Tracking Performance Limitations in LTI Multivariable Discrete-Time Systems

Onur Toker, Jie Chen, and Li Qiu

***Abstract*—In this paper, we investigate tracking properties of linear shift-invariant feedback control systems. We consider the standard unity feedback configuration, and use the energy of an error signal as a measure of tracking ability. Our main goal is to understand the fundamental limitation on tracking performance, which can arise due to the nonminimum phase zeros, unstable poles, and time delays in the plant, and which varies with input reference signals. We consider step, ramp, and sinusoidal signals, and for each type of the signals we derive a closed form expression for the minimum tracking error attainable by any stabilizing controller. Our results display an explicit dependence of the tracking error on nonminimum phase zeros, unstable poles, and in particular the coupling between the directions of the poles and zeros, and those of the input reference signal, upon which a number of useful conclusions can be drawn. One interesting outcome is that not only zero and pole locations affect tracking performance, but their directional properties also play an important role. The paper provides a nontrivial extension of the previously available results to discrete-time systems, with a consideration on broader classes of reference inputs.**

***Index Terms*—MIMO discrete-time systems, nonminimum phase zeros, optimal tracking error, time delays, tracking performance, unstable poles.**

I. Introduction

In THIS paper, we study a tracking performance problem for multi-input multi-output (MIMO), linear, shift-invariant systems posed in a unity feedback control scheme. The energy of an error signal, which is the difference between a given reference input and its output response, is used as performance measure. We are interested in the optimal performance that can be achieved by all stabilizing compensators, and more importantly, in how plant properties may limit the best performance achievable. We consider a class of benchmark reference inputs, including step, ramp, and sinusoidal signals. With respect to each of these signals, we show that the optimal performance depends critically upon the locations and directions of the unstable poles and nonminimum phase zeros in the plant transfer function matrix. We quantify the effects of these zeros and poles explicitly by deriving closed form expressions for the minimal tracking error.

Performance limitations resulting from plant nonminimum phase zeros and unstable poles have been known for a long time. For example, earlier studies of Bode and Poisson type integrals [10], [2], [15], [12], [3], [4], [19] show that they impose inherent limitations on a system's ability to reduce sensitivity and hence the ability to attenuate disturbance signals. Similarly, results in $\mathcal{H}_\infty$ optimal control suggest that such zeros and poles lead to irreducible lower bounds on the best achievable performance defined under $\mathcal{H}_\infty$ criteria [13], [22], [5]. Other pertinent results are found in problems concerning cheap control [14], [17], [20], LQG/LTR design [23], and optimal reference tracking [16], [7], pointing to fundamental constraints in attaining various feedback design objectives. The present paper continues these earlier studies, and in particular builds on the authors' recent work [7], which was focused on the tracking of a step reference signal in the continuous-time setting. Here we derive similar expressions for the best achievable tracking performance for MIMO discrete-time systems. Like their predecessors, these expressions demonstrate in a clear manner how the tracking performance may be limited by plant nonminimum phase zeros and unstable poles, and especially in how it may depend on the directions of such zeros and poles in a MIMO discrete-time system. Most notably, it will be seen that the relative orientation between the reference input direction and the directions of plant nonminimum phase zeros and unstable poles plays a central role to this effect, and that this orientation can be precisely quantified via an angular measure known as the *principal angle* between the directions. The results thus reinforce the existing work and extend it to discrete-time systems and to broader classes of reference input signals.

The remainder of this paper is organized as follows. In Section II, we formulate the tracking problem and state some preliminary facts concerning nonminimum phase systems. In Section III, we examine the tracking performance with respect to a generalized step reference input. Section IV addresses sinusoidal and ramp reference signals. Section V studies tracking performance limitations in time-delay systems. Section VI presents an illustrative example, and Section VII provides a number of concluding statements.

II. Preliminaries

We begin by summarizing briefly the notation used throughout this paper. For any complex number z, we denote its complex conjugate by $\overline{z}$. For any vector u, we denote its conjugate transpose by u^H, and its Euclidean norm by $\|u\|$.

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For a matrix A, we denote its conjugate transpose by A^H, and its column space by $\mathbb{R}[A]$. If A is a Hermitian matrix, we denote its largest and smallest eigenvalues by $\lambda_{\max}$ and $\lambda_{\min}$, respectively. All the vectors and matrices involved in the sequel are assumed to have compatible dimensions, and for simplicity their dimensions will be omitted. Let the open unit disc be denoted by $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$, the closed unit disc by $\overline{\mathbb{D}} := \{z \in \mathbb{C}\colon |z| \leq 1\}$, the unit circle by $\partial\mathbb{D} := \{z \in \mathbb{C}\colon |z| = 1\}$, and the complement of $\overline{\mathbb{D}}$ by $\overline{\mathbb{D}}^c := \{z \in \mathbb{C}\colon |z| > 1\}$. Define

$$\mathcal{L}_2 := \Bigg\{ f\colon\ f(z) \text{ measurable in } \partial\mathbb{D} \quad \|f\|_2 := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|f\left(e^{j\theta}\right)\right\|^2 d\theta\right)^{1/2} < \infty \Bigg\}.$$

Then, $\mathcal{L}_2$ is a Hilbert space with an inner product

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f^H\left(e^{j\theta}\right) g\left(e^{j\theta}\right) d\theta.$$

Next, define

$$\mathcal{H}_2 := \Bigg\{ f\colon\ f(z) \text{ analytic in } \overline{\mathbb{D}}^c \quad \|f\|_2 := \left(\sup_{r>1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|f\left(re^{j\theta}\right)\right\|^2 d\theta\right)^{1/2} < \infty \Bigg\}$$

and

$$\mathcal{H}_2^{\perp} := \Bigg\{ f\colon\ f(z) \text{ analytic in } \mathbb{D} \quad \|f\|_2 := \left(\sup_{r<1} \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|f\left(re^{j\theta}\right)\right\|^2 d\theta\right)^{1/2} < \infty \Bigg\}.$$

It is well-known that $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ are subspaces and form an orthogonal pair of $\mathcal{L}_2$. Similarly, define $\mathcal{H}_\infty$ as the space of all complex-valued matrix functions which are bounded and analytic in $\mathbb{D}^c$, and $\mathbb{R}\mathcal{H}_\infty$ the space of all rational matrix functions in $\mathcal{H}_\infty$. Note that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ defined here differ from the conventional Hardy spaces, which are usually defined over $\mathbb{D}$, instead of $\mathbb{D}^c$. However, this slight deviation in notation will prove more convenient in our later presentation. Note also that for each of the normed spaces $\mathcal{L}_2$, $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$ we have used the same notation $\|\cdot\|_2$ to denote the corresponding norm; this will be clear from the context as well. Finally, for any sequence $u[n]$, we define its $\mathcal{Z}$-transform by

$$\hat{u}(z) := \sum_{n=0}^{\infty} u[n] z^{-n}.$$

We shall consider the unity feedback control system depicted in Fig. 1, in which P represents a linear shift-invariant plant and K a stabilizing compensator. For a given input signal u, we define the tracking error as

$$J := \sum_{k=0}^{\infty} \|e[k]\|^2 = \sum_{k=0}^{\infty} \|y[k] - u[k]\|^2.$$

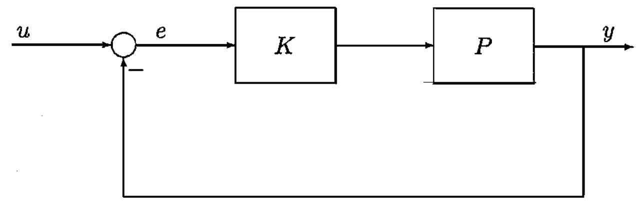

Fig. 1. The unity feedback system.

The best tracking performance is measured by the minimal possible tracking error achievable by all linear shift-invariant stabilizing compensators, determined as

$$J^* := \inf_{K \text{ stabilizes } P} J.$$

Let a right and left coprime factorization of the plant transfer function matrix $P(z)$ be given by

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N} \tag{2.1}$$

where $N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$ and satisfy the double Bezout identity

$$\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & Y \\ N & X \end{bmatrix} = I \tag{2.2}$$

for some $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$. Then, all the stabilizing compensators K can be characterized by the set [21]

$$\mathcal{K} := \Big\{ K\colon\ K = (Y - MQ)(NQ - X)^{-1} = \left(Q\tilde{N} - \tilde{X}\right)^{-1}\left(\tilde{Y} - Q\tilde{M}\right),\ Q \in \mathbb{R}\mathcal{H}_\infty \Big\}. \tag{2.3}$$

If $P(z)$ is stable, we may choose $N = \tilde{N} = P$, $\tilde{X} = M = I$, $X = \tilde{M} = I$, $Y = 0$, $\tilde{Y} = 0$, and $\mathcal{K}$ can be further simplified to

$$\mathcal{K} = \big\{ K\colon\ K = Q(I - PQ)^{-1} = (I - QP)^{-1}Q,\ Q \in \mathbb{R}\mathcal{H}_\infty \big\}. \tag{2.4}$$

Let the system sensitivity function be defined by

$$S(z) := [I + P(z)K(z)]^{-1}.$$

It follows that $\hat{e}(z) = S(z)\hat{u}(z)$, and further

$$J = \|S(z)\hat{u}(z)\|_2^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|S\left(e^{j\theta}\right)\hat{u}\left(e^{j\theta}\right)\right\|^2 d\theta. \tag{2.5}$$

Throughout this paper we shall impose the following assumptions.

Assumption 2.1: $P(z)$ has full row rank for at least one z.

