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TRACKING PERFORMANCE LIMITATIONS UNDER DISTURBANCE OR UNCERTAINTY [1]

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Abstract: This paper investigates tracking performance limitations of a linear time invariant (LTI) multivariable plant subject to external disturbance or model uncertainty. It attempts to extend some recent results on tracking performance limitation in which neither the disturbance nor the uncertainty is considered. The reference signal to be tracked is a step signal. The tracking performance is measured by the energy of the tracking error. The external disturbance is assumed to be energy bounded and to be injected into the input channels of the plant. The model uncertainty is assumed to have some special structure and to have certain induced norm bound. The performance limitation studied is the minimal attainable value, under any controller structure and parameters, of the maximum tracking error energy for all possible disturbance or uncertainty. It is shown that the tracking performance limit of the plant with the worst case disturbance can be decomposed as the sum of the tracking performance limit without the disturbance and the optimal disturbance attenuation performance without the reference. It is also found that, for an LTI plant with a nonlinear time-invariant uncertainty, the best attainable tracking performance of the system under the worst possible uncertainty is proportional to the tracking performance limit of the plant without uncertainty, magnified by a quantity related to the size of the uncertainty and certain optimal $\mathcal{H}_\infty$ gain. *Copyright ©2005 IFAC*

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1. INTRODUCTION

In this paper, tracking performance limitations are considered for a linear time invariant (LTI) multivariable plant subject to external disturbance or model uncertainty. The reference signal to be tracked is a step signal. The tracking performance is measured by the energy of the tracking error. Obviously, this tracking performance not only depends on the LTI plant and the reference signal but also the disturbance or the uncertainty. The fundamental limitations of the system under consideration are its minimal attainable tracking error energy for the worst case disturbance or uncertainty under all possible controllers. The aim of this paper is to find explicit formulas for these limitations in terms of the plant characteristics and structures.

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The tracking control problem for a given LTI plant with an external disturbance includes two tasks: 1) asymptotically tracking the reference signal and 2) attenuating the disturbance. In this paper, the best achievable tracking performance, under all possible controllers, of the system is considered for the worst disturbance input. It is referred as *minimax performance limit of asymptotical tracking/disturbance rejection*. Here we assume that the disturbance comes into the system from the plant input and a two degree of freedom (2DOF) controller is used. It is found that the minimax performance limit of asymptotical tracking/disturbance attenuation is the sum of two terms: one is the best tracking performance of the system without the disturbance input and the other is the minimal cost of optimal disturbance attenuation. If the transfer function from the plant input to its measurement output satisfies certain conditions, in particular, if state feedback is used in the control, the minimax performance limit of asymptotical tracking/disturbance attenuation of the system is exactly equal to the tracking performance limit without a disturbance.

The tracking problem is also considered for a plant with a nonlinear time invariant uncertainty which has a certain induced norm bound. This problem also includes two tasks: 1) asymptotically tracking the reference signal and 2) robustly stabilizing the system for all possible uncertainties. The best attainable tracking performance of the system in the problem is considered for the worst uncertainty. This is referred as *minimax performance limit of asymptotical tracking/robust stabilization*. It is assumed that the uncertainty is driven by the tracking error, and the output of the uncertainty is injected into the input channel of the plant. It is also assumed that a 2DOF controller is used in this system. Then it is come out that the minimax performance limit of asymptotical tracking/robust stabilization is proportional to the tracking performance limit of the plant without uncertainty, magnified by a quantity related to the norm bound of the uncertainty and the optimal $\mathcal{H}_\infty$ gain.

