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Topics on Multirate Systems: Frequency Response, Interpolation, Model Validation[1]

Li Chai and Li Qiu[2]
Department of Electrical and Electronic Engineering
Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Abstract

In this paper, we address several topics on multirate systems, mostly in a frequency domain point of view. We first study the frequency response of a multirate system and derive the aliasing component (AC) representation. We give the relationship between the AC representation and the ususal transfer function matrix of the lifted LTI system. Secondly, we propose a multirate version of the Nevanlina-Pick (NP) interpolation problem and give a necessary and sufficient solvability condition. This version of the NP interpolation problem is of interest mathematically and has potential applications in addressing other issues in control, signal processing and circuit theory. Finally, as an application of the multirate version of the NP interpolation problem, we formulate and solve the robust model validation problem for multirate systems with frequency domain experiment data.

1 Introduction

Multirate systems are finding more and more applications in control, signal processing, communication, econometrics and numerical mathematics. The reason for using multirate systems may be due to hardware considerations or due to the fact that multirate systems can often achieve objectives that cannot be achieved by single rate systems. Multirate signal processing is now one of the most vibrant areas of research in signal processing [18]. In control community, there has recently considerable research devoted to multirate controller design [2, 13]. In communication community, multirate sampling is used for blind system identification and equalization [9]. One of the standard techniques for the analysis and synthesis of multirate systems is blocking or lifting, which is largely a time domain tool. As for single rate systems, frequency domain analysis of multirate systems also plays an important role in their understanding. Some preliminary results have been given for periodic systems [15, 18] and dual rate systems [14]. In this paper, we first present the frequency response property of multirate systems using the aliasing component (AC) representation. We also give the relationship between the AC representation and the standard transfer function of the lifted equivalence, which can be used to obtain the frequency response of a multirate system by sinusoidal experiment. Secondly, motivated by the wide applications of the Nevanlinna-Pick (NP) interpolation in engineering problems including digital filter design [8], control [7] and circuit theory [4], we propose a constrained NP interpolation problem pertinent to multirate systems and give a necessary and sufficient solvability condition. More recently, much attention has been paid on validation of uncertain models consisting of a nominal model and a norm bounded modeling uncertainty [1, 11, 17]. Finally, we study the robust model validation problem of multirate systems from frequency domain experimental data using the solvability condition of the constrained NP interpolation.

2 General Multirate Systems

The setup of a general MIMO multirate system is shown in Figure 1. Here u_i, $i = 1, 2, \ldots, p$, are input signals whose sampling intervals are $m_i h$ respectively, and y_j, $j = 1, 2, \ldots, q$, are output signals whose sampling intervals are $n_j h$ respectively, where h is a real number called base sampling interval and m_i, n_j are natural numbers (positive integers). We will assume that all signals in the system are synchronized at time 0, i.e., the time 0 instances of all signals occur at the same time. In this paper, we will focus on those multirate systems that satisfy certain causal, linear, shift invariance properties which are to be defined below.

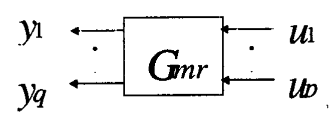

Figure 1: A general multirate system

Since we need to deal with signals with different rates,

[1]This work is supported by the Hong Kong Research Grants Council.

[2]Corresponding author, fax: 852-23581485, email: eeqiu@ee.ust.hk

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it is more convenient and clearer to associate each signal explicitly with its sampling interval. Let $\ell^r(\tau)$ denote the space of $\mathbb{R}^r$ valued sequences:

$$\begin{aligned}\ell^r(\tau) &= \{\{\ldots, x(-\tau), |x(0), x(\tau), x(2\tau), \ldots\} \\ &: \quad x(k\tau) \in \mathbb{R}^r\}.\end{aligned}$$

The system in Fig. 1 is a map from $\oplus_{i=1}^{p}\ell(m_i h)$ to $\oplus_{j=1}^{q}\ell(n_j h)$. It is said to be linear if this map is a linear map.

Let $l \in \mathbb{N}$ be a multiple of m_i and n_j, $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. Let $\bar{m}_i = l/m_i$ and $\bar{n}_j = l/n_j$. Denote the sets $\{m_i\}$ and $\{n_j\}$ by M and N respectively and the sets $\{\bar{m}_i\}$ and $\{\bar{n}_j\}$ by $\bar{M}$ and $\bar{N}$ respectively. Let $S : \ell^r(\tau) \longrightarrow \ell^r(\tau)$ be the forward shift operator, i.e.,

$$\begin{aligned}&S\{\ldots, x(-\tau), |x(0), x(\tau), \ldots\} \\ =\ &\{\ldots, x(-2\tau), |x(-\tau), x(0), x(\tau), \ldots\}.\end{aligned}$$

Define

$$S_{\bar{M}} = \operatorname{diag}\{S^{\bar{m}_1}, \ldots, S^{\bar{m}_p}\}, \ S_{\bar{N}} = \operatorname{diag}\{S^{\bar{n}_1}, \ldots, S^{\bar{n}_q}\}.$$