Assumption 2.2: $P(z)$ has only distinct poles in $\overline{\mathbb{D}}^c$.

Assumption 2.1 is standard and was made in, e.g., [14], [17], [7]. This assumption guarantees that the plant transfer function matrix be right invertible, which is necessary for insuring that the tracking error be finite. Assumption 2.2, on the other hand, is a technical one, intended mainly for simplifying our subsequent analysis. It can be relaxed at the expense of more complex expressions. On occasions we will also assume that $P(z)$ has only distinct zeros in $\overline{\mathbb{D}}^c$. This will be made clear as we proceed.

In the remainder of this section, we introduce a factorization formula for nonminimum phase discrete-time systems. Consider a right-invertible matrix function $P(z)$. A complex number $s \in \mathbb{C}$ is said to be a zero of $P(z)$ if $\eta^H P(s) = 0$ for some unitary vector η, where η is called an output direction vector associated with s, and $\|\eta\| = 1$. Suppose that $s \in \mathbb{D}^c$. Then s is said to be a nonminimum phase zero and $P(z)$ a nonminimum phase transfer function matrix. For such a zero, it is always true that $\eta^H N(s) = 0$, for some unitary vector η. On the other hand, a complex number $p \in \mathbb{C}$ is said to be a pole of $P(z)$ if $P(z)$ becomes unbounded at $z = p$. If $p \in \mathbb{D}^c$, i.e., p is an unstable pole of $P(z)$, then an equivalent statement is that $\tilde{M}(p)w = 0$ for some unitary vector w, $\|w\| = 1$. The unitary vector w may be conveniently termed an input pole direction vector associated with p. Throughout this paper, we shall assume that the plant does not have a nonminimum phase zero and an unstable pole at the same location.

It is well-known [2], [6] that any nonminimum phase, right invertible transfer function matrix $P(z)$ can be factorized in the form of

$$P(z) = L(z)P_m(z) \tag{2.6}$$

where $L(z)$ is an allpass factor containing all the nonminimum phase zeros of $P(z)$, and $P_m(z)$ has no nonminimum phase zero and hence is said to be minimum phase. Let $s_i \in \overline{\mathbb{D}}^c$, $i = 1, \ldots, k$, be the (finite) nonminimum phase zeros of $P(z)$. Then, one specific factorization can be constructed as

$$L(z) = \prod_{i=1}^{k} L_i(z)$$

$$L_i(z) := \frac{1-\overline{s_i}}{1-s_i}\,\frac{z-s_i}{1-\overline{s_i}z}\,\eta_i\eta_i^H + U_iU_i^H. \tag{2.7}$$

Here, η_i are unitary vectors obtained by factorizing the zeros one at a time, and U_i are matrices which together with η_i form a unitary matrix. Specifically, one can compute η_1 directly from $\eta_1^H P(s_1) = 0$, which gives rise to the factorized form $P(z) = L_1(z)P^{(1)}(z)$. Next, η_2 can be obtained from $\eta_2^H P^{(1)}(s_2) = 0$. This procedure is then continued until all the nonminimum phase zeros are factorized.

It follows rather evidently that a nonminimum phase, left invertible transfer function matrix $P(z)$ can be factorized in the form of $P(z) = \tilde{P}_m(z)\tilde{L}(z)$. Moreover, since the nonminimum phase zeros of $P(z)$ coincide with those of $N(z)$, the latter admits a similar factorization

$$N(z) = L(z)N_m(z) \tag{2.8}$$

where $L(z)$ is given in (2.7). Note that similar factorizations were used previously in [23], [6]. A special property with the present construction, however, is that $L(1) = I$. This property will facilitate our subsequent derivations. Finally, note that for a discrete-time system time delays can be interpreted as nonminimum phase zeros at the point of infinity; the factorization formula (2.7) accommodates these zeros as well in the limit.

We conclude this section by introducing the following angular measure. Given a unitary vector u, we call the one-dimensional subspace spanned by u the *direction* of u. For any two unitary vectors u, v, we define the angle $\angle(u, v)$ between their directions by

$$\cos\angle(u, v) := \left|u^H v\right|$$

which is often known as *principal angle* [1], [11] and has been shown to be useful in measuring geometrical orientations of zeros and poles in multivariable systems [3], [4], [6]. We say that the two directions are parallel if $\cos\angle(u, v) = 1$, and that they are orthogonal if $\cos\angle(u, v) = 0$.

III. Tracking Step Signals

In this section, we study the tracking performance problem pertaining to a vector version of the step signal, defined by

$$u[n] = \begin{cases} v, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0. \end{cases} \tag{3.1}$$

Here, v is a constant unitary vector. This signal may be interpreted as a generalization to the scalar unit step signal, or a unit step reference input with a specific direction determined by the vector v. The $\mathcal{Z}$-transform of $u[n]$ is

$$\hat{u}(z) = \frac{z}{z-1}\,v.$$

By virtue of (2.5), it is clear that the sensitivity function $S(z)$ must have a zero at $z = 1$ in such a way that $S(1)v = 0$, in order for J to be finite. In other words, it is necessary to have an integrator in the open loop system. This necessitates the following assumption on the plant transfer function matrix.

Assumption 3.1: $v \in \mathbb{R}[P(1)]$.

The condition follows from the expansion of $S(z)$ in terms of

$$S(z) = I - P(z)[I + K(z)P(z)]^{-1}K(z) \tag{3.2}$$

and the requirement $S(1)v = 0$. Note that Assumption 3.1 does not rule out the possibility that $P(z)$ may have a zero at $z = 1$; instead, it only requires that the input must enter from a direction lying in the column space of $P(1)$. This property is a fundamental one. While for single-input single-output (SISO) systems, one can track a step input only when the plant transfer function has no zero at $z = 1$, it is possible to do so in a multivariable system even when it does have a zero at $z = 1$, as long as the signal direction is properly aligned. Of course, if $P(z)$ has no zero at $z = 1$, such a condition is always satisfied for a right invertible plant.

A. Stable Plants

We begin our investigation with stable plants. According to (2.4) and (2.5), the minimal tracking error in this case can be expressed as

$$J^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\|\frac{(I - P(z)Q(z))v}{z-1}\right\|_2^2. \tag{3.3}$$

Our following theorem gives a closed form expression for J^*. This result and its derivation are most useful for highlighting the main conceptual insights, and for ushering in the key techniques used in the paper. We shall consider first plants such that $P(z)$ has no zero at ∞; in other words, $P(z)$ does not contain time delays. It will prove more advantageous to treat delays explicitly, and this is deferred to Section V.

Theorem 3.1: Let u be the step input signal defined by (3.1). Assume that $P(z)$ is stable, and that it has no zero at ∞. Let $P(z)$ be factorized as in (2.6). Furthermore, suppose that Assumptions 2.1 and 3.1 hold. Then

$$J^* = \sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} \cos^2 \angle(\eta_i, v). \tag{3.4}$$

Proof: In view of (3.3) and (2.6), we write first

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - L(z)P_m(z)Q(z)) \frac{v}{z-1} \right\|_2^2.$$

Noting that $L_i(z)$ is allpass, J^* can be further written as

$$\begin{aligned} J^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L_1^{-1}(z) - \left(\prod_{i=2}^{k} L_i(z) \right) P_m(z)Q(z) \right) \cdot \frac{v}{z-1} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L_1^{-1}(z) - I \right) \frac{v}{z-1} + \left(I - \left(\prod_{i=2}^{k} L_i(z) \right) P_m(z)Q(z) \right) \frac{v}{z-1} \right\|_2^2. \end{aligned}$$

Since $L_i(1) = I$, it follows that

$$\left(L_1^{-1}(z) - I \right) \frac{v}{z-1} \in \mathcal{H}_2^\perp.$$

Also, since $v \in \mathbb{R}[P(1)]$, it is possible to find a $Q \in \mathbb{R}\mathcal{H}_\infty$ such that $(I - P(1)Q(1))v = 0$; otherwise, J^* cannot be finite. Under this condition, we have

$$\left(I - \left(\prod_{i=2}^{k} L_i(z) \right) P_m(z)Q(z) \right) \frac{v}{z-1} \in \mathcal{H}_2.$$

By the fact that $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$ are orthogonal complements in $\mathcal{L}_2$, it follows that

$$J^* = \left\| \left(L_1^{-1}(z) - I \right) \frac{v}{z-1} \right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(I - \left(\prod_{i=2}^{k} L_i(z) \right) P_m(z)Q(z) \right) \frac{v}{z-1} \right\|_2^2$$

and accordingly

$$J^* = \sum_{i=1}^{k} \left\| \left(L_i^{-1}(z) - I \right) \frac{v}{z-1} \right\|_2^2 + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P_m(z)Q(z)) \frac{v}{z-1} \right\|_2^2. \tag{3.5}$$

Note now that under Assumption 2.1 $P_m(z)$ is right invertible. A well-known fact from [21] then dictates that

$$\inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P_m(z)Q(z)) \frac{v}{z-1} \right\|_2^2 = 0.$$

Consequently, we arrive at

$$\begin{aligned} J^* &= \sum_{i=1}^{k} \left\| \left(L_i^{-1}(z) - I \right) \frac{v}{z-1} \right\|_2^2 \\ &= \sum_{i=1}^{k} \left\| (I - L_i(z)) \frac{v}{z-1} \right\|_2^2. \end{aligned}$$