The performance limitations in various optimal control problems have been extensively studied during last three decades. It was obtained by Kwakernaak and Sivan (1972) and Francis (1979) that, for a right invertible LTI minimum phase systems, the performance limit of the optimal tracking or cheap control is zero. Since then the research on this issue was extended to nonminimum phase system (e.g. see Chen *et al.* (2000), Morari and Zafiriou (1989), Qiu and Chen (n.d.), Qiu and Davison (1993) and Su *et al.* (2003)). These works show that the performance limits of the optimal cheap control or optimal tracking problems are only dependent on the nonminimum phase zeros and the directional vectors associated with these zeros as well as the initial state or the reference signal. On the other hand, some efforts have also been made on this issue for LTI plants subject to external disturbance or model uncertainty. It was obtained by Davison and Scherzinger (1987) that, for a class of minimum phase LTI systems with an external disturbance in certain particular structure, the robust performance limit of the optimal cheap control is zero. This discussion was also extended to a type of nonminimum phase systems by Qiu and Davison (1993) and Jemaa and Davison (2003). Some interesting discussion on the robust performance limitation of an LTI system with uncertainties was presented by Xie and Petersen (2002) and Goodwin *et al.* (2003).

Finally, a note on the notation: A signal in the time domain is denoted by a lower case letter, such as r or $r(t)$. A system, viewed as an input/output operator, is denoted by a capital letter, such as G. The Laplace transform is denoted by a hat " ˆ ", i.e., $\hat{r}$ is the Laplace transform of r. If G is an LTI system, $\hat{G}$ represents the transfer function of G.

2. PROBLEM STATEMENTS AND PRELIMINARIES

In this paper, we first consider a feedback system shown in Figure 1. Here P is a given LTI plant

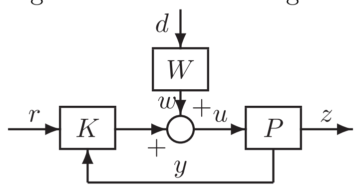

Fig. 1. A general control structure

whose measurement y and output z may not be the same, K is a 2DOF controller to be designed. The signal r is a step reference signal and the signal d is a disturbance with energy bounded by δ^2, i.e., $\|d\|_2^2 \leq \delta^2$. W is a known LTI stable, proper weight.

The asymptotical tracking/disturbance rejection problem for the system in Figure 1 is to design a controller K so that the closed loop system is internally stable and the plant output z asymptotically tracks a step signal $r(t) = v, t \geq 0$ for all disturbances $d \in \mathcal{L}_2$ with $\|d\|_2 \leq \delta$.

The tracking performance is measured by the energy of the tracking error $e(t)$,

$$J(v) = \int_0^\infty \|e(t)\|^2 dt,$$

which clearly depends on the disturbance d. The performance index we consider for the tracking/disturbance attenuation problem is the worst value of $J(v)$ over all possible d:

$$\sup_{\|d\|_2 \leq \delta} J(v).$$

The performance limit that we are interested in is therefore the minimum value of this performance index achievable by the choice of the controller K:

$$J_{opt}(v) = \inf_K \sup_{\|d\|_2 \leq \delta} J(v)$$

where K is chosen among all stabilizing 2DOF controllers.

Denote the transfer matrices from u to z by $\hat{G}$ and that from u to y by $\hat{H}$, i.e., $\hat{P} = \begin{bmatrix} \hat{G} \\ \hat{H} \end{bmatrix}$. In order for the tracking problem to be meaningful and solvable, we make the following assumption throughout the paper.

Assumption 1.

(1) $\hat{P}$, $\hat{G}$ and $\hat{H}$ have the same unstable poles.
(2) $\hat{G}$ has no zero at the origin.
(3) $\hat{G}$ is right-invertible.

The first item in the assumption means that the plant P is stabilizable by the measurement feedback from y and at the same time the measurement does not introduce any additional unstable modes. A simple interpretation of the assumption is that if $\hat{P} = \begin{bmatrix}\hat{N}\\ \hat{L}\end{bmatrix}\hat{D}^{-1}$, $\hat{D}$, $\hat{N}$, $\hat{L} \in \mathcal{R}H_\infty$ is a coprime factorization, then $\hat{N}\hat{D}^{-1}$ and $\hat{L}\hat{D}^{-1}$ are also coprime factorizations. The second and third items are necessary for the solvability of the tracking problem.

