \Delta A\otimes B + B\otimes\Delta A$$

is less than m^2+2. By Lemma 2, $\Lambda(A+\Delta A, B)$ has no element in $\mathbb{C}^0\setminus\{0\}$. Therefore, if $\Lambda(A+\Delta A,B)\cap\{\mathbb{C}^0\setminus\{0\}\}\neq\varnothing$, $\|\Delta A\|$ must be greater than or equal to $1/(2\|B\|)\sigma_{n^2-m^2-1}(A\otimes B + B\otimes A)$. This proves (17). Similar arguments can be used to prove (18) and (19). □

Compared to lower bound (10), the lower bounds (14)–(16) may produce better or worse results. This will be shown by examples in the next section.

It is of interest to give a comparison between the three lower bounds given in Theorem 2, i.e., it is desired to determine which bound is the best and which is the worst. We know that the tensor product space can be decomposed as the direct sum of the mutually orthogonal symmetric tensor space and skew-symmetric tensor space, and under this decomposition, we have the following representation:

$$A\otimes B + B\otimes A \simeq \begin{bmatrix} A\vee B + B\vee A & 0 \\ 0 & A\wedge B + B\wedge A\end{bmatrix}.$$

It is known that if (A,B) is stable and $|\Lambda(A,B)| = \text{rank}(B)$, then $\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)}(A\vee B + B\vee A)$ and $\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}m(m-1)}(A\wedge B + B\wedge A)$ are the smallest nonzero singular values of $(A\vee B + B\vee A)$ and $(A\wedge B + B\wedge A)$, respectively, and that $\sigma_{n^2-m^2-1}(A\otimes B + B\otimes A)$ is the second smallest nonzero singular value of $A\otimes B + B\otimes A$. Thus, $\sigma_{n^2-m^2-1}(A\otimes B + B\otimes A)$ must lie between $\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)}(A\vee B + B\vee A)$ and $\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}m(m-1)}(A\wedge B + B\wedge A)$. Examples show that both $\sigma_{\frac{1}{2}n(n+1)-\frac{1}{2}m(m+1)}(A\vee B + B\vee A)$ and $\sigma_{\frac{1}{2}n(n-1)-\frac{1}{2}m(m-1)}(A\wedge B + B\wedge A)$ can in fact be equal to the smallest nonzero singular value of $A\otimes B + B\otimes A$. The conclusion is that among the three bounds (14)–(16), both bound (15) or bound (16) can be the best or the worst, and bound (14) always lies between bounds (15) and (16).

When A, B are in $\mathbb{R}^{n\times n}$, the matrix representations of $A\otimes B + B\otimes A$, $A\vee B + B\vee A$, and $A\wedge B + B\wedge A$ are of $n^2\times n^2$, $\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)$, and $\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)$, respectively. If n is small, the singular values of these matrix representations can be computed easily. If n is large, the iteration method developed in [17] for the singular values of $A \otimes I + I \otimes A$, $A \vee I + I \vee A$, and $A \wedge I + I \wedge A$ can be easily extended to compute the required singular values of $A \otimes B + B \otimes A$, $A \vee B + B \vee A$, and $A \wedge B + B \wedge A$.

V. Examples

The first example shows that the lower bound of the real stability radius obtained by using the tensor product can have an arbitrary degree of improvement over the lower bound obtained from the complex stability radius.

Example 1: Let

$$A = \begin{bmatrix} -1 & k & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{where } k \geq 1.$$

Corollary 1 gives

$$r_{\mathbb{R}}(A, B) \geq \inf_{\omega \in \mathbb{R}} \underline{\sigma}\left(\begin{bmatrix} -1 - j\omega & k & 0 \\ -1 & -1 - j\omega & 0 \\ 0 & 0 & 5 \end{bmatrix}\right) = \inf_{\omega \in \mathbb{R}} \underline{\sigma}\left(\begin{bmatrix} -1 - j\omega & k \\ -1 & -1 - j\omega \end{bmatrix}\right)$$

where

$$\inf_{\omega \in \mathbb{R}} \underline{\sigma}\left(\begin{bmatrix} -1 - j\omega & k \\ -1 & -1 - j\omega \end{bmatrix}\right) \leq \underline{\sigma}\left(\begin{bmatrix} -1 - j\omega & k \\ -1 & -1 - j\omega \end{bmatrix}\right)\Bigg|_{\omega = \sqrt{k-1}} = \left[\frac{(1+k)^2 - \sqrt{(1+k)^4 - 16k}}{2}\right]^{\frac{1}{2}}$$

which goes to zero as $k \to \infty$.