Using the expression in (2.7), we may evaluate J^* explicitly as follows. First, we note that

$$(I - L_i(z)) \frac{v}{z-1} = \left(\eta_i^H v \right) \frac{|s_i|^2 - 1}{1 - s_i} \eta_i \frac{1}{1 - \bar{s}_i z}.$$

Since $1/(1 - \bar{s}_i z) \in \mathcal{H}_2$, it can be expanded as

$$\frac{1}{1 - \bar{s}_i z} = -\frac{1}{\bar{s}_i z} \sum_{n=0}^{\infty} \left(\frac{1}{\bar{s}_i z} \right)^n.$$

Therefore

$$\begin{aligned} \left\| (I - L_i(z)) \frac{v}{z-1} \right\|_2^2 &= \left| \eta_i^H v \right|^2 \frac{\left(|s_i|^2 - 1 \right)^2}{|1 - s_i|^2 |s_i|^2} \sum_{n=0}^{\infty} |s_i|^{-2n} \\ &= \frac{|s_i|^2 - 1}{|1 - s_i|^2} \left| \eta_i^H v \right|^2. \end{aligned}$$

The proof is now completed. ∎

Theorem 3.1 gives a complete characterization on how plant nonminimum phase zeros may affect the tracking performance with respect to step inputs. The result is appealing in several regards: not only is it unavailable previously even for SISO discrete-time systems, but also it exhibits important properties only found in multivariable systems. From this theorem, it is clear that zeros farther outside the unit circle have less significant an effect on the tracking performance. In fact, zeros close to the unit circle may not have a significant effect either; only those close to the point $z = 1$ will be most dominant. To illustrate this point, denote $s_i = r_i e^{j\theta_i}$ and rewrite J^* as

$$J^* = \sum_{i=1}^{k} \frac{r_i^2 - 1}{r_i^2 - 2r_i \cos\theta_i + 1} \cos^2 \angle(\eta_i, v).$$

Note that only when $\cos\theta_i > 1/r_i$,

$$\frac{r_i^2 - 1}{r_i^2 - 2r_i \cos\theta_i + 1} > 1.$$

This implies that the zeros in the first and fourth quadrants play a relatively more significant role. Furthermore, even for such zeros, one can see that J^* decreases monotonically with $|\theta_i|$. Hence, the zeros close to $z = 1$ have a more negative effect. In the limit

$$\frac{r_i^2 - 1}{r_i^2 - 2r_i \cos\theta_i + 1} \to \infty$$

when $s_i \to 1$, while as $s_i \to -1$

$$\frac{r_i^2 - 1}{r_i^2 - 2r_i \cos\theta_i + 1} \to 0.$$

More importantly, in a spirit similar to that of [7], the theorem shows that in a multivariable system the tracking performance depends not only on the zero locations, but also on their directional properties. The latter dependence is captured by the directional vectors η_i and their orientations relative to the input signal direction. Clearly, a nonminimum phase zero s_i will not affect significantly the tracking performance if the corresponding vector η_i is properly aligned with v, specifically when the subspace spanned by η_i is nearly orthogonal to the input direction. The implication of this phenomenon is a fundamental one. While perfect tracking is never possible for nonminimum phase SISO plants, it can be achieved in a MIMO system. This is immediately clear by examining the limiting case where the plant has one single nonminimum phase zero s with a direction vector η, for which the minimal tracking error becomes

$$J^* = \frac{|s|^2 - 1}{|s-1|^2} \cos^2 \angle(\eta, v). \tag{3.6}$$

It follows that $J^* = 0$ whenever the zero direction is orthogonal to that of the input signal, hence resulting in perfect tracking. This in fact is possible in more general situations. Indeed, a little thought indicates that J^* can be written alternatively as

$$J^* = v^H \left(\sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} \eta_i \eta_i^H \right) v. \tag{3.7}$$

This in turn suggests that J^* can always be made zero provided that the matrix

$$\sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} \eta_i \eta_i^H$$

is not full-rank, and that v is selected appropriately.

Since the minimal error depends upon the input direction, it is of interest to determine *a priori* the best and worst tracking performance possible. This is equivalent to determining

$$J^*_{\max} := \max_{\|v\|=1} J^* \tag{3.8}$$

and

$$J^*_{\min} := \min_{\|v\|=1} J^*. \tag{3.9}$$

From (3.7), it follows that

$$J^*_{\max} = \lambda_{\max} \left(\sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} \eta_i \eta_i^H \right)$$

and

$$J^*_{\min} = \lambda_{\min} \left(\sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} \eta_i \eta_i^H \right)$$

and that the least and most desirable signal directions lie in the eigenspaces corresponding to $\lambda_{\max}$ and $\lambda_{\min}$, respectively. Finally, consider any two nonminimum phase zeros s_i and s_j. A simple calculation yields

$$J_{\max} \geq \frac{|s_i|^2 - 1}{2|s_i - 1|^2} + \frac{|s_j|^2 - 1}{2|s_j - 1|^2} + \sqrt{\left(\frac{|s_i|^2 - 1}{2|s_i - 1|^2} + \frac{|s_j|^2 - 1}{2|s_j - 1|^2} \right)^2 - \frac{|s_i|^2 - 1}{|s_i - 1|^2} \cdot \frac{|s_j|^2 - 1}{|s_j - 1|^2} \sin^2 \angle(\eta_i, \eta_j).}$$

This expression shows that the zeros may themselves couple to affect the tracking performance.

B. Unstable Plants

For unstable plants, the doubly coprime factorizations (2.1) and Youla parameterization (2.3) lead to $S(z) = (X(z) - N(z)Q(z))\tilde{M}(z)$. Under Assumption 3.1, there exists a $Q \in \mathbb{R}\mathcal{H}_\infty$ such that $(X(1) - N(1)Q(1))\tilde{M}(1)v = 0$, and for such a Q

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(X(z)\tilde{M}(z) - N(z)Q(z)\tilde{M}(z) \right) \frac{v}{z-1} \right\|_2^2$$

is well-defined. Our following result extends Theorem 3.1, which shows how plant unstable poles may affect the tracking performance.

Theorem 3.2: Let u be the step input signal defined in (3.1). Assume that $P(z)$ has no zero at ∞. Furthermore, suppose that Assumptions 2.1 and 3.1 hold, and that $N(z)$ is factorized as in (2.8). Then

$$J^* = \sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} \cos^2 \angle(\eta_i, v) + v^H H v \tag{3.10}$$

where

$$H = \sum_{i,\, j \in \mathbb{N}} \frac{(|p_i|^2 - 1)(|p_j|^2 - 1)}{\overline{b}_i b_j (1 - \overline{p}_i)(1 - p_j)(\overline{p}_i p_j - 1)} \left(I - L^{-1}(p_i) \right)^H \cdot \left(I - L^{-1}(p_j) \right) \qquad b_i := \prod_{\substack{j \in \mathbb{N} \\ j \neq i}} \frac{p_i - p_j}{1 - p_i \overline{p}_j}$$

and $\mathbb{N}$ is the index set defined by $\mathbb{N} := \{i\colon \tilde{M}(p_i)v = 0\}$.

Proof: Using (2.8), we may first express J^* as

$$\begin{aligned} J^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(X(z)\tilde{M}(z) - L(z)N_m(z)Q(z)\tilde{M}(z) \right) \cdot \frac{v}{z-1} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L^{-1}(z)X(z)\tilde{M}(z) - N_m(z)Q(z)\tilde{M}(z) \right) \cdot \frac{v}{z-1} \right\|_2^2. \end{aligned}$$

Define

$$R_1(z) := L^{-1}(z)X(z)\tilde{M}(z) - L^{-1}(z) \tag{3.11}$$

and write

$$\begin{aligned} J^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L^{-1}(z) + R_1(z) - N_m(z)Q(z)\tilde{M}(z) \right) \cdot \frac{v}{z-1} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(L^{-1}(z) - I \right) \frac{v}{z-1} + \left(I + R_1(z) - N_m(z)Q(z)\tilde{M}(z) \right) \frac{v}{z-1} \right\|_2^2. \end{aligned}$$

Since $X(z)\tilde{M}(z) - N(z)\tilde{Y}(z) = I$, we have $R_1(z) = L^{-1}N(z)\tilde{Y}(z) = N_m(z)\tilde{Y}(z) \in \mathbb{R}\mathcal{H}_\infty$. Furthermore, since Q is required to satisfy

$$(X(1) - N(1)Q(1))\tilde{M}(1)v = \left(I + R_1(1) - N_m(1)Q(1)\tilde{M}(1)\right)v = 0$$

we have

$$\left(I + R_1(z) - N_m(z)Q(z)\tilde{M}(z)\right)\frac{v}{z-1} \in \mathcal{H}_2.$$

This, together with the fact that

$$\left(L^{-1}(z) - I\right)\frac{v}{z-1} \in \mathcal{H}_2^\perp$$

leads to

$$J^* = \left\|\left(L^{-1}(z) - I\right)\frac{v}{z-1}\right\|_2^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty}\left\|\left(I + R_1(z) - N_m(z)Q(z)\tilde{M}(z)\right)\frac{v}{z-1}\right\|_2^2.$$

Note that the first term in this expression has been evaluated explicitly in Section III-A. As such, the rest of the proof will proceed by evaluating the second term. To facilitate the evaluation, denote it by J_1^*. In addition, define $f(z) := \tilde{M}(z)v$. Since $f(p_i) = 0$ for all $i \in \mathbb{N}$, and since $f(z)$ is left invertible, it can be factorized in the form of $f(z) = m(z)b(z)$, where

$$b(z) := \prod_{i\in\mathbb{N}} \frac{z - p_i}{1 - \overline{p}_i z}$$

and $m(z)$ is left invertible in $\mathbb{R}\mathcal{H}_\infty$. Consequently,

$$J_1^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty}\left\|((I + R_1(z))v - N_m(z)Q(z)m(z)b(z)) \cdot \frac{1}{z-1}\right\|_2^2 = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty}\left\|\left(\frac{I + R_1(z)}{b(z)}v - N_m(z)Q(z)m(z)\right)\frac{1}{z-1}\right\|_2^2.$$

