Time Domain Characterizations of Performance Limitations of Feedback Control

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Dedicated to B. A. Francis and M. Vidyasagar.

1 Introduction

The last 20 years have seen remarkable progress in the evolution of optimization-based control theory and design techniques. The new theories, apart from their theoretic elegance, have proven effective in various applications. However, the solutions of control optimization problems, in most cases in terms of numerical algorithms, do not provide a clear picture on the relationship between the optimal performance of the controlled system and the characteristics of the plant to be controlled, nor do they provide a clear idea on the effect on optimal performance attainable, due to changes in plant parameters, allocation of actuators and sensors, and choice of control structures.

On the other hand, control practice has long furnished heuristic as well as empirical understanding of the difficulty in feedback control due to plant characteristics, given in terms of rules-of-thumb largely applicable to scalar systems. For example, it is known that nonminimum phase systems are difficult to control, and that unstable poles close to nonminimum phase zeros pose additional difficulty. There has been effort to quantify such rules-of-thumb and to extend them to multivariable systems, and the subject itself has matured into a fruitful research area. Good results, mostly in the frequency domain, have been obtained to quantify, and to explain various design limitations and tradeoffs in multivariable feedback control. See [7, 15] and the references therein for the state-of-the-art.

In this paper, we survey some recent results which discuss fundamental limitations in achieving time-domain performance objectives. In particular, we consider the limitations in achieving small mean-square errors which are

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to occur in tracking and in regulation. The materials are mainly based on [3, 16] but we put them in a more unified framework. Our purpose is to show a fundamental relationship between certain performance measures defined in the time domain, and such simple plant characteristics as poles and zeros. The results will complement those quantified in the frequency domain, obtained elsewhere previously. We present these results for both continuous-time and discrete-time systems, with an emphasis on the continuous-time case. The discrete-time case is included since, apart from its own interest, it is essential for the study of sampled-data systems.

Finally a note on the notation: A signal in the time domain is denoted by a lower case letter, such as r, and a system, viewed as an input/output operator, is denoted by a capital letter, such as F. The time domain to frequency domain transform (Laplace transform in the continuous time case and λ-transform in the discrete time case) is denoted by a hat " ^ ", i.e., $\hat{r}$ is the Laplace or λ-transform of r. If F is LTI, $\hat{F}$ represents the transfer function of F. For any two nonzero vectors u and v with the same dimension, an angular measure is provided by

$$\cos \angle(u,\ v) = \frac{|u^* v|}{\|u\|_2 \|v\|_2}.$$

2 Preliminaries

Let $\hat{F}$ be the real rational matrix transfer function of a continuous time FDLTI system F. Assume that $\hat{F}$ is right invertible. The poles and zeros of $\hat{F}$, including multiplicity, are defined according to its Smith-McMillan form. $\hat{F}$ is said to be minimum phase if all its zeros have a nonpositive real part; otherwise, it is said to be nonminimum phase. Moreover, $\hat{F}$ is said to be semistable if all its poles have a nonpositive real part, and otherwise strictly unstable. A pole is said to be antistable if it has a positive real part.

Suppose that $\hat{F}$ is stable and z is a nonminimum phase zero of $\hat{F}$. Then, there exists a unitary vector y such that

$$y^* \hat{F}(z) = 0.$$

We call y a (left or output) zero vector corresponding to the zero z. Let the nonminimum phase zeros of $\hat{F}$ be ordered as as $z_1, z_2, \ldots, z_\nu$. Let also η_1 be a zero vector corresponding to z_1. Define

$$\hat{F}_1(s) = V_1 \begin{bmatrix} \dfrac{j\omega_0 + z_1^*}{j\omega_0 - z_1} \dfrac{s - z_1}{s + z_1^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_1^*,$$

where $\omega_0 \in [0, \infty]$ and V_1 is a unitary matrix with the first column equal to η_1. Note that $\hat{F}_1$ is so constructed that it is inner, has only one zero at z_1 with η_1 as a zero vector, and additionally, $\hat{F}_1(j\omega_0) = I$. Note also that the choice of other columns in V_1 is immaterial. Now $\hat{F}_1^{-1}\hat{F}$ has zeros $z_2, z_3, \ldots, z_\nu$. Find a zero vector η_2 corresponding to the zero z_2 of $\hat{F}_1^{-1}\hat{F}$, and define

$$\hat{F}_2(s) = V_2 \begin{bmatrix} \frac{j\omega_0 + z_2^*}{j\omega_0 - z_2}\frac{s - z_2}{s + z_2^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_2^*,$$

where similarly, V_2 is a unitary matrix with the first column equal to η_2. It follows that $\hat{F}_2^{-1}\hat{F}_1^{-1}\hat{F}$ has zeros $z_3, z_4, \ldots, z_\nu$. Continue this process until $\eta_1, \ldots, \eta_\nu$ and $\hat{F}_1, \ldots, \hat{F}_\nu$ are obtained. Then we have one vector corresponding to each nonminimum zero, and the procedure yields a factorization of $\hat{F}$ in the form of

$$\hat{F} = \hat{F}_1 \cdots \hat{F}_\nu \hat{F}_0,$$

where $\hat{F}_0$ has no nonminimum phase zeros and

$$\hat{F}_i(s) = V_i \begin{bmatrix} \frac{j\omega_0 + z_i^*}{j\omega_0 - z_i}\frac{s - z_i}{s + z_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_i^*.$$

Since $\hat{F}_i$ is inner, has the only zero at z_i, and has η_i as a zero vector corresponding to z_i, it will be called a matrix Blaschke factor. Accordingly, the product

$$\hat{F}_z = \hat{F}_1 \cdots \hat{F}_\nu$$

will be called a matrix Blaschke product. The vectors $\eta_1, \ldots, \eta_\nu$ will be called Blaschke vectors of $\hat{F}$ at frequency ω_0. Keep in mind that these vectors depend on the order of the nonminimum zeros, and on ω_0. It can be shown that for a real rational $\hat{F}$ the Blaschke vectors corresponding to two complex conjugate zeros can be chosen as a complex conjugate pair provided that the two conjugate zeros are ordered consecutively.