Then the multirate system in Fig. 1 is said to be $(\bar{M}, \bar{N})$-shift invariant or lh periodic in real time if $G_{mr}S_{\bar{M}} = S_{\bar{N}}G_{mr}$. Now let $P_t : \ell^r(\tau) \longrightarrow \ell^r(\tau)$ be the truncation operator, i.e.,

$$\begin{aligned}&P_t\{\ldots, x((k-1)\tau), x(k\tau), x((k+1)\tau), \ldots\} \\ =\ &\{\ldots, x((k-1)\tau), x(k\tau), 0, \ldots\}\end{aligned}$$

if $k\tau \le t < (k+1)\tau$. Extend this definition to spaces $\oplus_{i=1}^{p}\ell(m_i h)$ and $\oplus_{j=1}^{q}\ell(n_j h)$ in an obvious way. Then the multirate system is said to be causal if

$$P_t u = P_t v \Rightarrow P_t G_{mr} u = P_t G_{mr} v$$

for all $t \in \mathbb{R}$. In this paper, we will concentrate on causal linear $(\bar{M}, \bar{N})$-shift invariant systems. Such general multirate system covers many familiar classes of systems as special cases. If m_i, n_j, l are all the same, then this is an LTI single rate system. If m_i, n_j are all the same but l is a multiple of them, then it is a single rate l-periodic system. If $p = q = 1$, this becomes the SISO dual rate system studied in [3]. If m_i are the same and n_j are the same, then this becomes the MIMO dual rate system studied in [12]. For systems resulted from discretizing LTI continuous time systems using multirate sample and hold schemes in [2, 13], l turns out to be the least common multiple of m_i and n_j. The study of multirate systems in such a generality as indicated above, however, has never been done before.

A standard way for the analysis of such systems is to use lifting or blocking. Define a lifting operator $L_r : \ell(\tau) \to \ell^r(r\tau)$ by

$$L_r\{\ldots |x(0), x(\tau), \ldots\} \to \left\{\cdots \left| \begin{bmatrix} x(0) \\ \vdots \\ x((r-1)\tau) \end{bmatrix}, \begin{bmatrix} x(r\tau) \\ \vdots \\ x((2r-1)\tau) \end{bmatrix}, \cdots \right.\right\}$$

and let

$$L_{\bar{M}} = \operatorname{diag}\{L_{\bar{m}_1}, \ldots, L_{\bar{m}_p}\}, \ L_{\bar{N}} = \operatorname{diag}\{L_{\bar{n}_1}, \ldots, L_{\bar{n}_q}\}.$$

Then the lifted system $G = L_{\bar{N}} G_{mr} L_{\bar{M}}^{-1}$ is an LTI system in the sense that $GS = SG$. Hence it has transfer function $\hat{G}$ in λ-transform. However, G is not an arbitrary LTI system, instead its direct feedthrough term $\hat{G}(0)$ is subject to a constraint that is resulted from the causality of G_{mr}. This constraint is best described using the language of nests and nest operators [12, 13].

Let $\mathcal{X}$ be a finite dimensional vector space. A nest in $\mathcal{X}$, denoted $\{\mathcal{X}_k\}$, is a chain of subspaces in $\mathcal{X}$, including $\{0\}$ and $\mathcal{X}$, with the nonincreasing ordering

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_{l-1} \supseteq \mathcal{X}_l = \{0\}.$$

Let $\mathcal{U}$, $\mathcal{Y}$ be both finite dimensional vector spaces. Denote by $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ the set of linear operators $\mathcal{U} \to \mathcal{Y}$. Assume that $\mathcal{U}$ and $\mathcal{Y}$ are equipped, respectively, with nests which have the same number of subspaces, say, $l + 1$ as above. A linear map $T \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is said to be a *nest operator* if $T\mathcal{U}_k \subseteq \mathcal{Y}_k$, $k = 0, 1, \ldots, l$. The set of all nest operators (with given nests) is denoted $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. If we decompose the spaces $\mathcal{U}$ and $\mathcal{Y}$ in the following way:

$$\mathcal{U} = (\mathcal{U}_0 \ominus \mathcal{U}_1) \oplus (\mathcal{U}_1 \ominus \mathcal{U}_2) \oplus \cdots \oplus (\mathcal{U}_{l-1} \ominus \mathcal{U}_l) \qquad (1)$$

$$\mathcal{Y} = (\mathcal{Y}_0 \ominus \mathcal{Y}_1) \oplus (\mathcal{Y}_1 \ominus \mathcal{Y}_2) \oplus \cdots \oplus (\mathcal{Y}_{l-1} \ominus \mathcal{Y}_l) \qquad (2)$$

then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has the following block lower triangular form

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ T_{21} & T_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{bmatrix}. \qquad (3)$$