The second problem that we will address is the minimax performance limit of asymptotical tracking/robust stabilization. In this problem, the disturbance in Figure 1 is replaced by a special uncertainty shown in Figure 2. The uncertainty is

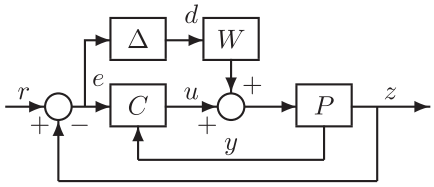

Fig. 2. Tracking system with uncertainty

unknown but is assumed to be a possibly nonlinear time variant causal operator with a induced norm bound:

$$\|\Delta\| = \sup_{e\in\mathcal{L}_2, e\neq 0} \frac{\|d\|_2}{\|e\|_2} \leq \delta. \qquad (1)$$

Note that C is simply another form of a 2DOF controller. The performance index of the system is the maximum value of $J(v)$ over all possible Δ

$$\sup_{\|\Delta\|\leq\delta} J(v).$$

The performance limit is then the minimum value of the performance index achievable by the choice of the controller:

$$\tilde{J}_{opt}(v) = \inf_C \sup_{\|\Delta\|\leq\delta} J(v).$$

We end this section with some preliminary materials. If $z \in \mathbb{C}_+$ is a nonminimum phase zero of right-invertible transfer function $\hat{G}$, then there exists a unitary vector η such that $\eta^*\hat{G}(z) = 0$. Suppose that $z_i \in \mathbb{C}_+$, $i = 1, \cdots, m$, are zeros of the nonminimum phase plant $\hat{G}$. The transfer matrix $\hat{G}$ can be factorized as follows:

$$\hat{G} = \hat{G}_{in}\hat{G}_0 \quad \text{and} \quad \hat{G}_{in} = \prod_{i=1}^{m}\hat{G}_i \qquad (2)$$

where $\hat{G}_i$ is inner with only one zero z_i in the following form

$$\hat{G}_i(s) = [\eta_i \;\; U_i]\begin{bmatrix}\frac{\bar{z}_i}{z_i}\frac{z_i - s}{\bar{z}_i + s} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\eta_i^*\\ U_i^*\end{bmatrix} \qquad (3)$$

with $\eta_i\eta_i^* + U_iU_i^* = I$, $\eta_i^*U_i = 0$; and $\hat{G}_0$ has no nonminimum phase zero.

3. THE TRACKING PERFORMANCE LIMIT UNDER DISTURBANCES

Consider Figure 1 again. Let $\hat{H} = \hat{\tilde{D}}^{-1}\hat{\tilde{L}}$ be a left coprime factorization of $\hat{H}$. Then there exist $\hat{X}, \hat{Y}, \hat{\tilde{X}}, \hat{\tilde{Y}} \in \mathcal{R}H_\infty$ satisfying the double Bezout identity (see Vidyasagar (1985))

$$\begin{bmatrix}\hat{\tilde{X}} & -\hat{\tilde{Y}}\\ -\hat{\tilde{L}} & \hat{\tilde{D}}\end{bmatrix}\begin{bmatrix}\hat{D} & \hat{Y}\\ \hat{L} & \hat{X}\end{bmatrix} = I. \qquad (4)$$

The set of all linear internally stabilizing 2DOF controllers K is given by (see Vidyasagar (1985))

$$\mathcal{K} = \Big\{\hat{K} = (\hat{Y} - \hat{R}\hat{\tilde{L}})^{-1}[\hat{Q} \quad (\hat{X} + \hat{R}\hat{\tilde{D}})] :$$
$$\hat{Q}, \hat{R} \in \mathcal{R}H_\infty \text{ and } |\hat{Y} - \hat{R}\hat{\tilde{L}}| \neq 0\Big\}.$$