For an unstable $\hat{F}$, there exist stable real rational matrix functions

$$\begin{bmatrix} \hat{\tilde{X}} & -\hat{\tilde{Y}} \\ -\hat{\tilde{N}} & \hat{\tilde{M}} \end{bmatrix}, \quad \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix}$$

such that

$$\hat{F} = \hat{N}\hat{M}^{-1} = \hat{\tilde{M}}^{-1}\hat{\tilde{N}}$$

and

$$\begin{bmatrix} \hat{\tilde{X}} & -\hat{\tilde{Y}} \\ -\hat{\tilde{N}} & \hat{\tilde{M}} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = I.$$

This is called a doubly coprime factorization of $\hat{F}$. Note that the nonminimum phase zeros of $\hat{F}$ are the nonminimum phase zeros of $\hat{N}$ and the antistable poles of $\hat{F}$ are the nonminimum phase zeros of $\hat{M}$. If we order the nonminimum phase zeros of F as $z_1, z_2, \ldots, z_\nu$ and the antistable poles of $\hat{F}$ as $p_1, p_2, \ldots, p_\mu$, then $\hat{N}$ and $\hat{M}$ can be factorized as

$$\begin{aligned} \hat{N} &= \hat{N}_1 \cdots \hat{N}_\nu \hat{N}_0, \\ \hat{M} &= \hat{M}_1 \cdots \hat{M}_\mu \hat{M}_0, \end{aligned}$$

with

$$\begin{aligned} \hat{N}_i(s) &= V_i \begin{bmatrix} \dfrac{j\omega_z + z_i^*}{j\omega_z - z_i} \dfrac{s - z_i}{s + z_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_i^*, \\ \hat{M}_i(s) &= U_i \begin{bmatrix} \dfrac{j\omega_p + p_i^*}{j\omega_p - p_i} \dfrac{s - p_i}{s + p_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} U_i^*. \end{aligned}$$

Here $\hat{N}_0$ and $\hat{M}_0$ have no nonminimum phase zeros, and ω_z need not be equal to ω_p; i.e., $\hat{N}$ and $\hat{M}$ may be factorized at different frequencies. Consequently, for any real rational matrix $\hat{F}$ with nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ and strictly unstable poles $p_1, p_2, \ldots, p_\mu$, it can always be factorized to

$$\hat{F} = \hat{F}_z \hat{F}_0 \hat{F}_p^{-1},$$

where

$$\begin{aligned} \hat{F}_z(s) &= \prod_{i=1}^{\nu} V_i \begin{bmatrix} \dfrac{j\omega_z + z_i^*}{j\omega_z - z_i} \dfrac{s - z_i}{s + z_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_i^*, \\ \hat{F}_p(s) &= \prod_{i=1}^{\mu} U_i \begin{bmatrix} \dfrac{j\omega_p + p_i^*}{j\omega_p - p_i} \dfrac{s - p_i}{s + p_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} U_i^*, \end{aligned}$$

and F_0 is a real rational matrix with neither nonminimum phase zero nor strictly unstable pole.

The relationship between zero vectors and zero Blaschke vectors is somewhat analogous to that between eigenvectors and Schur vectors of a square matrix. The eigenvectors are not in general completely defined in the sense that one may not find an eigenvector corresponding to each eigenvalue (with multiplicity counted) with desired property, say linear independence. However, a complete set of orthonormal Schur vectors exist as long as an order of eigenvalues is specified. Likewise, it is difficult to define a complete set of zero vectors corresponding to each nonminimum phase zero, and it is not clear what the desired property should be. Nevertheless, each Blaschke vector bears a natural correspondence to each nonminimum phase zero. The nice properties of the Blaschke vectors will become evident shortly.

The above factorization can be extended to transfer function matrices of discrete-time systems with much similarity and some differences. Consider a real rational transfer function matrix $\hat{F}$ of a discrete time FDLTI system under λ-transform ($\lambda = 1/z$). Let us assume that $\hat{F}$ is right invertible, and its poles and zeros with multiplicity included are defined according to its Smith-MacMillan form. Then $\hat{F}$ is said to be minimum phase if all its zeros have an absolute value no less than one, and otherwise nonminimum phase. It is said to be semistable if all its poles have an absolute value no less than one; otherwise, it is said to be strictly unstable. A pole is said to be antistable if it has an absolute value less than one.