Write $\underline{u} = L_{\bar{M}} u$, $\quad \underline{y} = L_{\bar{N}} y$. Then

$$\begin{aligned}\underline{u}(0) &= [u_1(0) \cdots u_1((\bar{m}_1 - 1)m_1 h) \cdots \\ &\quad u_p(0) \cdots u_p((\bar{m}_p - 1)m_p h)]^T, \\ \underline{y}(0) &= [y_1(0) \cdots y_1((\bar{n}_1 - 1)n_1 h) \cdots \\ &\quad y_q(0) \cdots y_q((\bar{n}_q - 1)n_q h)]^T.\end{aligned}$$

Note that $u_i(r)$ occurs at $t = r m_i h$, and $y_j(r)$ occurs at $t = r n_j h$. Define for $k = 0, 1, \ldots, l$,

$$\begin{aligned}\mathcal{U}_k &= \{\underline{u}(0) : u_i(r m_i h) = 0 \text{ if } r m_i h < kh\} \\ \mathcal{Y}_k &= \{\underline{y}(0) : y_j(r n_j h) = 0 \text{ if } r n_j h < kh\}.\end{aligned}$$

Then the lifted plant $\hat{G}$ will have

$$\hat{G}(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\}). \qquad (4)$$

Now we see that each multirate system has an equivalent single rate LTI system satisfying a causality constraint. This causality constraint is characterized by a nest operator constraint as in (4) on its transfer function.

3 Frequency Response

To make the frequency relation among all the input signals with different sampling rate clear, we shall use the Fourier transform of a sequence $\{u_i(0), u_i(m_ih), u_i(2m_ih), \ldots, \}$ with sampling interval m_ih, incorporating the real time frequency. For more detail, refer to Sec. 7.4 in [10]. The Fourier transform of the input signal $\{u_i(km_ih)\}$, $i = 1, \ldots, p$, is

$$U_i(e^{j\omega m_i h}) = \sum_{k=0}^{\infty} u_i(km_ih)e^{-j(\omega k m_i h)}.$$

It is well-known that $U_i(e^{j\omega m_i h})$ is periodic with period $\frac{2\pi}{m_i h}$. The length-$\bar{m}_i$ AC representation of u_i is defined as [16, 14]

$$U_i^{AC}(e^{j\omega l h}) = \begin{bmatrix} U_i\left(e^{j\omega m_i h}\right) \\ U_i(e^{j(\omega+\frac{2\pi}{lh})m_i h}) \\ \vdots \\ U_i(e^{j(\omega+\frac{2\pi}{lh}(l-m_i)h)}) \end{bmatrix}, \; 0 \le \omega < \frac{2\pi}{lh}.$$

In contrast to the time domain lifting in the last section, the AC representation of u_i can be considered as the frequency domain lifting of $U_i(e^{j\omega m_i h})$. Clearly, this frequency domain lifting is also a one-one correspondence and it gives a different representation to the signal u_i. Let $\underline{U}_i\left(e^{j\omega l h}\right), 0 \le \omega < \frac{2\pi}{lh}$, be the Fourier transform of $\underline{u}_i = L_{\bar{m}_i} u_i$. It is easy to check the following equation [14, 18]

$$U_i^{AC}(e^{j\omega l h}) = F_{\bar{m}_i} D_{\bar{m}_i}(e^{j\omega l h}) \underline{U}_i(e^{j\omega l h})$$

where $F_{\bar{m}_i}$ is the $\bar{m}_i$ dimensional DFT matrix and

$$D_{\bar{m}_i}(e^{j\omega l h}) = \text{diag}\left(1, e^{-j\omega m_i h}, \ldots, e^{-j\omega(\bar{m}_i-1)m_i h}\right).$$

For the input $u = \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix}'$ of the multirate system shown in Fig. 1, define its length-$\bar{M}$ AC representation as

$$U^{AC}(e^{j\omega l h}) = \begin{bmatrix} U_1^{AC}\left(e^{j\omega l h}\right) \\ \vdots \\ U_p^{AC}\left(e^{j\omega l h}\right) \end{bmatrix}.$$

Then the Fourier transform $\underline{U}\left(e^{j\omega l h}\right)$ of $\underline{u} = L_{\bar{M}} u$ and $U^{AC}(e^{j\omega l h})$ have the following relation

$$U^{AC}(e^{j\omega l h}) = F_{\bar{M}} D_{\bar{M}}(e^{j\omega l h}) \underline{U}(e^{j\omega l h}) \qquad (5)$$

where

$$\begin{aligned} F_{\bar{M}} &= \text{diag}\,(F_{\bar{m}_1}, \ldots, F_{\bar{m}_p}) \\ D_{\bar{M}}(e^{j\omega l h}) &= \text{diag}(D_{\bar{m}_1}(e^{j\omega l h}), \ldots, D_{\bar{m}_p}(e^{j\omega l h})). \end{aligned}$$

We can also define the length-$\bar{N}$ AC representation $Y^{AC}(e^{j\omega l h})$ of the output

$$Y^{AC}(e^{j\omega l h}) = F_{\bar{N}} D_{\bar{N}}(e^{j\omega l h}) \underline{Y}(e^{j\omega l h}). \qquad (6)$$