Plugging the parametrization of the 2DOF controllers into the system, we obtain the expression for the tracking error:

$$J(v) = \|\hat{e}\|_2^2 = \|\hat{r} - \hat{N}\hat{Q}\hat{r} - \hat{N}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2. \quad (5)$$

We will be able to show that the problem of minimizing the performance index $\sup_{\|d\|_2\leq\delta} J(v)$ by designing the free parameters $\hat{Q}$ and $\hat{R}$ is equivalent to two independent optimization problems: minimizing $\|\hat{r} - \hat{N}\hat{Q}\hat{r}\|_2^2$ by designing $\hat{Q}$ and minimizing

$$\|\hat{T}_{zd}\|_\infty = \|\hat{N}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\|_\infty$$

by designing $\hat{R}$. This is the key point in the following theorem.

Theorem 1. Let $\hat{G}$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$ with corresponding Blaschke vectors $\eta_1, \eta_2, \ldots, \eta_m$. Then the minimax tracking performance limit of asymptotical tracking/disturbance rejection of the system is given by

$$J_{opt}(v) = 2\sum_{i=1}^{m}\frac{\mathrm{Re}\,(z_i)}{|z_i|^2}\cos^2\angle(\eta_i, v) + \delta^2 \inf_{K\in\mathcal{K}}\|\hat{T}_{zd}\|_\infty^2.$$

The proof of this theorem is given in Appendix A.

The performance limit $J_{opt}(v)$ is a sum of two terms: The first is the tracking performance limit of the system without the disturbance input d and the second is the best achievable performance of disturbance attenuation of the system without the reference signal. The relationship between the second term $\delta^2 \inf_{K\in\mathcal{K}}\|\hat{T}_{zd}\|_\infty^2$ of $J_{opt}(v)$ and the characteristics of the plant was studied by Chang and Pearson (1984), and Vidyasagar (1985). Intuitively, one can see that if the $\hat{\tilde{L}}$ is left invertible in $\mathcal{R}H_\infty$, then $\|\hat{T}_{zd}\|_\infty$ can be made arbitrarily small by selecting a proper $\hat{R} \in \mathcal{R}H_\infty$. This means that almost disturbance decoupling can be achieved in this case. In light of this, we have the following corollary.

Corollary 1. If $\hat{H}$ has no bounded zeros in closed right half plane and is left invertible, the minimax tracking performance limit of asymptotical tracking/disturbance rejection of the system is given by

$$J_{opt}(v) = 2\sum_{i=1}^{m} \frac{\mathrm{Re}\,(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v).$$

Notice that the conditions in above corollary are always satisfied by the transfer function from the input to the state in the system. Hence the performance limitation in Corollary 1 is valid when the measurement y is the state variable of the plant.

4. THE TRACKING PERFORMANCE LIMIT UNDER UNCERTAINTIES

In this section, the minimax performance limit of asymptotical tracking/robust stabilization will be discussed for the system with a model uncertainty shown in Figure 2. Denote the transfer function from d to z by $\hat{T}_{zd}$. The uncertainty is a nonlinear time invariant operator and has certain induced norm bound given in (1).

Theorem 2. Let $\hat{G}$ have nonminimum phase zeros $z_1, z_2, \ldots, z_m$ with corresponding Blaschke vectors $\eta_1, \eta_2, \ldots, \eta_m$. Then the minimax performance limit of asymptotical tracking/robust stabilization of the system with a nonlinear time invariant uncertainty is given by

$$\tilde{J}_{opt}(v) = \frac{2}{1-\delta^2\rho^2} \sum_{i=1}^{m} \frac{\mathrm{Re}\,(z_i)}{|z_i|^2} \cos^2 \angle(\eta_i, v)$$

while $\rho = \inf_{C} \|\hat{T}_{zd}\|_\infty$.

The proof of this theorem is given in Appendix B.