At this point, we would like to emphasize that if z-transform is used for the transfer function instead, then the zeros at infinity should be considered nonminimum phase zeros. For example, transfer functions z^{-1} and $\frac{1}{z+0.5}$ represent nonminimum phase systems. This viewpoint is also more consistent with the definition of an outer function, i.e., a stable transfer function is outer iff it is minimum phase. Ambiguity often arises in this situation since in the continuous time case zeros at infinity are not considered nonminimum phase zeros. The reason is that in the continuous time case infinity is on the boundary of the stability region, whereas in the discrete time case when the z-transform is used, infinity is an interior point of the instability region and therefore should be considered the same as any other point in the same region. On the other hand, if λ-transform is used, the zeros at infinity are mapped to the origin, and so no confusion is likely to arise.

Based upon transfer functions under λ-transform, and using an analogous procedure, we may factorize $\hat{F}$ with nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ and antistable poles $p_1, p_2, \ldots, p_\mu$ as

$$\hat{F} = \hat{F}_z \hat{F}_0 \hat{F}_p^{-1},$$

where

$$\hat{F}_z(\lambda) = \prod_{i=1}^{\nu} V_i \begin{bmatrix} \frac{1-z_i^* e^{j\omega_z}}{e^{j\omega_z}-z_i}\frac{\lambda-z_i}{1-z_i^*\lambda} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_i^*,$$

$$\hat{F}_p(\lambda) = \prod_{i=1}^{\mu} U_i \begin{bmatrix} \frac{1-p_i^* e^{j\omega_p}}{e^{j\omega_p}-p_i}\frac{\lambda-p_i}{1-p_i^*\lambda} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} U_i^*,$$

and F_0 is a real rational matrix with no nonminimum phase zero or anti-stable pole. It thus becomes clear that any FDLTI system F, whether it is a continuous-time or discrete-time system, can be factorized into the cascade interconnection shown in Figure 1. In this factorization, F_0 is a minimum

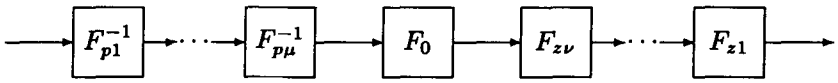

Figure 1: Cascade factorization

phase and semistable system, $\hat{F}_{zi}$ and $\hat{F}_{pi}$ are matrix Blaschke factors with certain special properties.

3 Frequency domain characterizations

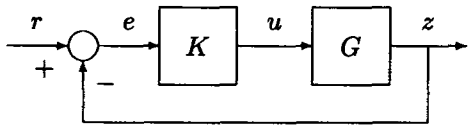

Figure 2: Unity feedback

Consider the unity feedback system shown in Figure 2. Assume that K and G are SISO LTI systems with real rational transfer functions $\hat{K}$ and $\hat{G}$ respectively. The loop gain is defined as $\hat{L} = \hat{G}\hat{K}$. The sensitivity and complementary sensitivity functions are defines as

$$\hat{S} = (1+\hat{L})^{-1} \quad \text{and} \quad \hat{T} = \hat{L}(1+\hat{L})^{-1}$$

respectively. Assume that $\hat{L}$ has antistable poles $p_1, p_2, \ldots, p_\mu$, and nonminimum phase zeros z_1, $z_2, \ldots, z_\nu$. Then, $\hat{L}$ has the factorization

$$\hat{L} = \hat{L}_z \hat{L}_0 \hat{L}_p^{-1},$$

with $\hat{L}_0$ being a real rational function of poles and zeros only in the closed left half plane, and $\hat{L}_z$ and $\hat{L}_p$ being the Blaschke products associated with the nonminimum phase zeros and the antistable poles, respectively:

$$\hat{L}_z(s) = \prod_{i=1}^{\nu} \frac{z_i - s}{z_i^* + s} \quad \text{and} \quad \hat{L}_p(s) = \prod_{i=1}^{\mu} \frac{p_i - s}{p_i^* + s}.$$

Suppose that $\hat{L}$ is proper ($\hat{L}(\infty)$ is finite), and that the feedback system is internally stable. Then we have

- Bode S-integral

$$\int_0^\infty \log \left| \frac{\hat{S}(\mathrm{j}\omega)}{\hat{S}(\infty)} \right| \mathrm{d}\omega = \frac{\pi}{2} \lim_{s \to \infty} \frac{s[\hat{S}(s) - \hat{S}(\infty)]}{\hat{S}(\infty)} + \pi \sum_{i=1}^{\mu} p_i, \tag{1}$$

- Bode T-integral

$$\int_0^\infty \log \left| \frac{\hat{T}(\mathrm{j}\omega)}{\hat{T}(0)} \right| \frac{\mathrm{d}\omega}{\omega^2} = \frac{\pi}{2} \lim_{s \to 0} \frac{\hat{T}(s) - \hat{T}(0)}{s\hat{T}(0)} + \pi \sum_{i=1}^{\nu} \frac{1}{z_i}, \tag{2}$$

- Poisson S-integrals

$$\int_{-\infty}^{\infty} \log |\hat{S}(\mathrm{j}\omega)| \frac{\operatorname{Re} z_i}{|\mathrm{j}\omega - z_i|^2} \mathrm{d}\omega = \pi \log |\hat{L}_p^{-1}(z_i)|, \quad i = 1, 2, \ldots, \nu, \tag{3}$$

- Poisson T-integrals

$$\int_{-\infty}^{\infty} \log |\hat{T}(\mathrm{j}\omega)| \frac{\operatorname{Re} p_i}{|\mathrm{j}\omega - p_i|^2} \mathrm{d}\omega = \pi \log |\hat{L}_z^{-1}(p_i)|, \quad i = 1, 2, \ldots, \mu. \tag{4}$$