Let

$$R_2(z) := \frac{I + R_1(z)}{b(z)} - \sum_{i\in\mathbb{N}}\frac{1 - \overline{p}_i z}{z - p_i}\,\frac{I + R_1(p_i)}{b_i}.$$

It is easy to show that $R_2(z) \in \mathbb{R}\mathcal{H}_\infty$. This allows us to write

$$J_1^* = \inf_{Q\in\mathbb{R}\mathcal{H}_\infty}\left\|\left(\sum_{i\in\mathbb{N}}\frac{1 - \overline{p}_i z}{z - p_i}\,\frac{I + R_1(p_i)}{b_i}v + R_2(z)v - N_m(z)Q(z)m(z)\right)\frac{1}{z-1}\right\|_2^2$$
$$= \inf_{Q\in\mathbb{R}\mathcal{H}_\infty}\left\|\left(\sum_{i\in\mathbb{N}}\left(\frac{1 - \overline{p}_i z}{z - p_i} - \frac{1 - \overline{p}_i}{1 - p_i}\right)\frac{I + R_1(p_i)}{b_i}v + R_3(z)v - N_m(z)Q(z)m(z)\right)\frac{1}{z-1}\right\|_2^2$$

where

$$R_3(z) := R_2(z) + \sum_{i\in\mathbb{N}}\left(\frac{1 - \overline{p}_i}{1 - p_i}\right)\frac{I + R_1(p_i)}{b_i}.$$

It is clear that

$$\frac{1}{z-1}\left(\frac{1 - \overline{p}_i z}{z - p_i} - \frac{1 - \overline{p}_i}{1 - p_i}\right) \in \mathcal{H}_2^\perp,$$

and

$$\frac{1}{z-1}\left(R_3(z)v - N_m(z)Q(z)m(z)\right) \in \mathcal{H}_2.$$

In particular, a straightforward calculation yields

$$\frac{1}{z-1}\left(\frac{1 - \overline{p}_i z}{z - p_i} - \frac{1 - \overline{p}_i}{1 - p_i}\right) = \frac{|p_i|^2 - 1}{(z - p_i)(1 - p_i)}.$$

Therefore, we have

$$J_1^* = \left\|\sum_{i\in\mathbb{N}}\frac{1}{z - p_i}\,\frac{\left(|p_i|^2 - 1\right)(I + R_1(p_i))}{b_i(1 - p_i)}v\right\|_2^2 + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty}\left\|\left(R_3(z)v - N_m(z)Q(z)m(z)\right)\frac{1}{z-1}\right\|_2^2$$
$$= \left\|\sum_{i\in\mathbb{N}}\frac{1}{z - p_i}\,\frac{\left(|p_i|^2 - 1\right)(I + R_1(p_i))}{b_i(1 - p_i)}v\right\|_2^2$$
$$= v^H\left(\sum_{i,\,j\in\mathbb{N}}\frac{\left(|p_i|^2 - 1\right)\left(|p_j|^2 - 1\right)}{\overline{b}_i b_j(1 - \overline{p}_i)(1 - p_j)}(I + R_1(p_i))^H \cdot (I + R_1(p_j))\left\langle\frac{1}{z - p_i}, \frac{1}{z - p_j}\right\rangle\right)v.$$

The proof can now be completed by noting that $R_1(p_i)v = -L^{-1}(p_i)v$ for any $i \in \mathbb{N}$, and by invoking Cauchy theorem, which gives rise to

$$\left\langle\frac{1}{z - p_i}, \frac{1}{z - p_j}\right\rangle = -\frac{1}{2\pi j}\int_{\partial\mathbb{D}}\frac{dz}{(1 - \overline{p}_i z)(z - p_j)} = \frac{1}{\overline{p}_i p_j - 1},$$

where in the contour integration the unit circle is positively oriented. ■

While one generally expects that plant unstable poles affect negatively the tracking performance, interestingly, this may or may not be true. Theorem 3.2 exhibits that such poles will have an effect on J^* only when the plant is also nonminimum phase. This is clear since for minimum phase plants $L(z) = I$. More interestingly, even for nonminimum phase plants, the unstable poles exert their effect in a rather distinctive manner, unlike the nonminimum phase zeros. Specifically, only those poles whose input directions are parallel to that of the reference signal have a toll on J^*, while other poles do not play any role. This brings us to another major distinction between MIMO and SISO systems: while in a MIMO system plant unstable poles may or may not affect the tracking performance and this depends on whether the pole and input directions are aligned, they always do in a SISO system, whenever the plant is also nonminimum phase.

Since the matrix H contains terms such as $L^{-1}(p_i)$, when some of the poles do have directions parallel to that of the input signal, one should expect a large J^* if there are nonminimum phase zeros lying in the close vicinity of the poles, and as such the performance limitation in these cases will become particularly acute. This can be seen by examining the following limiting case. Suppose that $P(z)$ has one zero $s \in \overline{\mathbb{D}}^c$ with a direction vector η, and that it has one pole $p \in \overline{\mathbb{D}}^c$ whose direction is parallel to the input direction. Then, it can be readily deduced from (3.10) that

$$J^* = \frac{|s|^2 - 1}{|s-1|^2}\left|\frac{1-\overline{s}p}{p-s}\right|^2 \cos^2 \angle(\eta,\, v) \qquad (3.12)$$

which indicates that J^* can become exceedingly large when s and p are located nearby. Note, however, that such a scenario will never arise in a MIMO system if the pole and signal directions are not completely aligned, although it always occurs in SISO systems.

We conclude this section by pointing out that Theorem 3.2 may be extended to cases where $P(z)$ has multiple poles. Under this more complex circumstance, $b(z)$ will generally possess the form

$$b(z) = \prod_{i \in \mathbb{N}} \left(\frac{z-p_i}{1-\overline{p}_i z}\right)^{n_i}$$

where n_i is the multiplicity of the unstable pole p_i. Accordingly, $R_2(z)$ is to be constructed as

$$R_2(z) = \frac{I + R_1(z)}{b(z)} - \sum_{i\in\mathbb{N}} \sum_{k=1}^{n_i} \left(\frac{1-\overline{p}_i z}{z-p_i}\right)^k A_{ik}$$

with A_{ik} determined via the formula

$$A_{ik} = \frac{1}{(n_i-k)!} \left.\frac{d^{n_i-k}\left[\left(\frac{z-p_i}{1-\overline{p}_i z}\right)^{n_i} \frac{I+R_1(z)}{b(z)}\right]}{dz^{n_i-k}}\right|_{z=p_i}.$$

With these modifications, it can be shown similarly that

$$J_1^* = \left\|\frac{1}{z-1}\sum_{i\in\mathbb{N}}\sum_{k=1}^{n_i}\left[\left(\frac{1-\overline{p}_i z}{z-p_i}\right)^k - \left(\frac{1-\overline{p}_i}{1-p_i}\right)^k\right] A_{ik} v\right\|_2^2$$

which can be further evaluated to fine details. Unfortunately, it will only yield a rather cumbersome expression.

IV. Sinusoidal and Ramp Signals

We now extend our results to other typical classes of signals. Of main interest herein are vector versions of real sinusoids and unit ramp signals. The extension helps reveal the intricate nature of tracking performance limitations in relation to different signals. It also points to a general technique that can be applied to solve analytically tracking performance problems of similar kinds.

A. Real Sinusoids

The real sinusoids in question are defined by

$$u[n] = \begin{cases} v \sin(n\omega_0), & n \geq 0 \\ 0, & n < 0 \end{cases} \qquad (4.1)$$

whose $\mathcal{Z}$-transform is given by

$$\hat{u}(z) = \frac{v \sin(\omega_0) z}{(z - e^{j\omega_0})(z - e^{-j\omega_0})}.$$

Here ω_0 is the frequency, and v is a real constant unitary vector. Note that for the tracking error to be finite, the system is required to meet the requirements $S(e^{j\omega_0})v = 0$ and $S(e^{-j\omega_0})v = 0$. This necessitates

Assumption 4.1: $v \in \mathbb{R}[P(e^{j\omega_0})]$.