The performance limit $\tilde{J}_{opt}(v)$ is proportional to the tracking performance limit of the system without uncertainty while it is a inverse ratio of $1-\delta^2\rho^2$. The factor $1-\delta^2\rho^2$ is determined by $\inf_{C} \|\hat{T}_{zd}\|_\infty$ which gives the largest magnitude stable margin of the system for the worst uncertainty. The former term is determined by selecting the free parameter $\hat{Q}$ of the controller C while the latter term is determined by selecting the free parameter $\hat{R}$ in the controller since the transfer function $\hat{R}_{zd}$ is given

$$\hat{T}_{zd} = \hat{N}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}.$$

The key point in the problem is to construct a worst uncertainty associated with the maximum value of $J(v)$. This is closely related to the works about the necessary and sufficient conditions of robust stabilization of LTI systems by Qiu *et al.* (1995) and Shamma (1994). Here the difficulty is that the LTI system under consideration is driven by a given external signal which is usually assumed to be zero in the existing works. To solve this problem, a new method in constructing a worst uncertainty is discussed in the proof of Theorem 2.

It is worth to note that the result in Theorem 2 also holds for a linear time varying uncertainty with certain norm bound.

5. CONCLUSION

In this paper, we discussed tracking performance limitation for an LTI plant with an external disturbance or uncertainty. It is shown that, if the external disturbance is injected into the system at the input of the plant, then the tracking performance limit under the worst disturbance is a sum of the tracking performance limit of the system without the disturbance and the disturbance attenuation performance limit of the system without the reference and tracking. Then, the minimax performance limit of asymptotical tracking/robust stabilzation is considered for the system with a nonlinear time invariant uncertainty in a special structure. The minimax performance limit of asymptotical tracking/robust stabilzation under the worst uncertainty is dependent on the optimal $\mathcal{H}_\infty$ gain from disturbance input to the output of the system, the size of the worst uncertainty and the tracking performance limit of the system without uncertainties.

APPENDIX A: PROOF OF THEOREM 1

Applying a controller from $\mathcal{K}$ into the system shown in Figure 1, we have

$$\hat{e} = \hat{r} - \hat{N}\hat{Q}\hat{r} - \hat{N}\hat{Y}\hat{W}\hat{d} + \hat{N}\hat{R}\hat{\tilde{L}}\hat{W}\hat{d}. \tag{6}$$

The tracking performance is given by

$$J(v) = \|\hat{r} - \hat{N}\hat{Q}\hat{r} - \hat{N}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2.$$

Due to the fact that an inner-outer factorization of $\hat{N}$ is given by $\hat{N} = \hat{G}_{in}\hat{N}_{out}$, $J(v)$ is written as:

$$\begin{aligned} J(v) &= \|\hat{r} - \hat{G}_{in}\hat{N}_{out}\hat{Q}\hat{r} - \hat{G}_{in}\hat{N}_{out}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2 \\ &= \|\hat{G}_{in}^{-1}\hat{r} - \hat{N}_{out}\hat{Q}\hat{r} - \hat{N}_{out}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2. \end{aligned}$$

Notice that $\hat{G}_{in}^{-1}\hat{r} - \hat{r} \in \mathcal{H}_2^\perp$ and $\hat{r} - \hat{N}_{out}\hat{Q}\hat{r} - \hat{N}_{out}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d} \in \mathcal{H}_2$ by selecting proper $\hat{Q}$. Then, we have

$$\begin{aligned} J(v) = \|\hat{G}_{in}^{-1}\hat{r} - \hat{r}\|_2^2 &+ \|\hat{r} - \hat{N}_{out}\hat{Q}\hat{r}\|_2^2 \\ &+2\langle \hat{r} - \hat{N}_{out}\hat{Q}\hat{r}, -\hat{N}_{out}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\rangle \\ &+\|\hat{N}_{out}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d})\|_2^2. \end{aligned} \tag{7}$$

For $\|d\|_2 \le \delta$, in the worst case, the third term on the far right is positive. Therefore, for any $\hat{Q} \in \mathcal{R}H_\infty$,

$$\sup_{\|d\|_2 \le \delta} J(v) \ge \|\hat{G}_{in}^{-1}\hat{r} - \hat{r}\|_2^2 + \sup_{\|d\|_2 \le \delta_0} \|\hat{N}_{out}(\hat{Y} - \hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2.$$