In the discrete time case, $\hat{L}$ can be factorized similarly as

$$\hat{L} = \hat{L}_z \hat{L}_0 \hat{L}_p^{-1},$$

with $\hat{L}_0$ being a real rational function with no poles and zeros inside the unit circle, and $\hat{L}_z$ and $\hat{L}_p$ being the Blaschke products associated with the strictly nonminimum phase zeros and the strictly unstable poles, respectively:

$$\hat{L}_z(\lambda) = \prod_{i=1}^{\nu} \frac{\lambda - z_i}{z_i^* \lambda - 1} \quad \text{and} \quad \hat{L}_p(\lambda) = \prod_{i=1}^{\mu} \frac{\lambda - p_i}{p_i^* \lambda - 1}.$$

Under the condition that $\hat{L}$ is proper ($\hat{L}(0)$ is finite), and that the feedback system is internally stable, we have

- Bode S-integral

$$\int_0^{\pi} \log \left|\hat{S}(e^{j\omega})\right| d\omega = \pi \lim_{\lambda \to 0} \log |\hat{S}(\lambda)| + \pi \sum_{i=1}^{\mu} \log \frac{1}{|p_i|}, \tag{5}$$

- Bode T-integral

$$\int_0^{\pi} \log \left|\frac{\hat{T}(e^{j\omega})}{T(1)}\right| \frac{d\omega}{1-\cos\omega} = \frac{\pi}{T(1)} \lim_{\lambda \to 1} \frac{\hat{T}(\lambda) - \hat{T}(1)}{\lambda - 1} + \pi \sum_{i=1}^{\nu} \frac{1+z_i}{1-z_i}, \tag{6}$$

- Poisson S-integrals

$$\int_{-\pi}^{\pi} \log |\hat{S}(e^{j\omega})| \frac{|z_i|^2 - 1}{|e^{j\omega} - p_i|^2} d\omega = 2\pi \log |\hat{L}_p^{-1}(z_i)|, \quad i = 1, 2, \ldots, \nu, \tag{7}$$

- Poisson T-integrals

$$\int_{-\pi}^{\pi} \log |\hat{T}(e^{j\omega})| \frac{|p_i|^2 - 1}{|e^{j\omega} - p_i|^2} d\omega = 2\pi \log |\hat{L}_z^{-1}(p_i)|, \quad i = 1, 2, \ldots, \mu. \tag{8}$$

The performance limitations characterized by the above integral relations (except (5) and (6)) exhibit an interesting symmetry between sensitivity function and complementary sensitivity function, poles and zeros, etc.; see [9] for more details. These frequency domain characterizations have the following features, which may be undesirable in certain applications. The performance limitations characterized by the above integral relations (except (5) and (6)) exhibit an interesting symmetry between sensitivity function and complementary sensitivity function, poles and zeros, etc.; see [9] for more details. These frequency domain characterizations have the following features, which may be undesirable in certain applications.

- Sometimes it may not be desirable to characterize performance of a feedback system in terms of logarithmic integrals of $\hat{S}$ and/or $\hat{T}$, or pointwise in frequency; such is the case, for example, when the minimal $\mathcal{H}_\infty$ norm is sought after. Under this circumstance, the integral formulas give only indirect quantifications of the performance limitations and their interpretations must be carefully and delicately done. Nevertheless, one should note that the logarithmic integrals can be weakened to yield bounds on the performance.

- Since $\hat{L}$ contains both the plant $\hat{G}$ and the controller $\hat{K}$, the limitations expressed by the logarithmic integrals depend on both the plant and controller. While this may be advantageous in some cases, often one also desires to know the *a priori*, intrinsic performance achievable by designing the best controller possible. The latter, therefore, should depend on the plant only. Again, it should be pointed out that the integrals can also be weakened to lead to inequality versions depending upon solely on the plant.

- It is not clear from the integrals, even conceptually, what should be the time-varying and nonlinear generalizations of the limitations. (See [14] for an attempt.)

- The limitations are insensitive to controller used. For example, if the plant G is given with certain zero and pole pattern, then the integrals have the same values no matter what stabilizing controller K is used, as long as it does not introduce additional nonminimum phase zeros or antistable poles. Therefore, the Bode and Poisson integrals above may be more appropriately called performance invariances.

4 Minimum error tracking

We first consider a minimum error tracking problem. Let an FDLTI plant P be given with $\hat{P} = \begin{bmatrix} \hat{G} \\ \hat{H} \end{bmatrix}$, where G has output z and H has output y. Assume that we wish to design a feedback controller K in the structure shown in Figure 3 so that the closed loop system is internally stable (in any reasonable sense) and the output of the control system z tracks a vector step signal r with $r(t) = v$ when $t > 0$.

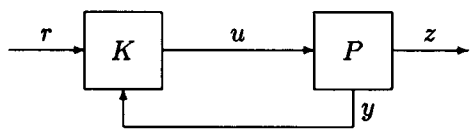

Figure 3: A general two-parameter control structure

In order for the problem to be solvable, we assume that $\hat{P}, \hat{G}, \hat{H}$ have the same unstable poles, and that $\hat{G}(0)$ has full row rank.

Let the tracking performance be measured by the energy of the tracking error

$$J(v) = \int_0^\infty \|r(t) - z(t)\|_2^2 dt.$$

This performance depends on v. A performance measure free of v can be obtained by averaging $J(v)$ over a reasonable set of v:

$$J_a = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv') = I\}.$$

Here $\boldsymbol{E}$ is the expectation operator. The best tracking performance achievable by designing K is then given, in the two cases, respectively by

$$J^*(v) = \inf_K J(v)$$

and

$$J_a^* = \inf_K J_a,$$

where K is chosen among all internally stabilizing (possibly nonlinear time-varying) controllers.