Theorem 4.1: Let u be the real sinusoid defined in (4.1). Suppose that Assumptions 2.1 and 4.1 hold. Furthermore, suppose that $P(z)$ has only distinct zeros and poles in $\overline{\mathbb{D}}^c$, and that $P(z)$ has no zero at ∞. Let $N(z)$ be factorized as in (2.8). Then,

$$J^* = v^H H_1 v + v^H H_2 v \qquad (4.2)$$

where H_1 and H_2 are defined as follows:

$$H_1 := \sin^2 \omega_0 \sum_{i,\, j=1}^{k} \overline{\alpha}_i \alpha_j \overline{\beta}_i \beta_j \left(\frac{\zeta_i^H \zeta_j}{\overline{s}_i s_j - 1}\right) w_i w_j^H$$

$$H_2 := \sin^2 \omega_0 \sum_{i,\, j \in \mathbb{N}} \frac{\overline{\mu}_i \mu_j}{\overline{b}_i b_j (\overline{p}_i p_j - 1)} \left(\Phi(p_i) - L^{-1}(p_i)\right)^H \cdot \left(\Phi(p_j) - L^{-1}(p_j)\right)$$

$$\alpha_i := \frac{1-s_i}{1-\overline{s}_i}$$

$$\beta_i := \frac{|s_i|^2 - 1}{s_i^2 - 2 s_i \cos\omega_0 + 1}$$

$$\mu_i := \frac{|p_i|^2 - 1}{p_i^2 - 2 p_i \cos\omega_0 + 1}$$

$$\zeta_i := \left(\prod_{j=i+1}^{k} L_j(s_i)\right)^{-1} \eta_i$$

$$w_i^H := \eta_i^H \left(\prod_{j=1}^{i-1} L_j(s_i)\right)^{-1}$$

$$\Phi(z) := \frac{z - e^{-j\omega_0}}{2j \sin\omega_0} L^{-1}(e^{j\omega_0}) - \frac{z - e^{j\omega_0}}{2j\sin\omega_0} L^{-1}\left(e^{-j\omega_0}\right).$$

The constants b_i and the set $\mathbb{N}$ are defined as in Theorem 3.2.

Proof: The proof is similar to that for Theorem 3.2, but with some additional nontrivial derivations. We begin with the characterization

$$J^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\|\left(L^{-1}(z) + R_1(z) - N_m(z) Q(z) \tilde{M}(z)\right)\hat{u}(z)\right\|_2^2$$

where $R_1(z)$ is defined by (3.11); this characterization is found in the proof for Theorem 3.2. By construction, we have $\Phi(e^{j\omega_0}) = L^{-1}(e^{j\omega_0})$, and $\Phi(e^{-j\omega_0}) = L^{-1}(e^{-j\omega_0})$. Hence, it follows that

$$\left(L^{-1}(z) - \Phi(z)\right)\hat{u}(z) \in \mathcal{H}_2^\perp$$

and

$$\left(\Phi(z) + R_1(z) - N_m(z) Q(z) \tilde{M}(z)\right)\hat{u}(z) \in \mathcal{H}_2.$$

The proof then proceeds as

$$\begin{aligned} J^* &= \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \Big\| \left(L^{-1}(z) - \Phi(z)\right) \hat{u}(z) \\ &\qquad + \Big(\Phi(z)+R_1(z)-N_m(z)Q(z)\tilde{M}(z)\Big)\hat{u}(z)\Big\|_2^2 \\ &= \left\|\left(L^{-1}(z) - \Phi(z)\right) \hat{u}(z)\right\|_2^2 \\ &\quad + \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \Big\|\Big(\Phi(z)+R_1(z)-N_m(z)Q(z)\tilde{M}(z)\Big)\hat{u}(z)\Big\|_2^2. \end{aligned}$$

Denote

$$J_1^* := \left\|\left(L^{-1}(z) - \Phi(z)\right) \hat{u}(z)\right\|_2^2$$

and

$$J_2^* := \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \Big\|\Big(\Phi(z)+R_1(z)-N_m(z)Q(z)\tilde{M}(z)\Big)\hat{u}(z)\Big\|_2^2.$$

It suffices to show that $J_1^* = v^H H_1 v$, and $J_2^* = v^H H_2 v$. For this purpose, we calculate J_1^* and J_2^* explicitly. First, note that we may expand $L^{-1}(z)$ as

$$L^{-1}(z) = Y + \sum_{i=1}^{k} L_k^{-1}(s_i)\cdots L_{i+1}^{-1}(s_i)L_i^{-1}(z)L_{i-1}^{-1}(s_i)\cdots L_1^{-1}(s_i) \quad (4.3)$$

where Y is some constant matrix. This is possible whenever the zeros s_i are distinct, and it can be readily observed by conducting a partial fraction expansion on $L^{-1}(z)$. Let

$$\Phi_i(z) := \frac{z-e^{-j\omega_0}}{2j\sin\omega_0} L_i^{-1}(e^{j\omega_0}) - \frac{z-e^{j\omega_0}}{2j\sin\omega_0} L_i^{-1}(e^{-j\omega_0}).$$

Then, it follows that

$$\Phi(z) = Y + \sum_{i=1}^{k} L_k^{-1}(s_i)\cdots L_{i+1}^{-1}(s_i)\Phi_i(z)L_{i-1}^{-1}(s_i)\cdots L_1^{-1}(s_i).$$

A straightforward calculation yields

$$L_i^{-1}(z) - \Phi_i(z) = -\left(\frac{1-s_i}{1-\overline{s}_i}\right) \frac{(|s_i|^2-1)(z-e^{j\omega_0})(z-e^{-j\omega_0})}{(z-s_i)(s_i^2-2s_i\cos\omega_0+1)} \eta_i\eta_i^H.$$

As a result, we have

$$\left(L^{-1}(z) - \Phi(z)\right)\hat{u}(z) = -\sin\omega_0 \sum_{i=1}^{k} \alpha_i\beta_i\left(\zeta_i w_i^H\right) v \frac{z}{z-s_i}.$$

Therefore

$$\begin{aligned} J_1^* &= \sin^2\omega_0 \left\| \sum_{i=1}^{k} \alpha_i\beta_i\left(\zeta_i w_i^H\right) v \frac{z}{z-s_i}\right\|_2^2 \\ &= \left(\sin^2\omega_0\right) v^H \left(\sum_{i,j=1}^{k} \overline{\alpha}_i\alpha_j\overline{\beta}_i\beta_j\left(\zeta_i^H\zeta_j\right) w_i w_j^H \cdot \left\langle \frac{1}{z-s_i}, \frac{1}{z-s_j}\right\rangle\right) v \\ &= \left(\sin^2\omega_0\right) v^H \left(\sum_{i,j=1}^{k} \overline{\alpha}_i\alpha_j\overline{\beta}_i\beta_j \left(\frac{\zeta_i^H\zeta_j}{\overline{s}_i s_j - 1}\right) w_i w_j^H\right) v. \end{aligned}$$

This proves $J_1^* = v^H H_1 v$. Next, we evaluate J_2^*. Toward this end, we begin with the characterization

$$J_2^* = \sin^2\omega_0 \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \left\| \left(\frac{\Phi(z)+R_1(z)}{b(z)} v - N_m(z)Q(z)m(z)\right) \cdot \frac{1}{(z-e^{j\omega_0})(z-e^{-j\omega_0})}\right\|_2^2.$$

As in the proof of Theorem 3.2, we may write

$$\frac{\Phi(z)+R_1(z)}{b(z)} v = \sum_{i\in\mathbb{N}} \left(\frac{1-\overline{p}_i z}{z-p_i}\right) \frac{\Phi(p_i)-L^{-1}(p_i)}{b_i} v + R_2(z)v$$

where $R_2(z) \in \mathbb{R}\mathcal{H}_\infty$. Define

$$\phi_i(z) := \frac{z-e^{-j\omega_0}}{2j\sin\omega_0}\left(\frac{1-\overline{p}_i e^{j\omega_0}}{e^{j\omega_0}-p_i}\right) - \frac{z-e^{j\omega_0}}{2j\sin\omega_0}\left(\frac{1-\overline{p}_i e^{-j\omega_0}}{e^{-j\omega_0}-p_i}\right)$$

and

$$R_3(z) := R_2(z) + \sum_{i\in\mathbb{N}} \phi_i(z) \frac{\Phi(p_i)-L^{-1}(p_i)}{b_i}.$$

It follows that

$$\begin{aligned} J_2^* &= \sin^2\omega_0 \inf_{Q\in\mathbb{R}\mathcal{H}_\infty} \cdot \Bigg\| \sum_{i\in\mathbb{N}} \left(\frac{1-\overline{p}_i z}{z-p_i} - \phi_i(z)\right) \frac{\Phi(p_i)-L^{-1}(p_i)}{b_i(z-e^{j\omega_0})(z-e^{-j\omega_0})} v \\ &\qquad + \frac{R_3(z)v - N_m(z)Q(z)m(z)}{(z-e^{j\omega_0})(z-e^{-j\omega_0})}\Bigg\|_2^2 \\ &= \sin^2\omega_0 \left\| \left(\sum_{i\in\mathbb{N}} \left(\frac{1-\overline{p}_i z}{z-p_i} - \phi_i(z)\right) \frac{\Phi(p_i)-L^{-1}(p_i)}{b_i} v\right) \cdot \frac{1}{(z-e^{j\omega_0})(z-e^{-j\omega_0})}\right\|_2^2. \end{aligned}$$

Since

$$\frac{1-\overline{p}_i z}{z-p_i} - \phi_i(z) = -\frac{(|p_i|^2-1)(z-e^{j\omega_0})(z-e^{-j\omega_0})}{(z-p_i)(p_i^2-2p_i\cos\omega_0+1)}$$

J_2^* becomes

$$\begin{aligned} J_2^* &= \sin^2\omega_0 \left\| \sum_{i\in\mathbb{N}} \left(\frac{|p_i|^2-1}{p_i^2-2p_i\cos\omega_0+1}\right) \cdot \left(\frac{\Phi(p_i)-L^{-1}(p_i)}{b_i}\right) \frac{v}{z-p_i}\right\|_2^2 \\ &= \left(\sin^2\omega_0\right) v^H \left(\sum_{i,j\in\mathbb{N}} \frac{\overline{\mu}_i\mu_j}{\overline{b}_i b_j(\overline{p}_i p_j - 1)} \left(\Phi(p_i)-L^{-1}(p_i)\right)^H \cdot \left(\Phi(p_j)-L^{-1}(p_j)\right)\right) v. \end{aligned}$$