It is easy to verify that

$$\underline{\sigma}(A) = \underline{\sigma}\left(\begin{bmatrix} -1 & k \\ -1 & -1 \end{bmatrix}\right) = \left[\frac{3 + k^2 - \sqrt{(k^2+1)^2 - 8k + 4}}{2}\right]^{\frac{1}{2}} > 1$$

that R_{22}, given by the decomposition (1), is equal to 5, and that

$$\frac{1}{2}\sigma_7(A \otimes B + B \otimes A) = \frac{1}{2}\sigma_5(A \wedge B + B \wedge A) = 1$$

for all $k \geq 1$. Therefore, inequalities (14) and (16) in Theorem 2 give that

$$r_{\mathbb{R}}(A, B) \geq 1.$$

This shows that the lower bounds (14) and (16) can have an arbitrary degree of improvement over the lower bound (10).

The second example gives an application of the stability robustness analysis of matrix pairs, and also shows that the lower bound (10) can be better than the lower bounds (14)–(16).

Example 2 (Application to Singularly Perturbed Systems): A homogeneous singularly perturbed system is described by a state-space equation in the form [8]

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{20}$$

where $x \in \mathbb{R}^{n_1}$, $z \in \mathbb{R}^{n_2}$, and A_{ij}, $i, j = 1, 2$ are matrices of compatible sizes. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

It is known that if A_{22} is stable and (A, E) is stable, then there exists $\epsilon^* > 0$ such that for all $\epsilon \in [0, \epsilon^*]$, system (20) is asymptotically stable. Now suppose that matrix A is subject to an unstructured real parameter perturbation ΔA. Then the perturbed system is given by

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = (A + \Delta A)\begin{bmatrix} x \\ z \end{bmatrix}.$$

Hence, we can see that the perturbed system has the previous stability property for all ΔA with $\|\Delta A\| < \delta$ if and only if

$$\delta \leq \min\{r_{\mathbb{R}}(A_{22}, I), r_{\mathbb{R}}(A, E)\}.$$

Here we include a numerical example to illustrate the bounds obtained in this note. The following singularly perturbed system represents a voltage regulator controlled by a so-called corrected near-optimal state feedback law [8]:

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} -0.2 & 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1.6 & 0 & 0 \\ 0 & 0 & -1.429 & 8.571 & 0 \\ 0 & 0 & 0 & -2.5 & 7.5 \\ -2.754 & -0.57 & -0.033 & -0.114 & -1.0861 \end{bmatrix} \cdot \begin{bmatrix} x \\ z \end{bmatrix}$$

where $x \in \mathbb{R}^2$, $z \in \mathbb{R}^3$. Corollary 1 yields

$$r_{\mathbb{R}}(A_{22}, I) \geq \inf_{\omega \in \mathbb{R}} \underline{\sigma}(A_{22} - j\omega I) = 0.1094$$

and it follows from Theorem 2 that

$$r_{\mathbb{R}}(A_{22}, I) \geq \min\left\{\underline{\sigma}(A_{22}), \frac{1}{2}\sigma_8(A_{22} \otimes I + I \otimes A_{22})\right\} = \min\{0.1094, 0.2282\} = 0.1094$$

$$r_{\mathbb{R}}(A_{22}, I) \geq \frac{1}{2}\underline{\sigma}(A_{22} \vee I + I \vee A_{22}) = 0.012$$

$$r_{\mathbb{R}}(A_{22}, I) \geq \min\left\{\underline{\sigma}(A_{22}), \frac{1}{2}\underline{\sigma}(A_{22} \wedge I + I \wedge A_{22})\right\} = \min\{0.1094, 0.2282\} = 0.1094.$$

In this case, lower bounds (10), (14), and (16) produce the same result, namely that $r_{\mathbb{R}}(A_{22}, I) \geq 0.1094$. Since $r_{\mathbb{R}}(A_{22}, I) \leq \underline{\sigma}(A_{22}) = 0.1094$ by Proposition 2d), we actually have that $r_{\mathbb{R}}(A_{22}, I) = 0.1094$.