Let a doubly coprime factorization of $\hat{H}$ be

$$\hat{H} = \hat{N}\hat{M}^{-1} = \hat{\tilde{M}}^{-1}\hat{\tilde{N}}, \quad \begin{bmatrix} \hat{\tilde{X}} & -\hat{\tilde{Y}} \\ -\hat{\tilde{N}} & \hat{\tilde{M}} \end{bmatrix} \begin{bmatrix} \hat{M} & \hat{Y} \\ \hat{N} & \hat{X} \end{bmatrix} = I. \tag{9}$$

Then by the standard stabilization theory [19], the set of all stabilizing controllers is given by

$$K = \begin{bmatrix} Y + MQ & MR \end{bmatrix} \begin{bmatrix} 0 & I \\ X + NQ & NR \end{bmatrix}^{-1}, \tag{10}$$

where Q, R are arbitrary causal stable (possibly nonlinear time-varying) controllers. With this class of controllers applied to the system, the map from r to the error $e = r - z$ is given by $I - GMQ$. Let $\hat{G}$ have nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding Blaschke vectors at frequency 0. Since $\hat{P}, \hat{G}, \hat{H}$ have the same unstable poles, GM is stable and it has the factorization

$$\hat{G}\hat{M} = \hat{G}_1\hat{G}_2 \cdots \hat{G}_\nu \hat{G}_0,$$

where

$$\hat{G}_i(s) = V_i \begin{bmatrix} -\frac{z_i^*}{z_i}\frac{s - z_i}{s + z_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} V_i^* = I - \frac{2\,\mathrm{Re}\, z_i}{z_i}\frac{s}{s + z_i^*}\eta_i\eta_i^*,$$

and $\hat{G}_0$ is outer in $\mathcal{H}_\infty$. Using the Parseval's identity, we obtain

$$\begin{aligned} J(v) &= \|\hat{r} - \hat{G}_1\hat{G}_2 \cdots \hat{G}_\nu \hat{G}_0 \widehat{Qr}\|_2^2 \\ &= \|\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1}\hat{r} - \hat{G}_0\widehat{Qr}\|_2^2 \\ &= \|(\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1} - I)\hat{r} + (\hat{r} - \hat{G}_0\widehat{Qr})\|_2^2. \end{aligned}$$

Here the second equality follows from the fact that G_i are unitary operators in $\mathcal{L}_2$. When r is a step signal, $(\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1} - I)\hat{r} \in \mathcal{H}_2^\perp$. Hence $J(v)$ is finite only if $\hat{r} - \hat{G}_0\widehat{Qr} \in \mathcal{L}_2$. Since Q is causal, we must have $\hat{r} - \hat{G}_0\widehat{Qr} \in \mathcal{H}_2$. Therefore,

$$J(v) = \|(\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1} - I)\hat{r}\|_2^2 + \|\hat{r} - \hat{G}_0\widehat{Qr}\|_2^2 \geq \|(\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1} - I)\hat{r}\|_2^2.$$

On the other hand, since G_0 has a dense image in $\mathcal{H}_2$, we can find an LTI system Q such that $\hat{G}(0)\hat{Q}(0) = I$ and $\|[I - \hat{G}_0(s)\hat{Q}(s)]\frac{1}{s}\|_2$ is arbitrarily small. Therefore,

$$J^*(v) = \|(\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1} - I)\hat{r}\|_2^2.$$

Straightforward computation then shows that

$$\begin{aligned} J^*(v) &= \|(\hat{G}_\nu^{-1} \cdots \hat{G}_2^{-1}\hat{G}_1^{-1} - I)\hat{r}\|_2^2 = v^* \left(2\sum_{i=1}^{\nu} \frac{1}{z_i}\eta_i\eta_i^*\right) v \\ &= 2\,\|v\|_2^2 \sum_{i=1}^{\nu} \frac{1}{z_i} \cos^2 \angle(v, \eta_i). \end{aligned}$$

Since a nearly optimal Q, i.e., a Q such that $\|[I - \hat{G}_0(s)\hat{Q}(s)]\frac{1}{s}\|_2$ is vanishingly small, can be chosen independently of v, we have

$$\begin{aligned} J_a^* &= \inf_K \{\boldsymbol{E}J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \\ &= \{\boldsymbol{E}J^*(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv^*) = I\} \\ &= \operatorname{tr}\left(2\sum_{i=1}^{\nu} \frac{1}{z_i}\eta_i\eta_i^*\right) \\ &= 2\sum_{i=1}^{\nu} \frac{1}{\lambda_i}. \end{aligned}$$

We have thus established the following theorem.

Theorem 1 *Let $\hat{G}$ have nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding Blaschke vectors at frequency* 0. *Then*

$$J^*(v) = 2\|v\|_2^2 \sum_{i=1}^{\nu} \frac{1}{z_i} \cos^2 \angle(v, \eta_i)$$

and

$$J_a^* = 2\sum_{i=1}^{\nu} \frac{1}{z_i}.$$

Remarks:

1. The limiting performance does not change if the controller is chosen from the set of LTI controllers or the set of nonlinear time-varying controllers,

2. The limiting performance does not depend on the poles of the plant,

3. The limiting performance does not depend on how the measurement is taken as long as the measurement does not introduce additional unstable poles and the stabilization can be accomplished by the measurement. This makes sense since the measurement does not provide any extra information on the behavior of the system when no uncertainty or disturbance is present.