This establishes the fact $J_2^* = v^H H_2 v$. The proof is now completed. ■

In much the same spirit, Theorem 4.1 demonstrates that nonminimum phase zeros and unstable poles can each exert a significant effect on a system's performance in tracking a real sinusoidal signal. This effect, however, manifests itself in a significantly more complex fashion, leading to a rather complicated expression of the minimal tracking error. Unlike in the case of tracking step signals, the zero effects are now seen to be coupled, which obscures the relationship between J^* and the zeros, and renders the analysis more difficult; the interaction between the zeros is captured by the cross terms involving ζ_i and w_i. In spite of this difficulty notwithstanding, one may be sure that certain conceptual statements remain valid. This can be partly observed by examining a number of special instances. For example, when $P(z)$ is stable and has only one zero $s \in \overline{\mathbb{D}}^c$ with a direction vector η, one obtains

$$J^* = \frac{|s|^2 - 1}{|s - e^{j\omega_0}||s - e^{-j\omega_0}|} \sin^2 \omega_0 \cos^2 \angle(\eta, v).$$

In addition, if $P(z)$ has one pole $p \in \overline{\mathbb{D}}^c$ whose direction is parallel to that of v, then

$$J^* = \frac{|s|^2 - 1}{|s - e^{j\omega_0}||s - e^{-j\omega_0}|} \left|\frac{1 - \overline{p}s}{s - p}\right|^2 \sin^2 \omega_0 \cos^2 \angle(\eta, v).$$

Both expressions follow readily from (4.2). It is clear that the nonminimum phase zeros close to $z = e^{\pm j\omega_0}$ can be particularly problematic. Other additional interpretations follow analogously.

B. Ramp Signals

The ramp signal under consideration is described by

$$u[n] = \begin{cases} nv, & n \geq 0 \\ 0, & n < 0. \end{cases} \tag{4.4}$$

Likewise, we assume that $\|v\| = 1$. The $\mathcal{Z}$-transform of $u[n]$ is given by

$$\hat{u}(z) = \frac{z}{(z-1)^2}\, v. \tag{4.5}$$

As in the previous analysis, it is easy to see that the sensitivity function must meet the requirements that $S(1)v = 0$, and $S'(1)v = 0$. Thus, the following condition is necessary.

Assumption 4.2: $v \in \mathbb{R}[P(1)] \cap \mathbb{R}[P'(1)]$.

This assumption implies that the open loop system must be of a type no less than two. For that to be possible, whenever $P(z)$ has a multiple zero at $z = 1$, v must lie in the column spaces of $P(1)$ and $P'(1)$.

Theorem 4.2: Let u be the ramp signal given in (4.4). Suppose that Assumptions 2.1 and 4.2 hold. Suppose also that $P(z)$ has only distinct zeros and poles in $\overline{\mathbb{D}}^c$, and that $P(z)$ has no zero at ∞. Let $N(z)$ be factorized as in (2.8). Then

$$J^* = v^H H_1 v + v^H H_2 v, \tag{4.6}$$

where

$$H_1 := \sum_{i,j=1}^{k} \left(\frac{|s_i|^2 - 1}{|s_i - 1|^2}\right)\left(\frac{|s_j|^2 - 1}{|s_j - 1|^2}\right)\left(\frac{\zeta_i^H \zeta_j}{\overline{s}_i s_j - 1}\right) w_i w_j^H$$

$$H_2 := \sum_{i,j \in \mathbb{N}} \frac{(|p_i|^2 - 1)(|p_j|^2 - 1)}{\overline{b}_i b_j (\overline{p}_i p_j - 1)(1 - \overline{p}_i)^2 (1 - p_j)^2} \cdot \left(\Psi(p_i) - L^{-1}(p_i)\right)^H \left(\Psi(p_j) - L^{-1}(p_j)\right)$$

$$\Psi(z) := I + (z-1) \sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2}\, \eta_i \eta_i^H.$$

Furthermore, b_i, w_i, ζ_i, and $\mathbb{N}$ are defined as in Theorem 4.1.

Proof: The proof follows the essential steps in the proof for Theorem 4.1, with $\Phi(z)$ replaced by $\Psi(z)$, and $\phi_i(z)$ by

$$\psi_i(z) := \frac{1 - \overline{p}_i}{1 - p_i} + (z-1)\, \frac{|p_i|^2 - 1}{(1 - p_i)^2}.$$

Since $L^{-1}(1) = L_i^{-1}(1) = I$, and since

$$\left.\frac{dL^{-1}(z)}{dz}\right|_{z=1} = \sum_{i=1}^{k} \left.\frac{dL_i^{-1}(z)}{dz}\right|_{z=1} = \sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2}\, \eta_i \eta_i^H$$

$\Psi(z)$ can be written alternatively as

$$\Psi(z) = L^{-1}(1) + (z-1) \left.\frac{dL^{-1}(z)}{dz}\right|_{z=1}.$$

In light of (4.3), we further have

$$L^{-1}(1) = Y + \sum_{i=1}^{k} L_k^{-1}(s_i) \cdots L_{i+1}^{-1}(s_i) L_{i-1}^{-1}(s_i) \cdots L_1^{-1}(s_i)$$

for some constant matrix Y, and additionally

$$\begin{aligned} \left.\frac{dL^{-1}(z)}{dz}\right|_{z=1} &= \sum_{i=1}^{k} L_k^{-1}(s_i) \cdots L_{i+1}^{-1}(s_i) \\ &\quad \cdot \left(\left.\frac{dL_i^{-1}(z)}{dz}\right|_{z=1}\right) L_{i-1}^{-1}(s_i) \cdots L_1^{-1}(s_i) \\ &= \sum_{i=1}^{k} L_k^{-1}(s_i) \cdots L_{i+1}^{-1}(s_i) \\ &\quad \cdot \left(\frac{|s_i|^2 - 1}{|s_i - 1|^2}\, \eta_i \eta_i^H\right) L_{i-1}^{-1}(s_i) \cdots L_1^{-1}(s_i). \end{aligned}$$

Consequently

$$\begin{aligned} \Psi(z) = Y &+ \sum_{i=1}^{k} L_k^{-1}(s_i) \cdots L_{i+1}^{-1}(s_i) \\ &\cdot \left(I + (z-1)\frac{|s_i|^2 - 1}{|s_i - 1|^2}\, \eta_i \eta_i^H\right) L_{i-1}^{-1}(s_i) \cdots L_1^{-1}(s_i) \end{aligned}$$

and

$$\begin{aligned} &L^{-1}(z) - \Psi(z) \\ &= \sum_{i=1}^{k} L_k^{-1}(s_i) \cdots L_{i+1}^{-1}(s_i) \\ &\quad \cdot \left(L_i^{-1}(z) - I - (z-1)\frac{|s_i|^2 - 1}{|s_i - 1|^2}\, \eta_i \eta_i^H\right) \\ &\quad \cdot L_{i-1}^{-1}(s_i) \cdots L_1^{-1}(s_i) \\ &= -(z-1)^2 \sum_{i=1}^{k} \frac{|s_i|^2 - 1}{|s_i - 1|^2} L_k^{-1}(s_i) \cdots L_{i+1}^{-1}(s_i) \\ &\quad \cdot \left(\eta_i \eta_i^H\right) L_{i-1}^{-1}(s_i) \cdots L_1^{-1}(s_i) \frac{1}{z - s_i}. \end{aligned}$$

Finally, in a similar manner, we also obtain

$$\frac{1-\overline{p}_i z}{z-p_i} - \psi_i(z) = -\frac{(|p_i|^2-1)(z-1)^2}{(1-p_i)^2(z-p_i)}.$$

The proof may then be completed via similar manipulations as those in the proof for Theorem 4.1. ■

Theorem 4.2 also shows that the tracking of a ramp signal depends in a rather complex way on plant nonminimum phase zeros. This result is very similar to Theorem 4.1. In fact, a close inspection reveals that the former may be regarded as the limit of the latter, in the sense described below. Denote $u_s[n]$ as the real sinusoid (4.1), and $u_r[n]$ the ramp signal (4.4). Furthermore, denote J_s^* and J_r^* as the corresponding tracking errors. It can be readily verified that

$$\Psi(z) = \lim_{\omega_0 \to 0} \Phi(z).$$

A comparison of (4.2) and (4.6) then shows that

$$J_r^* = \lim_{\omega_0 \to 0} \frac{J_s^*}{\sin^2 \omega_0} = \lim_{\omega_0 \to 0} \frac{J_s^*}{\omega_0^2}.$$

Interestingly, a similar relationship exists between $u_r[n]$ and $u_s[n]$

$$u_r[n] = \lim_{\omega_0 \to 0} \frac{u_s[n]}{\omega_0}.$$

On another account, it is of interest to compare Theorem 4.2 with Theorem 3.2. Suppose that $P(z)$ has only one zero $s \in \overline{\mathbb{D}}^c$, and one pole $p \in \overline{\mathbb{D}}^c$, so that its direction parallels to that of the signal. Under this circumstance, Theorem 4.2 gives

$$J^* = \frac{|s|^2-1}{|s-1|^4} \left| \frac{1-\overline{p}s}{p-s} \right|^2 \cos^2 \angle(\eta, v). \tag{4.7}$$

In comparison to (3.6) and (3.12), it becomes clear that a nonminimum phase zero close to $z=1$, specifically when $0 < |s-1| < 1$, has a more serious effect in tracking a ramp signal than in the case of a step input. On the other hand, for a zero farther away from $z=1$, so that $|s-1| > 1$, the error in tracking a ramp signal is relatively smaller. Since ramp signals vary faster with time, this seemingly suggests that zeros close to $z=1$ are more serious toward the tracking error in steady state, while those farther away from $z=1$ play a more significant role on the error in transience.