As shown in Section II, an (M, N) shift-invariant multirate system has a transfer function $\hat{G}$. Then we have

$$\underline{Y}(e^{j\omega l h}) = \hat{G}(e^{j\omega l h}) \underline{U}(e^{j\omega l h}). \qquad (7)$$

It follows from (5 - 6) that

$$\begin{aligned} Y^{AC}(e^{j\omega l h}) &= F_{\bar{N}} D_{\bar{N}}(e^{j\omega l h}) \hat{G}(e^{j\omega l h}) \\ &\cdot D_{\bar{M}}^{-1}(e^{j\omega l h}) F_{\bar{M}}^{-1} U^{AC}(e^{j\omega l h}). \end{aligned} \qquad (8)$$

Denote

$$G^{AC}\left(e^{j\omega l h}\right) := F_{\bar{N}} D_{\bar{N}}(e^{j\omega l h}) \hat{G}(e^{j\omega l h}) D_{\bar{M}}^{-1}(e^{j\omega l h}) F_{\bar{M}}^{-1},$$

then (8) becomes

$$Y^{AC}(e^{j\omega l h}) = G^{AC}\left(e^{j\omega l h}\right) U^{AC}(e^{j\omega l h}). \qquad (9)$$

We call $G^{AC}\left(e^{j\omega l h}\right)$ the AC matrix of the multirate system. This gives us the following interpretation of the frequency response for an $(\bar{M}, \bar{N})$ shift-invariant system:

Let $L_1 = \max_i\{\bar{m}_i\}$ and $L_2 = \max_j\{\bar{n}_j\}$. Let $\mathcal{U}_\omega$ be the set of all signals that consist of sinusoidal components $\exp(j\omega t + jk\frac{2\pi}{lh}t)$ of the input, $k = 0, \ldots, L_1$, and $\mathcal{Y}_\omega$ be the set of all signals that consist of sinusoidal components $\exp(j\omega t + jk\frac{2\pi}{lh}t)$ of the output, $k = 0, \ldots, \max_i\{\bar{n}_j\}$, where $w \in [0, \frac{2\pi}{lh})$. That is

$$\begin{aligned} \mathcal{U}_\omega &= \{a_0 e^{j\omega t} + a_1 e^{j(\omega+2\pi/lh)t} + \cdots : a_i \in C^p\} \\ \mathcal{Y}_\omega &= \{b_0 e^{j\omega t} + b_1 e^{j(\omega+2\pi/lh)t} + \cdots : b_i \in C^q\} \end{aligned}$$

Then the system maps $\mathcal{U}_\omega$ into $\mathcal{Y}_\omega$ in the steady state. Note that t takes the value of m_ih or n_jh which depends on the specified signals. Compared to the LTI system $\hat{G}$ via lifting defined in the last section, the AC matrix is convenient in dealing with the specifications in frequency domain [14].

The AC matrix and the lifted transfer function matrix of a multirate system are two complementary representations of the multirate system. The lifted system transfer function exhibits the time domain features, such as causality, clearly but obscure the frequency domain features. On the other hand, the AC matrix exhibits the frequency response properties more clearly but the causality becomes obscure. One may be more advantageous than the other in a particular application.

4 Constrained Nevanlinna-Pick interpolation

Let $\mathcal{X}_i$, $i = 1, \ldots, n$, be finite dimensional Hilbert spaces. Also let $\mathcal{U}$ and $\mathcal{Y}$ be finite dimensional Hilbert spaces with nests $\{\mathcal{U}_k\}$ and $\{\mathcal{Y}_k\}$ respectively. Let U_i and Y_i be linear operators from $\mathcal{X}_i$ to $\mathcal{U}$ and from $\mathcal{X}_i$ to $\mathcal{Y}$ respectively. Let $\lambda_i, i = 1, \ldots, n$, be n complex numbers on the open unit disc $\mathbb{D}$. Denote $H_\infty(\mathcal{U}, \mathcal{Y})$ the Hardy class of all uniformly bounded analytic functions on $\mathbb{D}$ with values in $\mathcal{L}(\mathcal{U}, \mathcal{Y})$. Denote by $H_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ the set of functions $\hat{G} \in H_\infty(\mathcal{U}, \mathcal{Y})$

satisfying $G(0) \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. The tangential NP interpolation problem with constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data λ_i, U_i, Y_i, $i = 1, \ldots, n$, is to find (if possible) a function $\hat{G}$ in $H_\infty(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that $\|\hat{G}\|_\infty \leq 1$, and $Y_i = \hat{G}(\lambda_i)U_i$ for $i = 1, \ldots, n$. Denote $U := [\, U_1 \ \cdots \ U_n \,]$ and $Y := [\, Y_1 \ \cdots \ Y_n \,]$.