The performance limitation exhibited in Theorem 1 is a fundamental one imposed by the plant. Since two-parameter control is the most general control structure, no other control scheme can do better. The use of other less general control structure can only introduce additional limitation. For example, robustness consideration motivates the use of error feedback in the tracking problem. If the one-parameter unity error feedback structure as in Figure 2 is used, it is shown in [3] that if the plant P is strictly unstable, then the best achievable performance will be worse than that given in Theorem 1. In this case, additional limitation on the tracking performance is introduced by the control structure. This provides further quantitative support to the observation made in [19] that one-parameter controller does not have enough freedom to accomplish both stabilization and tracking effectively. To take advantages of error feedback and two-parameter control, we may use the control structures shown in Figure 4 and Figure 5.

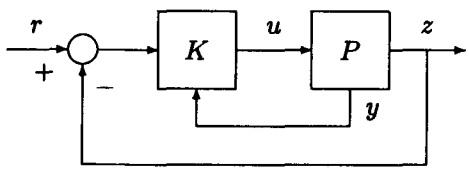

Figure 4: Separating stabilization and tracking error feedback

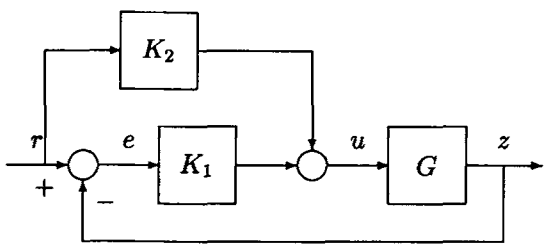

Figure 5: Feedback plus feedforward tracking

For discrete-time systems, analogous results can be obtained. Consider again the feedback controller structure in Figure 3 and a discrete-time FDLTI plant P, with $\hat{P} = \begin{bmatrix} \hat{G} \\ \hat{H} \end{bmatrix}$. Assume that we wish to design a feedback controller

K so that the closed loop system is internally stable (in any reasonable sense) and the output of the control system z tracks a vector step signal r with $r(k) = v$ when $k \geq 0$. Define the tracking error by

$$J(v) = \sum_{k=0}^{\infty} \|r(k) - z(k)\|_2^2$$

and the average tracking error by

$$J_a = \boldsymbol{E}\{J(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv') = I\}.$$

The best tracking performances achievable by designing K are then given by

$$J^*(v) = \inf_K J(v)$$

and

$$J_a^* = \inf_K J_a,$$

where K is chosen among all internally stabilizing (possibly time-varying, nonlinear) controllers. Similarly, we assume that $\hat{P}, \hat{G}, \hat{H}$ have the same unstable poles, and that $\hat{G}(1)$ has full row rank.

Theorem 2 *Let $\hat{G}$ has nonminimum phase zeros $z_1, z_2, \ldots, z_\nu$ with $\eta_1, \eta_2, \ldots, \eta_\nu$ being the corresponding zero Blaschke vectors at frequency* 0. *Then*

$$J^*(v) = \|v\|_2^2 \sum_{i=1}^{\nu} \frac{1 + z_i}{1 - z_i} \cos^2 \angle(v, \eta_i)$$

and

$$J_a^* = \sum_{i=1}^{\nu} \frac{1 + z_i}{1 - z_i}.$$

5 Minimum energy regulation

Next, we consider a minimum energy regulation problem. Let G be a given plant. Assume that we wish to design a feedback controller K in the structure shown in Figure 6 so that the closed loop system is stable. Assume that d is a vector impulse signal $d(t) = v\delta(t)$. The input energy is given by

$$E(v) = \int_0^{\infty} \|u(t)\|_2^2 dt.$$

A normalized average input energy independent of v can be obtained as

$$E_a = \boldsymbol{E}\{E(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv') = I\}.$$

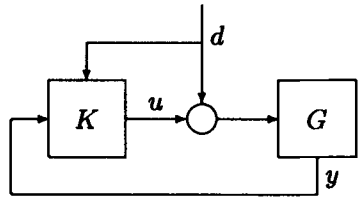

Figure 6: Feedback plus feedforward regulation

The minimum energy required in stabilizing the system is then determined as

$$E^*(v) = \inf_K E(v)$$

and

$$E_a^* = \inf_K E_a,$$

respectively, where K is chosen among all internally stabilizing (possibly nonlinear time-varying) controllers.