V. Time Delay Systems

As we have pointed out in Section II, time delays in a discrete-time system can be treated as nonminimum phase zeros at ∞. In principle, it is thus possible to generalize the preceding results directly to time-delay systems by resorting to a limiting argument. Nevertheless, in view of the complexity of zero directions, it is more instructive to study delay effects explicitly. For this purpose, we shall consider plants with measurement delays, by which we mean that the plant transfer function matrix can be expressed as

$$P_d(z) = \Lambda(z)P(z), \tag{5.1}$$

where $P(z)$ contains no delay, and

$$\Lambda(z) = \begin{bmatrix} z^{-d_1} I_{k_1} & & \\ & \ddots & \\ & & z^{-d_m} I_{k_m} \end{bmatrix}.$$

The integers d_i indicate the units of delay time, and $\sum_{i=1}^m k_i$ represents the number of output channels. To simplify our analysis, we shall also assume that $P(z)$ is stable. For a given reference input $u[n]$, denote the tracking error with respect to $P_d(z)$ by

$$J_d := \left\| [I + P_d(z)K(z)]^{-1} \hat{u}(z) \right\|_2^2$$

and accordingly, the minimal error by

$$J_d^* := \inf_{K \text{ stabilizes } P_d} J_d.$$

Furthermore, J^* denotes the minimal error in the absence of delay.

Let the right and left coprime factorizations of $P(z)$ be given by (2.1). When $P(z)$ is stable, the right and left coprime factors for $P_d(z)$ can be constructed such that

$$N_d(z) = \Lambda(z)N(z) = \Lambda(z)P(z) = P_d(z)$$
$$\tilde{N}_d(z) = \Lambda(z)\tilde{N}(z) = \Lambda(z)P(z) = P_d(z)$$

and

$$M_d(z) = \tilde{M}(z) = I.$$

It then follows that

$$\begin{aligned} J_d^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \|(I - P_d(z)Q(z))\hat{u}(z)\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \|(I - \Lambda(z)P(z)Q(z))\hat{u}(z)\|_2^2. \end{aligned}$$

Since $\Lambda(1) = I$, it is clear that Assumption 3.1 is necessary when $u[n]$ is the step signal (3.1), and Assumption 4.2 needs to be imposed when $u[n]$ is the ramp signal (4.4).

We first give the following formula for J_d^* in the case of step signals.

Theorem 5.1: Let u be the step input signal defined in (3.1), and v be partitioned compatibly with $\Lambda(z)$. Assume that $P(z)$ is stable. Furthermore, suppose that Assumptions 2.1 and 3.1 hold. Then

$$J_d^* = \sum_{i=1}^m d_i \|v_i\|^2 + J^* \tag{5.2}$$

where J^* is given by (3.4).

Proof: First, note that $\Lambda(z)$ is allpass. Hence, with the step signal (3.1), we may write

$$\begin{aligned} J_d^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(\Lambda^{-1}(z) - P(z)Q(z)\right) \frac{v}{z-1} \right\|_2^2 \\ &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(\Lambda^{-1}(z) - I\right) \frac{v}{z-1} + (I - P(z)Q(z)) \frac{v}{z-1} \right\|_2^2. \end{aligned}$$

It is clear that

$$\left(\Lambda^{-1}(z) - I\right) \frac{v}{z-1} \in \mathcal{H}_2^\perp.$$

As a result

$$\begin{aligned} J_d^* &= \left\| \left(\Lambda^{-1}(z) - I\right) \frac{v}{z-1} \right\|_2^2 \\ &\quad + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (I - P(z)Q(z)) \frac{v}{z-1} \right\|_2^2 \\ &= \left\| \left(\Lambda^{-1}(z) - I\right) \frac{v}{z-1} \right\|_2^2 + J^*. \end{aligned}$$

Since for any integer $n \geq 1$

$$\frac{z^n - 1}{z-1} = \sum_{k=0}^{n-1} z^k$$

and as such

$$\left\| \frac{z^n - 1}{z-1} \right\|_2^2 = n$$

we obtain

$$\left\| \left(\Lambda^{-1}(z) - I\right) \frac{v}{z-1} \right\|_2^2 = \sum_{i=1}^m d_i \|v_i\|^2$$

thus completing the proof. ■

Theorem 5.1 shows indeed that time delays impose also a fundamental performance limit irreducible by feedback, much as we expect of nonminimum phase zeros. One particularly appealing feature about this result dwells on the fact that when tracking a step signal, the delay effects are completely independent of those due to plant nonminimum phase zeros. This phenomenon, however, cannot be observed in general, as evidenced by our next result.

Theorem 5.2: Let u be the ramp input signal defined in (4.4), and v be partitioned compatibly with $\Lambda(z)$. Assume that $P(z)$ is stable, and that it is factorized in (2.6). Furthermore, suppose that Assumptions 2.1 and 4.2 hold. Then

$$J_d^* = \tfrac{1}{6} \sum_{i=1}^m (d_i - 1) d_i (2d_i - 1) \|v_i\|^2 + v^H H v \tag{5.3}$$

where

$$\begin{aligned} H &:= \sum_{i,j=1}^k \left(\frac{|s_i|^2 - 1}{|s_i - 1|^2} \right) \left(\frac{|s_j|^2 - 1}{|s_j - 1|^2} \right) \left(\frac{\zeta_i^H \zeta_j}{\overline{s}_i s_j - 1} \right) \\ &\quad \cdot [I + D(\overline{s}_i - 1)] w_i w_j^H [I + D(s_j - 1)] \\ D &:= \begin{bmatrix} d_1 I_{k_1} & & \\ & \ddots & \\ & & d_m I_{k_m} \end{bmatrix} \end{aligned}$$

and w_i, ζ_i are defined in Theorem 4.1.

Proof: Define $G(z) := I + D(z-1)$. It follows readily that

$$\left(\Lambda^{-1}(z) - G(z)\right) \frac{v}{(z-1)^2} \in \mathcal{H}_2^\perp.$$

Hence, we have

$$\begin{aligned} J_d^* &= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left(\Lambda^{-1}(z) - G(z)\right) \frac{v}{(z-1)^2} \right. \\ &\qquad \left. + (G(z) - P(z)Q(z)) \frac{v}{(z-1)^2} \right\|_2^2 \\ &= \left\| \left(\Lambda^{-1}(z) - G(z)\right) \frac{v}{(z-1)^2} \right\|_2^2 \\ &\quad + \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| (G(z) - P(z)Q(z)) \frac{v}{(z-1)^2} \right\|_2^2. \end{aligned}$$

We claim that

$$\begin{aligned} &\left\| \left(\Lambda^{-1}(z) - G(z)\right) \frac{v}{(z-1)^2} \right\|_2^2 \\ &\qquad = \tfrac{1}{6} \sum_{i=1}^m (d_i - 1) d_i (2d_i - 1) \|v_i\|^2. \end{aligned}$$

This follows by noting that for any integer $n \geq 1$

$$z^n - 1 - n(z-1) = (z-1)^2 \sum_{k=1}^{n-1} (n-k) z^{k-1}$$

and hence

$$\begin{aligned} \left\| \frac{z^n - 1 - n(z-1)}{(z-1)^2} \right\|_2^2 &= \left\| \sum_{k=1}^{n-1} (n-k) z^{k-1} \right\|_2^2 \\ &= \sum_{k=1}^{n-1} k^2 = \tfrac{1}{6}(n-1)n(2n-1). \end{aligned}$$

The rest of the proof then proceeds as in that for Theorem 4.2. ■

Despite that, time delays and nonminimum phase zeros will generally interact to affect the tracking error, an interesting outcome from (5.3) is that partial delay effects can be independently characterized, which are captured by the first term in (5.3). For a minimum phase plant, it follows that

$$J_d^* = \tfrac{1}{6} \sum_{i=1}^m (d_i - 1) d_i (2d_i - 1) \|v_i\|^2. \tag{5.4}$$

This in turn furnishes a complete characterization of the delay effects. Note that for $d_i \geq 3$,

$$\tfrac{1}{6}(d_i - 1) d_i (2d_i - 1) > d_i$$

which indicates that in a time delay system it is generally more difficult to track a ramp input than a step signal. Note also that if $d_i = 1$, then the delay z^{-d_i} contributes no effect on J_d^*. A deeper investigation reveals that this observation can be extended to a more general conclusion, that for a minimum phase plant, a delay unit will generally have no effect on the tracking performance whenever the $\mathcal{Z}$-transform of the reference input is a rational function with an order greater than the delay time of that unit. As such, in order to achieve perfect tracking, the variation speed of the reference input will necessarily impose a limit on the allowable delay time.