Consequently, it holds

$$\inf_{\hat{Q}\in\mathcal{R}H_\infty}\sup_{\|d\|_2\le\delta} J(v) \ge \|\hat{G}_{in}^{-1}\hat{r}-\hat{r}\|_2^2 + \sup_{\|d\|_2\le\delta}\|\hat{N}_{out}(\hat{Y}-\hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2. \quad (8)$$

On the other hand, selecting $\hat{Q}$ such that

$$\|\hat{r}-\hat{N}_{out}\hat{Q}\hat{r}\| \to 0 \quad (9)$$

results in

$$\sup_{\|d\|_2\le\delta} J(v) = \|\hat{G}_{in}^{-1}\hat{r}-\hat{r}\|_2^2 + \sup_{\|d\|_2\le\delta}\|\hat{N}_{out}(\hat{Y}-\hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2. \quad (10)$$

It follows from (8)-(10) that

$$\inf_{\hat{Q}\in\mathcal{R}H_\infty}\sup_{\|d\|_2\le\delta} J(v) = \|\hat{G}_{in}^{-1}\hat{r}-\hat{r}\|_2^2 + \sup_{\|d\|_2\le\delta}\|\hat{N}_{out}(\hat{Y}-\hat{R}\hat{\tilde{L}})\hat{W}\hat{d}\|_2^2. \quad (11)$$

Denote the operator from d to z by T_{zd}. Then,

$$\hat{T}_{zd}\hat{d} = \hat{G}_{in}\hat{N}_{out}(\hat{Y}-\hat{R}\hat{\tilde{L}})\hat{W}\hat{d}.$$

From the definition of $\mathcal{H}_\infty$ norm, (11) is written

$$J_{opt}(v) = \|\hat{r}-\hat{G}_{in}\hat{r}\|_2^2 + \inf_{K\in\mathcal{K}}\|\hat{T}_{zd}\|_\infty^2\delta^2. \quad (12)$$

Applying the result by Chen *et al.* (2000) into (12) leads to

$$J_{opt} = \sum_{i=1}^{m} 2\mathrm{Re}(z_i)\left|\frac{\langle\eta_i, v\rangle}{z_i}\right|^2 + \inf_{K\in\mathcal{K}}\|\hat{T}_{zd}\|_\infty^2\delta^2.$$

APPENDIX B: PROOF OF THEOREM 2

For the simplicity, we only consider SISO systems in this proof. But it can be easily extended to the multivariable case. Here, all the discussion is carried on in the time domain and all the converted functions in the system are converted into operators in the time domain.

For this system in Figure 2, the set of all 2DOF stabilizing controller C is given by

$$\mathcal{C} = \Big\{\hat{C} = (\hat{Y}-\hat{R}\hat{\tilde{L}})^{-1}[\hat{Q} \quad (\hat{X}+\hat{R}\hat{\tilde{D}})+\hat{Q}] : \hat{Q},\hat{R}\in\mathcal{R}H_\infty \text{ and } |\hat{Y}-\hat{R}\hat{\tilde{L}}|\neq 0\Big\}.$$

Then the tracking error of the system is given by

$$e = \tilde{r} + r_Q - T_{zd}d \quad (13)$$

where $\tilde{r} = r - G_{in}r$, $r_Q = G_{in}(r - N_0Qr)$ while it follows from the discussion in Appendix A that the integral square of this tracking error is given by

$$\|e\|_2^2 = \|\tilde{r}\|_2^2 + \|r_Q - T_{zd}d\|_2^2. \quad (14)$$

Select Q such that (9) holds. Then, we have

$$\|e\|_2^2 = \|\tilde{r}\|_2^2 + \|T_{zd}d\|_2^2. \quad (15)$$

Due to $d = \Delta e$ and $\|\Delta\|_\infty \le \delta$, (15) is written

$$\begin{aligned}\|e\|_2^2 &= \|\tilde{r}\|_2^2 + \|T_{zd}\Delta e\|_2^2 \\ &\le \|\tilde{r}\|_2^2 + \|T_{zd}\|_\infty^2\delta^2\|e\|_2^2.\end{aligned} \quad (16)$$