Let a doubly coprime factorization of $\hat{G}$ be given as that of $\hat{H}$ in (9), and the set of all stabilizing controllers K be as in (10). The map from d to the input u is then given by

$$M(Y - R\tilde{N}) + MQ - I.$$

Write

$$Q = Q_0 - (Y - R\tilde{N}).$$

Then, according to the Parseval's identity, and in light of the fact that the map from Q_0 to Q is bijective over the set of all causal stable systems, we have

$$E^*(v) = \inf_{Q_0 \text{ stable}} \|\hat{M}\widehat{Q_0 d} - \hat{d}\|_2^2$$

Let $\hat{G}$ have antistable poles $p_1, p_2, \ldots, p_\mu$ with $\zeta_1, \zeta_2, \ldots, \zeta_\mu$ being the corresponding Blaschke vectors at the frequency ∞. Then $\hat{M}$ has the factorization

$$\hat{M} = \hat{M}_1 \hat{M}_2 \cdots \hat{M}_\mu \hat{M}_0,$$

where

$$\hat{M}_i(s) = U_i \begin{bmatrix} \frac{s - z_i}{s + z_i^*} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} U_i^* = I - \frac{2\,\mathrm{Re}\, p_i}{s + p_i^*} \eta_i \eta_i^*,$$

and $\hat{M}_0$ is outer in $\mathcal{H}_\infty$. It follows that

$$\begin{aligned} \|\hat{M}\widehat{Q_0 d} - \hat{d}\|_2^2 &= \|\hat{M}_0 \widehat{Q_0 d} - \hat{M}_\mu^{-1} \cdots \hat{M}_2^{-1} \hat{M}_1^{-1} \hat{d}\| \\ &= \|(\hat{M}_0 \widehat{Q_0 d} - \hat{d}) + (I - \hat{M}_\mu^{-1} \cdots \hat{M}_2^{-1} \hat{M}_1^{-1}) \hat{d}\|_2^2 \end{aligned}$$

Since $(I-\hat{M}_\mu^{-1}\cdots\hat{M}_1^{-1})\hat{d}\in\mathcal{H}_2^\perp$, $\|\hat{M}\widehat{Q_0 d}-\hat{d}\|_2^2$ is finite only if $\hat{M}_0\widehat{Q_0 d}-\hat{d}\in\mathcal{L}_2$. Since Q_0 is causal, we mush have $\hat{M}_0\widehat{Q_0 d}-\hat{d}\in\mathcal{H}_2$. Therefore,

$$E(v)=\|\hat{M}_0\widehat{Q_0 d}-\hat{d}\|_2^2+\|(I-\hat{M}_\mu^{-1}\cdots\hat{M}_2^{-1}\hat{M}_1^{-1})\hat{d}\|_2^2\geq\|(I-\hat{M}_\mu^{-1}\cdots\hat{M}_1^{-1})\hat{d}\|_2^2.$$

On the other hand, since M_0 has a dense image in $\mathcal{H}_2$, we can find an LTI system Q_0 such that $\|\hat{M}_0\hat{Q}_0-I\|_2$ is arbitrarily small. Therefore,

$$E^*(v)=\|(I-\hat{M}_\mu^{-1}\cdots\hat{M}_2^{-1}\hat{M}_1^{-1})\hat{d}\|_2^2.$$

Straightforward computation shows that

$$E^*(v)=v^*\left(2\sum_{i=1}^{\mu}p_i\zeta_i\zeta_i^*\right)v=2\|v\|_2^2\sum_{i=1}^{\mu}p_i\cos^2\angle(v,\zeta_i).$$

Since a nearly optimal Q_0, i.e., a Q_0 such that $\|\hat{M}_0\hat{Q}_0-I\|_2$ is vanishingly small, can be chosen independently of v, we have

$$E_a^*=2\sum_{i=1}^{\mu}p_i.$$

This proves the following theorem.

Theorem 3 *Let $\hat{G}$ have antistable poles $p_1,p_2,\ldots,p_\mu$ with $\zeta_1,\zeta_2,\ldots,\zeta_\mu$ being the corresponding pole Blaschke vectors at the frequency ∞. Then,*

$$E^*(v)=2\|v\|_2^2\sum_{i=1}^{\mu}p_i\cos^2\angle(v,\zeta_i)$$

and

$$E_a^*=2\sum_{i=1}^{\mu}p_i.$$

Remarks:

1. The limiting input energy does not change if the controller is chosen from the set of LTI controllers or the set of nonlinear time-varying controllers,
2. The limiting input energy does not depend on the zeros of the plant,
3. The limiting input energy does not depend on how the measurement is taken as long as the stabilization can be accomplished by the measurement. This makes sense since the measurement does not provide any extra information on the behavior of the system when no uncertainty or disturbance is present.

In practical situations, the excitation (considered as a disturbance) signal may not be accessible. A natural question is then if the measurement feedback structure in Figure 7 would lead to the same performance limitation? The answer is yes if and only G is minimum phase. Consequently, in order to achieve good performance in regulation, measurement variables should be selected, whenever possible, in such a way that the input to measurement transfer function is minimum phase. This is the case when the measurement vector contains all states.

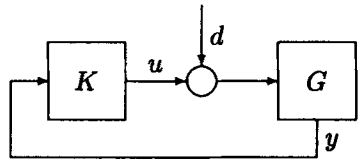

Figure 7: Measurement feedback regulation

Finally, for discrete time systems, the same problem can be studied but the result takes a different form. Indeed, assume that d is a vector impulse signal $d(k) = v\delta(k)$ and define the input energy measures similarly by

$$E(v) = \sum_{k=1}^{\infty} \|u(k)\|_2^2,$$

and

$$E_a = \boldsymbol{E}\{E(v) : \boldsymbol{E}(v) = 0, \boldsymbol{E}(vv') = I\}.$$

Furthermore, define the optimal versions of $E(v)$ and E_a as

$$E^*(v) = \inf_K E(v)$$

and

$$E_a^* = \inf_K E,$$

respectively. Here K is, likewise, chosen among all internally stabilizing (possibly time-varying, nonlinear) controllers. It turns out a clean formula for $E^*(v)$ as in Theorems 1-3 is not available in this context.