VI. An Illustrative Example

We now construct an illustrative example. Consider a plant whose transfer function matrix is given by

$$P(z) = \begin{bmatrix} \dfrac{0.1a}{z+0.1} & 0 & \dfrac{1}{z^2+0.1z+0.2} \\ \dfrac{z-1.1}{z+0.1} & 0 & 0 \\ 0 & \dfrac{z-1.2}{(z-\epsilon)(z+0.2)} & \dfrac{1}{z^2+0.1z+0.2} \end{bmatrix}.$$

This plant is invertible, but is nonminimum phase. The two nonminimum phase zeros are located at $s_1 = 1.1$ and $s_2 = 1.2$, with output zero direction vectors as

$$\tilde{\eta}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \tilde{\eta}_2 = \frac{1}{\sqrt{a^2+2}} \begin{bmatrix} -1 \\ a \\ 1 \end{bmatrix}$$

respectively. When factorized sequentially, the corresponding direction vectors are

$$\eta_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \eta_2 = \frac{1}{\sqrt{a^2+0.1953}} \begin{bmatrix} 0.3125 \\ a \\ -0.3125 \end{bmatrix}.$$

It follows that

$$\cos\angle(\tilde{\eta}_1, \tilde{\eta}_2) = \frac{|a|}{\sqrt{a^2+2}}$$
$$\cos\angle(\eta_1, \eta_2) = \frac{|a|}{\sqrt{a^2+0.1953}}.$$

As such, both angles vary from zero to $\pi/2$ as a takes values in $[0, \infty)$. Moreover, depending on the value of ϵ, the plant may or may not be stable. For any ϵ such that $|\epsilon| > 1$, it has an unstable pole at $z = \epsilon$ with a pole direction vector

$$w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

With the plant so constructed, our purpose is to demonstrate and verify several highlights found in the preceding results concerning how zero-pole coupling and how the relative orientation between zero and pole directions, and that between zero and input signal directions, may affect the tracking performance. For this purpose, we focus on step input signals only. In all cases discussed below, we formulate the tracking problem as one of optimal $\mathcal{H}_2$ control, and use MATLAB's μ-Toolbox to compute the optimal $\mathcal{H}_2$ cost. The computational results all match precisely the minimal tracking errors evaluated using the expressions obtained in Section III.

Let us fix $a = 1$ and select the input vector v as

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad v_2 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

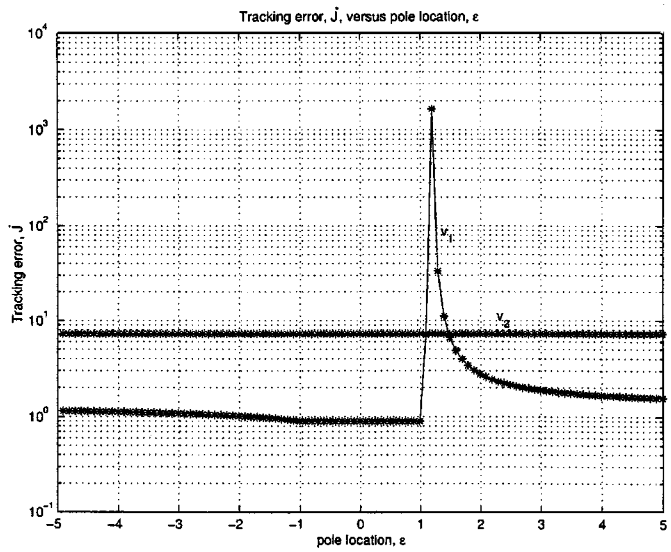

Fig. 2. J^* with respect to v_1 and v_2.

Note that for $v = v_1$, the pole and input directions are parallel, but for $v = v_2$, they are not. Hence, if $|\epsilon| > 1$, one expects that in the former case the tracking error will be larger due to the effect from the unstable pole $z = \epsilon$, and that it will become excessively large when ϵ approaches $s_1 = 1.1$ or $s_2 = 1.2$. On the other hand, when $v = v_2$, the pole $z = \epsilon$ does not play any role and hence one expects no change in the tracking error regardless of the value of ϵ. The computational results shown in Fig. 2 clearly confirm these observations. Additionally, we also observe that with $v = v_1$, J^* is a constant for $|\epsilon| < 1$, and that a gap exists between this value and the J^* corresponding to v_2; this can clearly be attributed to the different alignments between zero and signal directions in the two respective cases.

Now, fix $\epsilon = 0.5$ and construct

$$v = \frac{v_3 + \lambda v_4}{||v_3 + \lambda v_4||}$$

with

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad v_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

For $\lambda = 0$, the signal direction is orthogonal to both the zero directions, and further to the directions spanned by η_1 and η_2. Fig. 3 shows that in this case perfect tracking is achieved. When λ departs from zero, the alignment changes. As shown in the figure, the change in the value of J^* is substantial.

Finally, let us examine how $J^*_{\max}$ may vary with the alignment between $\tilde{\eta}_1$ and $\tilde{\eta}_2$, or that between η_1 and η_2. For this purpose, set $\epsilon = 0.5$ and change a from -10 to 10. When $a = 0$, the directions spanned by η_1 and η_2 are orthogonal (so are the zero directions), and Fig. 4 shows that $J^*_{\max}$ achieves its minimum. If $a \neq 0$, $J^*_{\max}$ increases with $|a|$ monotonically, to its maximum value when $a \to \pm\infty$, at which η_1 and η_2 are perfectly aligned. Note that a computation of J^* with respect to $v = v_2$ shows that not only $J^*_{\max}$, but also J^* itself can change substantially with the alignment between the direction vectors.

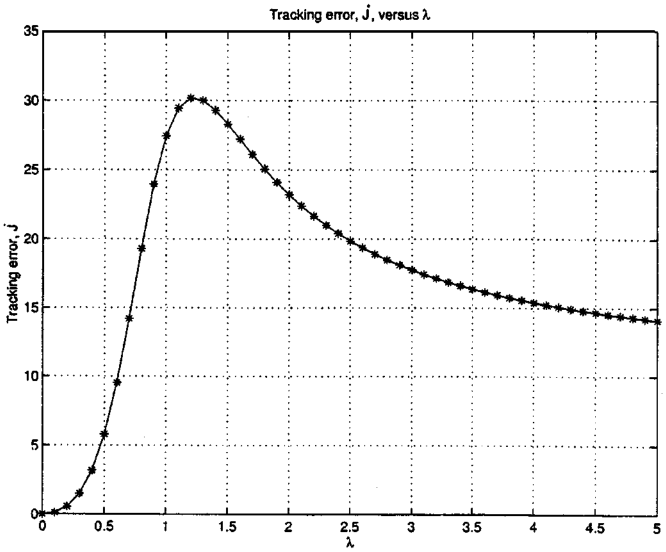

Fig. 3. J^* vs. zero and signal direction alignment.

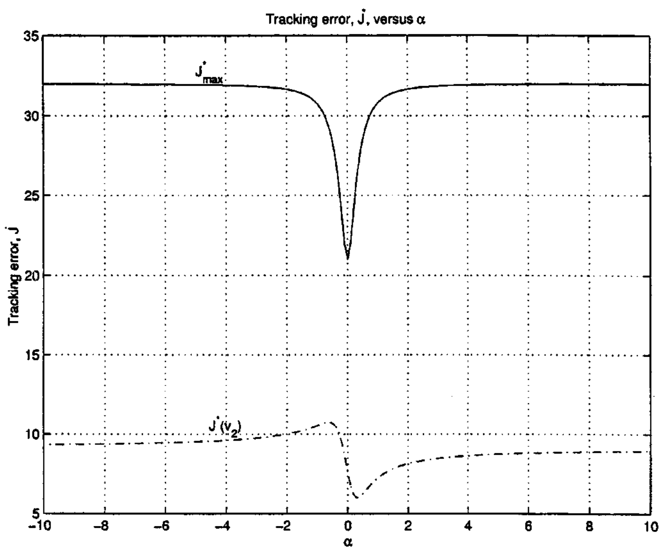

Fig. 4. $J^*_{\max}$ and J^* vs. zero direction alignment.

VII. Conclusion

Optimal tracking performance is a classical issue and has been long under examination for its intrinsic appeal and fundamental implication. This paper furnishes a study on this issue for linear, shift-invariant, MIMO discrete-time systems. The problem is investigated in conjunction with several classes of reference input signals, including step, ramp, and sinusoidal signals, which are widely held as benchmark signals for testing system performance. For each class of signals, we derived a closed form expression for the minimal tracking error. The results characterize explicitly how plant properties such as nonminimum phase zeros, unstable poles, and time delays may lead to a performance limit that cannot be overcome via the means of feedback. While in a SISO system this limit is solely determined by the location of nonminimum phase zeros and unstable poles, modulo to the effect from time delays, in a MIMO system it depends on the directional properties of such zeros and poles also. The relative orientations of the zero, pole, and input signal directions were seen to play a major role. As a fundamental consequence, it becomes clear that perfect tracking may be achieved in a MIMO nonminimum phase system, which, on the other hand, can never be possible for a SISO nonminimum phase plant. The analytic quantification of these important facts reinforces the previously known results and is the main contribution of this paper.

The tracking performance considered herein is the best possible under the use of causal, one-parameter feedback controller. It is known that tracking quality can be further improved using more general control structures and strategies. In this vein, our work can be directly generalized to two-parameter tracking systems, in a way similar to [7], which will rid of the effect by plant unstable poles. The effect by the nonminimum phase zeros may be circumvented by adopting a noncausal feedforward (e.g., preview) compensation scheme [17], [8]. Additionally, the techniques and results herein can in principle be extended to polynomial type reference inputs whose $\mathcal{Z}$-transforms may be rational functions of a higher order; it is clear from Section IV that central to our development is to construct a polynomial interpolant [$\Phi(z)$ and $\Psi(z)$] that interpolates certain values at the poles of these rational functions. Generally, such signals vary faster with time, and as a result the analysis is expected to be more complex and difficult. Finally, the optimal tracking problems can be interpreted and tackled from an optimal function interpolation perspective, as suggested in [18]. We leave these extensions and perspectives to the reader's discretion.

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