Before going into the solvability conditions of the constrained NP interpolation, we need to state a result on matrix positive completion. The matrix positive completion problem is as follows [5]: Given $B_{ij}, |j-i| \leq q$, satisfying $B_{ij} = B_{ji}^*$, find the remaining matrices $B_{ij}, |j-i| > q$, such that the block matrix $B = [B_{ij}]_{i,j=1}^n$ is positive definite. The matrix positive completion problem was first proposed by Dym and Gohberg [5], who gave the following result:

Lemma 1 *The matrix positive completion problem has a solution if and only if*

$$\begin{bmatrix} B_{ii} & \cdots & B_{i,i+q} \\ \vdots & & \vdots \\ B_{i+q,i} & \cdots & B_{i+q,i+q} \end{bmatrix} \geq 0, \ i = 1, \ldots, n-q. \quad (10)$$

Theorem 1 *There exists a solution to the NP interpolation problem with constraint $\mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ for the data λ_i, U_i, Y_i, $i = 1, \ldots, n$, if and only if*

$$\left[\frac{U_i^* U_j - Y_i^* Y_j}{1 - \bar{\lambda}_i \lambda_j} - U_i^* \Pi_{\mathcal{U}_k} U_j + Y_i^* \Pi_{\mathcal{Y}_k} Y_j\right]_{i,j=1}^n \geq 0 \quad (11)$$

for all $k = 1, \ldots, l$.

Proof: The nest operator constraint on the interpolation function $\hat{G}$ can be considered as an additional interpolation condition $\hat{G}(0) I = T$ for some $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$. If we set $\lambda_0 = 0$, $U_0 = I$ and $Y_0 = T$. By the solvability condition of the standard NP interpolation problem [6], the NP interpolation problem with nest operator constraint has a solution if and only if there exists $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ such that

$$\left[\, \frac{U_i^* U_j - Y_i^* Y_j}{1 - \bar{\lambda}_i \lambda_j} \,\right]_{i,j=0}^n \geq 0. \quad (12)$$

Let $P = \left[\, \frac{U_i^* U_j - Y_i^* Y_j}{1 - \bar{\lambda}_i \lambda_j} \,\right]_{i,j=1}^n$, which is the Pick matrix corresponding to the NP interpolation problem with data λ_i, U_i, Y_i, $i = 1, \ldots, n$, without constraint. Then (12) can be rewritten as

$$\begin{bmatrix} I - T^* T & U - T^* Y \\ U^* - Y^* T & P \end{bmatrix} \geq 0. \quad (13)$$

By Schur complement, (13) is equivalent to

$$\begin{bmatrix} I & U & T^* \\ U^* & P + Y^* Y & Y^* \\ T & Y & I \end{bmatrix} \geq 0. \quad (14)$$

If we decompose the spaces $\mathcal{U}$ and $\mathcal{Y}$ as in (1-3), then a nest operator $T \in \mathcal{N}(\{\mathcal{U}_k\}, \{\mathcal{Y}_k\})$ has a block lower triangular form shown in (3). Therefore, the constrained NP interpolation problem has a solution if and only if (14) holds for a block lower triangular matrix T. This is a matrix positive completion problem. By Lemma 1, (14) holds for some block lower triangular T if and only if

$$\begin{bmatrix} I & \Pi_{\mathcal{U}_k} U & 0 \\ (\Pi_{\mathcal{U}_k} U)^* & P + Y^* Y & (\Pi_{\mathcal{Y}_k^\perp} Y)^* \\ 0 & \Pi_{\mathcal{Y}_k^\perp} Y & I \end{bmatrix} \geq 0 \quad (15)$$

for $k = 0, \ldots, l$. Using Schur complement twice, we can easily show that (15) is equivalent to

$$P + (\Pi_{\mathcal{Y}_k} Y)^* (\Pi_{\mathcal{Y}_k} Y) - (\Pi_{\mathcal{U}_k} U)^* (\Pi_{\mathcal{U}_k} U) \geq 0 \quad (16)$$

for $k = 0, \ldots, l$. We claim that inequalities (16) when $k = 0$ is implied by (16) when $k = l$. In fact, when $k = l$, inequalities (16) gives

$$P \geq 0. \quad (17)$$

When $k = 0$, inequalities (16) gives

$$\begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{\lambda}_n \end{bmatrix} P \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \geq 0. \quad (18)$$

It is obvious that (17) implies (18). The proof is then completed by noticing that (16) is exactly the same as (11). ∎

5 Frequency Domain Model Validation of Multirate Systems

In this section, we extend the results in [1] to multirate systems. The setup is shown in Fig 2, where P_{mr} and Δ_{mr} are both multirate systems, and they together form a multirate uncertain system model with P_{mr} fixed and Δ_{mr} unknown. Here, u_i, $i = 1, \ldots, p$, are input signals whose sampling intervals are $m_i' h$ and y_j, $j = 1, \ldots, q$, are output signals whose sampling intervals are $n_j' h$. Also v_i, $i = 1, \ldots, r$, and w_j, $j = 1, \ldots, s$, are the auxiliary signals whose sampling intervals are $m_i h$ and $n_j h$ respectively.