Hence, one can see that

$$\|e\|_2^2 \le \frac{\|\tilde{r}\|_2^2}{1-\|T_{zd}\|_\infty^2\delta^2} \quad (17)$$

and $e\in\mathcal{L}_2$.

Next, we will proof that, for any given positive scalar ε_0, there exists a nonlinear time invariant uncertainty with norm bound δ, $\|\Delta\|_\infty \le \delta$ such that

$$\frac{\|\tilde{r}\|_2^2 - \varepsilon_0}{1-\|T_{zd}\|_\infty^2\delta^2} \le \|e\|_2^2. \quad (18)$$

Define $[f]_{[T_1,T_2]}$ by

$$[f]_{[T_1,T_2]} = \begin{cases} f(t), & t\in[T_1,T_2) \\ 0, & t\notin[T_1,T_2)\end{cases}$$

for any function $f(t)$.

Notice the fact that T_{zd} is a stable LTI system. Suppose ω_0 be a peak frequency of T_{zd} and let $s_k(t) = [A_k\sin\omega_0 t]_{[kT,(k+1)T]}$ where A_k is a real amplitude. Then, for any given $\varepsilon > 0$, there exists a positive T_0 such that if $T \ge T_0$, it holds

$$0 \le \|T_{zd}\|_\infty^2 - \frac{\left\|[T_{zd}s_k]_{[kT,(k+1)T]}\right\|_2^2}{\|s_k\|_2^2} \le \varepsilon. \quad (19)$$

To seek the simplicity, it is assumed that T is integral times of $\dfrac{\pi}{\omega_0}$.

Construct an uncertainty $\Delta = \Delta_2\Delta_1$. The first part Δ_1 generates a sequence of impulse signals at $t = kT$, $k = 1, 2, \cdots, \infty$ and the output of Δ_1 is

$$\Delta_1 e = \sum_{k=1}^{\infty} C_k\delta(t-kT)$$

where $C_k = \left\|[e]_{[(k-1)T,kT]}\right\|_2^2$, $k = 1, 2, \cdots, \infty$ and $\delta(t-kT)$, $k = 1, 2, \cdots, \infty$ are a unit impulse function. The second part Δ_2 is a sinusoid generator as follows:

$$[\Delta_2\Delta_1 e](t) = A_k\sin\omega_0 t, \quad t\in[kT,(k+1)T)$$

where $A_k = \delta\sqrt{\dfrac{2C_k}{T}}$. That is, the output $d(t)$ of the uncertainty is given by

$$d(t) = [\Delta_2\Delta_1 e](t) = \sum_{k=1}^{\infty} s_k(t). \quad (20)$$

After a simple calculation, we can see that

$$\left\|[d]_{[kT,(k+1)T]}\right\|_2^2 = \delta^2 C_k.$$

It is clear that, for any $e(t)\in\mathcal{H}_2$, there holds

$$\|d\|_2^2 = \|\Delta_2\Delta_1 e\|_2^2 = \delta^2\|e\|_2^2 \quad (21)$$

and $\|\Delta_2\Delta_1\|_\infty = \delta$. To show that the inequality (18) holds for this uncertainty, a lower bound of $\mathcal{L}_2$ norm of $T_{zd}d = \sum_{k=1}^{\infty} T_{zd}s_k$ is considered. To do this, $T_{zd}s_k$ is partitioned in time domain as below:

$$T_{zd}s_k = \sum_{i=1}^{\infty} [T_{zd}s_k]_{[(k+i-1)T,(k+i)T]}. \qquad (22)$$

For the stable LTI system T_{zd}, there exists a positive constant ρ_i such that

$$\|[T_{zd}s_k]_{[(k+i)T,(k+i+1)T]}\|_2 \leq \rho_i \|[T_{zd}s_k]_{[(k+i-1)T,(k+i)T]}\|_2. \qquad (23)$$