Theorem 4 *Let $\hat{G}$ have antistable poles $p_1, p_2, \ldots, p_\mu$. Then,*

$$E_a^* = 1 - \prod_{i=1}^{\mu} p_i^2.$$

6 Concluding remarks

It is observed (first in [16]) that in the continuous-time case the sum of the reciprocal of the nonminimum phase zeros, interestingly, shows up in both the Bode T-integral and the expression of the minimum tracking error, and that the sum of the antistable poles shows up in both the Bode S-integral and the expression of the minimum regulation energy. Notice that in the special case when G is stable and the unity feedback in Figure 2 is used, the minimum tracking error can also be defined, via the Parseval's identity, in the frequency domain as

$$J_a^* = \inf_{K \text{stabilizing}} \frac{1}{\pi} \int_0^\infty \|S(\mathrm{j}\omega)\|_F^2 \frac{\mathrm{d}\omega}{\omega^2}. \tag{11}$$

Furthermore, notice that in the special case when G is minimum phase and the feedback control structure in Figure 7 is used, the minimum regulation energy can be defined, via the Parseval's identity, in the frequency domain as

$$E_a^* = \inf_{K \text{stabilizing}} \frac{1}{\pi} \int_0^\infty \|T(\mathrm{j}\omega)\|_F^2 \mathrm{d}\omega. \tag{12}$$

This leads to the speculation that there may be a deep connection between the square integrals (11-12) and the Bode type logarithmic integrals. Investigation is being undertaken to clarify this issue.

In the continuous-time case, we again observe a nice symmetry between the tracking problem and the regulation problem, which complements the symmetry between the Bode type sensitivity and complementary sensitivity integrals.

In the discrete time case, the asymmetry between the tracking problem and the regulation problem (cf. Theorem 2 and Theorem 3) is not an isolated phenomenon. A similar asymmetry occurs between (5) and (6).

Several other extensions of the problems and results in this paper have been studied recently or are currently under study, including:

1. Minimum tracking error for sinusoidal signals [4].
2. Minimum tracking error in systems with delays [3].
3. Tracking and regulation performance limitation of sampled-data systems.
4. Time domain performance limitation in filtering and estimation problems [17].

References

[1] J. Chen, "Sensitivity integral relations and design trade-offs in linear multivariable feedback systems", *IEEE Trans. Autom. Contr.*, vol. 40, pp. 1700-1716, 1995.

[2] J. Chen, "On logarithmic integrals and performance bounds for MIMO system, part I and part II", *Preprints*, 1997.

[3] J. Chen, L. Qiu, and O. Toker, "Limitations on maximal tracking accuracy", *Proc. 35th IEEE Conf. on Decision and Control*, pp. 726–731, 1996, submitted to *IEEE Trans. Autom. Contr.*.

[4] J. Chen, L. Qiu, and O. Toker, "Limitation on maximal tracking accuracy, part 2: tracking sinusoidal and ramp signals", *Proc. 1997 American Control Conf.*, pp. 1757-1761, 1997.

[5] E. J. Davison and B. M. Scherzinger, "Perfect control of the robust servomechanism problem", *IEEE Trans. Autom. Contr.*, vol. 32, pp. 689-702, 1987.

[6] B. A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory*, Springer-Verlag, 1987.

[7] J. Freudenberg and D. Looze, *Frequency Domain Properties of Scalar and Multivariable Feedback Systems*, Springer-Verlag, 1988.

[8] G. C. Goodwin and M. M. Seron, "Fundamental design tradeoffs in filtering, prediction, and smoothing", *IEEE Trans. Autom. Contr.*, vol. 42, pp. 1240-1251, 1997.

[9] H. Kwakernaak, "Symmetries in control system design", *Trends in Control: A European Perspective*, A. Isidori, editor, pp. 17–51, Springer, 1995.

[10] H. Kwakernaak and R. Sivan, "The maximal achievable accuracy of linear optimal regulators and linear optimal filters", *IEEE Trans. Autom. Contr.*, vol. 17, pp. 79-86, 1972.

[11] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley Interscience, New York, 1972.

[12] M. Morari and E. Zafiriou, *Robust Process Control*, Prentice Hall, Englewood Cliffs, NJ, 1989.

[13] L. Qiu and E. J. Davison, "Performance limitations of non-minimum phase systems in the servomechanism problem", *Automatica*, vol. 29, pp. 337-349, 1993.

[14] J. Shamma, "Performance limitations in sensitivity reduction for nonlinear plants", *System & Control Letters*, vol. 27, pp. 249-254, 1991.

[15] M. M. Seron, J. H. Braslavsky, and G. C. Goodwin, *Fundamental Limitations in Filtering and Control*, Springer, London, 1997.

[16] M. M. Seron, J. H. Braslavsky, P. V. Kokotović, and D. Q. Mayne, "Feedback limitations in nonlinear systems: from Bode integrals to cheap control", *IEEE Trans. on Automatic Control*, 1998, to appear.

[17] M. M. Seron, J. H. Braslavsky, D. G. Mayne, and P. V. Kokotović, "Limiting performance of optimal linear filters", *Preprints*, 1997.

[18] O. Toker, J. Chen, and L. Qiu, "Tracking performance limitations for multivariable discrete time systems", *Proc. 1997 American Control Conf.*, pp. 3887-3891, 1997. Also submitted to *Automatica*.

[19] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, The MIT Press, 1985.