The model validation problem considered in this paper is as follows. Given P_{mr}, an uncertainty set which Δ_{mr} belongs to, a set of time domain experimental data on u_i and y_i, and a set $\mathcal{E}$ of noise signals, find out if there exists a Δ_{mr} in the uncertainty set such that the experimental data can be reproduced with P_{mr} and Δ_{mr} together with the noises $\mathcal{E}$.

Assume that both P_{mr} and Δ_{mr} are lh periodic in real time for some integer l. Let $\bar{m}_i' = l/m_i'$, $\bar{n}_j' = l/n_j'$, $\bar{m}_i = l/m_i$, $\bar{n}_j = l/n_j$. And let $\underline{y} = L_{\bar{N}'} y$, $\underline{u} = L_{\bar{M}'} u$,

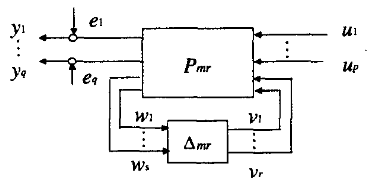

Figure 2: A general multirate LFT uncertain model

$\underline{v} = L_{\bar{M}} v$, $\underline{w} = L_{\bar{N}} w$, and

$$\begin{aligned} P &= \begin{bmatrix} L_{\bar{N}'} & 0 \\ 0 & L_{\bar{N}} \end{bmatrix} P_{mr} \begin{bmatrix} L_{\bar{M}'} & 0 \\ 0 & L_{\bar{M}} \end{bmatrix}^{-1}. \\ \Delta &= L_{\bar{N}} \Delta_{mr} (L_{\bar{M}})^{-1} \end{aligned}$$

where $L_{\bar{N}'}$, $L_{\bar{M}'}$, $L_{\bar{M}}$, $L_{\bar{N}}$ are appropriately defined as in Section II. Then the multirate uncertain system in Fig. 2 is converted to an equivalent LTI uncertain system with a causality constraints shown in Fig. 3.

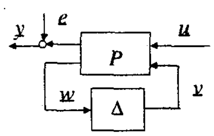

Figure 3: The equivalent LTI uncertain model

Denote

$$\begin{aligned} \underline{v}(0) &= [v_1(0)^T, \ldots, v_1((\bar{m}_1 - 1)m_1 h)^T, \ldots, \\ &\quad v_r(0)^T, \ldots, v_r((\bar{m}_r - 1)m_r h)^T]^T \\ \underline{w}(0) &= [w_1(0)^T, \ldots, w_1((\bar{n}_1 - 1)n_1 h)^T, \ldots, \\ &\quad w_s(0)^T, \ldots, w_s((\bar{n}_s - 1)n_s h)^T]^T \\ \underline{u}(0) &= [u_1(0)^T, \ldots, u_1((\bar{m}'_1 - 1)m'_1 h)^T, \ldots, \\ &\quad u_p(0)^T, \ldots, u_p((\bar{m}'_p - 1)m'_p h)^T]^T \\ \underline{y}(0) &= [y_1(0)^T, \ldots, y_1((\bar{n}'_1 - 1)n'_1 h)^T, \ldots, \\ &\quad y_q(0)^T, \ldots, y_q((\bar{n}'_q - 1)n'_q h)^T]^T. \end{aligned}$$

Define for $k = 0, 1, \ldots, l$,

$$\begin{aligned} \mathcal{V}_k &= \{\underline{v}(0) : v_i(rm_i h) = 0 \text{ if } rm_i h < kh\} \\ \mathcal{W}_k &= \{\underline{w}(0) : w_j(rn_j h) = 0 \text{ if } rn_j h < kh\} \\ \mathcal{U}_k &= \{\underline{u}(0) : u_i(rm'_i h) = 0 \text{ if } rm'_i h < kh\} \\ \mathcal{Y}_k &= \left\{\underline{y}(0) : y_j(rn'_j h) = 0 \text{ if } rn'_j h < kh\right\}. \end{aligned}$$

Then the causality constraints that P and Δ satisfy are $\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{V}_k\}, \{\mathcal{Y}_k \oplus \mathcal{W}_k\})$ and $\hat{\Delta}(0) \in \mathcal{N}(\{\mathcal{W}_k\}, \{\mathcal{V}_k\})$, where $\hat{P}$ and $\hat{\Delta}$ are transfer functions of P and Δ respectively. The model validation for multirate systems P_{mr} and Δ_{mr} are then converted to that for LTI system P and Δ satisfying the above causality constraints.