Since, for different k, the differences on $s_k(t)$ are its amplitude and the time interval which it is defined in, ρ_i is independent from k. Moreover, ρ_i is dominantly determined by an exponential function of the most slowly decaying mode of T_{zd} and T. So $\rho_i \to 0$ as $T \to \infty$. Denote the upper bound of ρ_i, $i = 1, 2, \cdots, \infty$ by ρ_T. By the same reason, $\rho_T \to 0$ as $T \to \infty$.

From (20) and (22), we have

$$T_{zd}\Delta e = \sum_{k=1}^{\infty} \left\{ [T_{zd}s_k]_{[kT,(k+1)T]} + [T_{zd}s_k]_{[(k+1)T,\infty]} \right\}. \qquad (24)$$

Following (23) and the triangular inequality, $\left\|[T_{zd}s_k]_{[(k+1)T,\infty]}\right\|$ is bounded by

$$\left\|[T_{zd}s_k]_{[(k+1)T,\infty]}\right\| \leq \sum_{i=2}^{\infty} \rho_T^{i-1} \left\|[T_{zd}s_k]_{[kT,(k+1)T]}\right\| \leq \frac{\rho_T}{1-\rho_T} \left\|[T_{zd}s_k]_{[kT,(k+1)T]}\right\|. \qquad (25)$$

On the other other, from the triangle inequality and (24), the $\mathcal{L}_2$ norm of $T_{zd}\Delta e$ is bounded by

$$\|T_{zd}\Delta e\|_2 \geq \left\| \sum_{k=1}^{\infty} [T_{zd}s_k]_{[kT,(k+1)T]} \right\|_2 - \sum_{k=1}^{\infty} \left\|[T_{zd}s_k]_{[(k+1)T,\infty]}\right\|_2. \qquad (26)$$

Substituting (25) into (26) and noticing the orthogonality among the items $[T_{zd}s_k]_{[kT,(k+1)T]}$, $k = 1, 2, \cdots, \infty$ in time domain, we have

$$\|T_{zd}\Delta e\|_2 \geq \frac{1-2\rho_T}{1-\rho_T} \sum_{k=1}^{\infty} \left\|[T_{zd}s_k]_{[kT,(k+1)T]}\right\|_2. \qquad (27)$$

It follows from (19) that

$$\left(\|T_{zd}\|_\infty^2 - \varepsilon\right) \sum_{k=1}^{\infty} \|[s_k]\|_2^2 \leq \sum_{k=1}^{\infty} \left\|[T_{zd}s_k]_{[kT,(k+1)T]}\right\|_2^2. \qquad (28)$$

From (27) and (28), we have that, for any given positive ε_0, it holds

$$\|T_{zd}\Delta e\|_2^2 \geq \|T_{zd}\|_\infty^2 \sum_{k=1}^{\infty} \|[s_k]\|_2^2 - \varepsilon_0, \quad \text{as } T \to \infty. = \|T_{zd}\|_\infty^2 \|\Delta e\|_2^2 - \varepsilon_0 \qquad (29)$$

where the second equality follows the orthogonality among the items s_k, $k = 1, 2, \cdots, \infty$ in the time domain. Substituting (21) into (29) leads to

$$\|T_{zd}\Delta e\|_2^2 \geq \|T_{zd}\|_\infty^2 \delta^2 \|e\|_2^2 - \varepsilon_0, \quad \text{as } T \to \infty. (30)$$

With (17), (16) and (30), we have

$$\|e\|_2^2 \to \frac{\|\tilde{r}\|_2^2}{1 - \delta^2 \|T_{zd}\|_\infty^2}, \quad \text{as } T \to \infty.$$

The proof is completed.

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