From Corollary 1, we obtain

$$r_{\mathbb{R}}(A, E) \geq \inf_{\omega \in \mathbb{R}} (A - j\omega E) = 0.1094$$

whereas from the bounds given in Theorem 2, we obtain

$$r_{\mathbb{R}}(A, E) \geq \min\left\{\underline{\sigma}(A), \frac{1}{2}\sigma_{15}(A \otimes E + E \otimes A), \underline{\sigma}(A_{22})\right\} = \min\{0.5047, 0.0919, 0.1094\} = 0.0919$$

$$r_{\mathbb{R}}(A, E) \geq \min\left\{\frac{1}{2}\sigma_9(A \vee E + E \vee A), \underline{\sigma}(A_{22})\right\} = \min\{0.0919, 0.1094\} = 0.0919$$

$$r_{\mathbb{R}}(A, E) \geq \min\left\{\underline{\sigma}(A), \frac{1}{2}\sigma_7(A \wedge E + E \wedge A)\right\} = \min\{0.5047, 0.0120\} = 0.0120.$$

In this case, the result obtained from lower bound (10) is the best. In fact, since we also have $r_{\mathbb{R}}(A, E) \leq \underline{\sigma}(A_{22}) = 0.1094$ by Proposition 2e), the bound is exact, i.e., $r_{\mathbb{R}}(A, E) = 0.1094$.

We conclude that the uncertain singularly perturbed system maintains the desired stability property for all real unstructured perturbations ΔA with $\|\Delta A\| < \delta$ if and only if $\delta \leq 0.1094$.

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A Simple Method for Deriving *J*-Spectral Factors

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***Abstract*—A simple method is presented for deriving the *J*-spectral factor of a transfer function matrix explicitly in terms of the parameters of this matrix. This method provides closed-form expressions for the *J*-spectral factor and its inverse which only require a solution of a single linear Sylvester equation. This method can be easily applied in the solution of the H_∞-optimal regulation problem and it provides a useful geometric insight to the structure of the optimal regulator return difference matrix.**

Manuscript received March 10, 1989; revised October 19, 1990. This work was supported by Cella and Marcos Maus Chair of Computer Systems Engineering, Tel-Aviv University, Israel.

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IEEE Log Number 9107966.

I. Introduction

J-spectral factorization [1] has been recognized lately [2]–[4] as an important tool for solving and investigating H_∞-optimization problems. Similar to the application of the standard spectral factorization in the L_2-optimization problem, the *J*-spectral factor is a transfer function matrix of a causal system whose inverse is analytic in the right half plane. It readily yields the return difference matrix of the H_∞-optimal regulator and it can therefore be used in the derivation of the optimal regulator gain without solving the modified Riccati equation of [5].

Unlike the standard spectral factorization where the spectral factor can be found for almost all para-Hermitian transfer function matrices [6], the *J*-spectral factor does not always exist. Its existence depends on the special structure of the transfer function matrix to be factorized; in fact, it exists whenever the Hankel norm of a related transfer function matrix is less than one [1]. In the H_∞-optimization problems the latter condition depends on a positive scalar γ that is related to the H_∞-norm of the transfer function to be minimized. For large enough values of $\gamma > 0$ there always exists a solution to the H_∞-optimization problem. As we reduce the value of γ we arrive at a critical point γ_o under which there exists no solution to the problem.

The problem of *J*-spectral factorization is closely related to the BGK Wiener–Hopf factorization theory of [7]. The latter theory can be applied (see the recent paper [8]) to obtain the *J*-spectral factor in terms of the modified Riccati equation of [5] (a closely related result appears also in [3]). *J*-spectral factors can be also computed using the observability and controllability Gramians that can be computed by solving a couple of Lyapunov equations [1].

In the present note, an alternative method is presented to derive *J*-spectral factors. The partial fraction expansion of the transfer function matrix to be factored is used as a starting point, and then a solution of a single Sylvester equation is shown to provide the required *J*-spectral factor. Our derivation is direct and does not use the Riccati equation approach of [8]. We explore, however, the relation of the latter to our method. We also show how our method copes with the existence problem of the *J*-spectral factor and what happens at the critical point of γ_o.

II. The *J*-Spectral Factorization

We consider the following *J*-spectral factorization problem. Given

$$\Phi(s) = J + G(s)G'(-s) \tag{1}$$

where $G(s)$ is a strictly proper asymptotically stable $(m + l) \times q$ transfer function matrix and

$$J = \operatorname{diag}\{I_m, -I_l\}. \tag{2}$$

It is desired to find the (left) *J*-spectral factor $\Delta(s)$ that satisfies the following:

$$\Delta(s)J\Delta'(-s) = \Phi(s) \tag{3}$$

where $\Delta^{-1}(s)$ and $\Delta(s)$ are analytic in the RHP.

We denote the minimal state-space realization of $\Phi(s)$ by $S(A, B, C, J)$. Since J is nonsingular, we readily find that the state-space realization of $\Phi^{-1}(s)$ is $S(A - BJC, BJ, -JC, J)$. We assume the following.

Assumption 1: $\Phi^{-1}(s)$ has no eigenvalues on the imaginary axis.