To state the frequency domain model validation problem, we need to introduce a few definitions. Let $\mathbb{D}_\rho := \{\lambda : |\lambda| < \rho\}$, define

$$\begin{aligned} H_\infty(\rho) &:= \{\hat{F}(\lambda) : \hat{F} \text{ is analytic in } \mathbb{D}_\rho \\ &\qquad \text{and } \sup_{\lambda \in \mathbb{D}_\rho} \bar{\sigma}(\hat{F}) < \infty\} \\ H_\infty(\rho, \gamma) &:= \{\hat{F}(\lambda) : \hat{F} \text{ is analytic in } \mathbb{D}_\rho \\ &\qquad \text{and } \sup_{\lambda \in \mathbb{D}_\rho} \bar{\sigma}(\hat{F}) \le \gamma\} \end{aligned}$$

where $\bar{\sigma}(\cdot)$ denotes the largest singular value. Assume that an uncertain model of the lifted LTI equivalence of a multirate system is represented by the LFT $\mathcal{F}(\hat{P}, \hat{\Delta})$, where the nominal model $\hat{P} \in H_\infty(\rho)$ is given satisfying $\hat{P}_{22} \in H_\infty(\rho, \frac{1}{\gamma})$ and the uncertainty $\hat{\Delta}$ is known *a priori* to satisfy $\hat{\Delta} \in H_\infty(\rho, \gamma)$. By carrying out a series of steady state frequency response experiments on the multirate system, we can obtain the frequency response data for the LTI equivalent system $\underline{U}_i$ and $\underline{Y}_i$ at different frequency points $\omega_i \in [0, \pi)$, $i = 1, \ldots, n$, by equations (5) and (6). Note that $\underline{U}_i$ and $\underline{Y}_i$ can also be obtained by the Discrete Fourier Transform from time-domain data of $\underline{u}$ and $\underline{y}$. The model validation problem is to test whether the uncertain model is consistent with the experimental data, i.e. whether there exists a $\hat{\Delta} \in H_\infty(\rho, \gamma)$ with $\hat{\Delta}(0) \in \mathcal{N}(\{\mathcal{W}_k\}, \{\mathcal{V}_k\})$ such that

$$\underline{Y}_i = \mathcal{F}(\hat{P}(e^{j\omega_i}), \hat{\Delta}(e^{j\omega_i}))\underline{U}_i + \underline{E}_i, \; i = 1, \ldots, n \quad (19)$$

for some $\underline{E}_i \in \mathcal{E}$, where $\mathcal{E}$ is a compact convex set representing a bound on the measurement noise. From Fig. 3, we see that (19) implies

$$\begin{aligned} \underline{Y}_i &= \hat{P}_{11}(\lambda_i)\underline{U}_i + \hat{P}_{12}(\lambda_i)\underline{V}_i + \underline{E}_i & (20) \\ \underline{W}_i &= \hat{P}_{21}(\lambda_i)\underline{U}_i + \hat{P}_{22}(\lambda_i)\underline{V}_i & (21) \\ \underline{V}_i &= \hat{\Delta}(\lambda_i)\underline{W}_i & (22) \end{aligned}$$

for some $\underline{V}_i$, $\underline{W}_i$, $i = 1, \ldots, n$, where $\lambda_i = e^{j\omega_i}$.

Theorem 2 *For data* $\underline{U}_1, \ldots, \underline{U}_n$ *and* $\underline{Y}_1, \ldots, \underline{Y}_n$, *Define*

$$\Omega_i := \left\{\underline{V}_i : \underline{Y}_i = \hat{P}_{11}(\lambda_i)\underline{U}_i + \hat{P}_{12}(\lambda_i)\underline{V}_i + \underline{E}_i, \underline{E}_i \in \mathcal{E}\right\}$$

The uncertain model is not invalidated if and only if there exists a sequence $\underline{V} := (\underline{V}_1, \ldots, \underline{V}_n)$ *with* $\underline{V}_i \in \Omega_i, i = 1, \ldots, n$, *such that* $H_k(\underline{V}) \ge 0$ *for all* $k = 1, \ldots, l$, *where*

$$H_k(\underline{V}) = \begin{bmatrix} H_{k11}(\underline{V}) & H_{k12}(\underline{V}) \\ H_{k21}(\underline{V}) & I \end{bmatrix}$$

$$H_{k11}(\underline{V}) =$$

$$\left[\frac{\underline{U}_i^* \hat{P}_{21}^*(\lambda_i)\hat{P}_{21}(\lambda_j)\underline{U}_j}{1 - (1/\rho^2)\bar{\lambda}_i \lambda_j} + \underline{U}_i^* \hat{P}_{21}^*(\lambda_i)\Pi_{\mathcal{W}_k}\hat{P}_{21}(\lambda_j)\underline{U}_j\right]_{i,j=1}^n$$

$$+ \left[\frac{\underline{U}_i^* \hat{P}_{21}^*(\lambda_i)\hat{P}_{22}(\lambda_j)\underline{V}_j}{1 - (1/\rho^2)\bar{\lambda}_i \lambda_j} + \underline{U}_i^* \hat{P}_{21}^*(\lambda_i)\Pi_{\mathcal{W}_k}\hat{P}_{22}(\lambda_j)\underline{V}_j\right]_{i,j=1}^n$$

$$+\left[\frac{\underline{V}_i^*\hat{P}_{22}^*(\lambda_i)\hat{P}_{21}(\lambda_j)\underline{U}_j}{1-(1/\rho^2)\lambda_i\lambda_j}+\underline{V}_i^*\hat{P}_{22}^*(\lambda_i)\Pi_{\mathcal{W}_k}\hat{P}_{21}(\lambda_j)\underline{U}_j\right]_{i,j=1}^n$$

$$H_{k12}(\underline{V}) = D_{\underline{V}}^*Q_k = \left[\underline{V}_i^*Q_{kij}\right]_{i,j=1}^n$$

$$H_{k21}(\underline{V}) = Q_kD_{\underline{V}} = \left[Q_{kij}\underline{V}_j\right]_{i,j=1}^n$$

$$D_{\underline{V}} = \text{diag}(\underline{V}_1,\ldots,\underline{V}_n)$$

$$Q_k^2 =$$

$$\left[\frac{\frac{1}{\gamma^2}I-\hat{P}_{22}^*(\lambda_i)\hat{P}_{22}(\lambda_j)}{1-(1/\rho^2)\lambda_i\lambda_j}-\frac{1}{\gamma^2}\Pi_{\mathcal{V}_k}+\hat{P}_{22}^*(\lambda_i)\Pi_{\mathcal{W}_k}\hat{P}_{22}(\lambda_j)\right]_{i,j=1}^n.$$

Proof: We know that $\hat{P}_{22}(0) \in \mathcal{N}(\{\mathcal{V}_k\},\{\mathcal{W}_k\})$ since $\hat{P}(0) \in \mathcal{N}(\{\mathcal{U}_k \oplus \mathcal{V}_k\},\{\mathcal{Y}_k \oplus \mathcal{W}_k\})$. Define $\hat{F}(\lambda) = \gamma\hat{P}_{22}(\rho\lambda)$, then $\|\hat{F}\|_\infty \le 1$, with $\hat{F}(\lambda_i/\rho) = \gamma\hat{P}_{22}(\lambda_i)$ and $\hat{F}(0) \in \mathcal{N}(\{\mathcal{V}_k\},\{\mathcal{W}_k\})$. By Theorem 1, we get $\gamma^2Q_k^2 \ge 0$ for all $l = 1,\ldots,l$. This shows that Q_k is well-defined. Consider now the problem of finding an analytic $\hat{F} \in H_\infty(\{\mathcal{W}_k\},\{\mathcal{V}_k\})$ such that

$$\hat{F}(\lambda_i/\rho)\underline{W}_i = \frac{1}{\gamma}\underline{V}_i, \quad \forall i = 1,\ldots,n \tag{23}$$

and $\|\hat{F}\|_\infty \le 1$. This is a tangential NP interpolation problem with constraint $\mathcal{N}(\{\mathcal{W}_k\},\{\mathcal{V}_k\})$. By Theorem 1, there exists a solution if and only if

$$\left[\frac{\underline{W}_i^*\underline{W}_j-\frac{1}{\gamma^2}\underline{V}_i\underline{V}_j}{1-(1/\rho^2)\lambda_i\lambda_j}-\underline{W}_i^*\Pi_{\mathcal{W}_k}\underline{W}_j+\frac{1}{\gamma^2}\underline{V}_i^*\Pi_{\mathcal{V}_k}\underline{V}_j\right]_{i,j=1}^n \ge 0$$

for all $k = 1,\ldots,l$. Substituting (21) into the above inequality yields

$$H_{k11}(\underline{V}) - D_{\underline{V}}^*Q_k^2D_{\underline{V}} \ge 0. \tag{24}$$

It follows by Schur complement that (24) is equivalent to $H_k(\underline{V}) \ge 0$. Hence, there exists a function $\hat{F} \in H_\infty(\{\mathcal{W}_k\},\{\mathcal{V}_k\})$ such that (23) holds and $\|\hat{F}\|_\infty \le 1$ if and only if $H_k(\underline{V}) \ge 0$ for all $k = 1,\ldots,l$. On the other hand, if we set $\hat{\Delta}(\lambda) = \gamma\hat{F}(\lambda/\rho)$, then $\hat{F}$ has the above property if and only if $\hat{\Delta}(\lambda) \in H_\infty(\rho,\gamma)$ with $\hat{\Delta}(0) \in N(\{\mathcal{W}_k\},\{\mathcal{V}_k\})$ such that (19) holds . Therefore, the uncertain model is not invalidated if and only if $H_k(\underline{V}) \ge 0$, $k = 1,\ldots,l$, for some $\underline{V} = (\underline{V}_1,\ldots,\underline{V}_n)$ with $\underline{V}_i \in \Omega_i$. This completes the proof. ∎

6 Conclusion

In this paper, a frequency domain method for the analysis and model validation is presented for general multirate systems. First, we generalize the frequency response property of LTI systems to multirate systems. we then propose and study a muiterate version of the NP interpolation problem and give a necessary and sufficient solvability condition. Finally we formulate and solve the robust model validation problem for multirate systems with frequency experimental data.

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