Proceedings of the 41st IEEE
Conference on Decision and Control
Las Vegas, Nevada USA, December 2002

ThM09-3

Uncertainty Equivalence Principle and $\mathcal{H}_\infty$-Based Robust Adaptive Control: Stable Plants *

Guoxiang Gu[§] and Li Qiu[†]

Abstract

A novel idea, termed as *uncertainty equivalence principle*, is proposed, based on which an equivalent measure to the $\mathcal{H}_\infty$-norm is adopted for unmodeled dynamics using time-domain measurement data. Such an equivalent description for modeling errors is consistent with $\mathcal{H}_\infty$-based robust control, and allows $\mathcal{H}_\infty$ optimization to be successfully used in adaptive control to achieve robust stability and performance comparable to $\mathcal{H}_\infty$ control. Specifically a new adaptive control systems is proposed in this paper, focusing on stable plants. It employs the recursive least-squares (RLS) algorithm for adaptive model estimation, and weighted sensitivity minimization plus robust stabilization for adaptive controller design. Our results show that the proposed adaptive control system admits robust stability and performance asymptotically, provided that the estimated plant model converges.

1 Introduction

The intellectual appealing of the notion of *self-tuning*, abundant algorithms for adaptive model estimation and controller design, and wide practice in industrial applications have made adaptive feedback control one of the central methodologies in the research community of control. Yet the rich theory and great success of adaptive control is built upon the *certainty equivalence principle* under which adaptively estimated model is taken as the true one for on-line controller design. While the *certainty equivalence principle* was instrumental to the success of adaptive control, it failed to accomplish the same in robust adaptive control. Indeed it was shown in [11] that instability may occur if unmodeled dynamics or persistent disturbances appear near the crossover frequency where the loop transfer function has a phase angle of 180° or magnitude of 0 dB. The lack of stability robustness has stimulated research activities in robust adaptive control for almost two decades. See [1, 2, 5, 6, 8, 12, 13, 14], and references therein. Robust stability was established for adaptive control systems in presence of unmodeled dynamics provided that the size of the uncertainty is suitably small. The progress in robust adaptive control is notwithstanding. There lacks a uniform lower bound on the stability margin in terms of the size of the uncertainty for several robust adaptive control schemes using normalizations [7]. It is still far away from achieving stability margins (by any adaptive control system) comparable to those achievable by $\mathcal{H}_\infty$-based robust control [3, 15, 16], which aims to design a single feedback controller having the maximum stability margin, and optimal disturbance rejection, both measured by $\mathcal{H}_\infty$-norm. Consequently robustification via various modifications of conventional adaptive laws appears to be limited.

Lack of adequate stability margins for adaptive control seems to have its root in *certainty equivalence principle* whose premise is the accurate representation of physical processes by difference/differential equations of finite and known order. Such a premise does not hold in most engineering practice. It is now widely accepted that physical processes involve uncertainties in both parameters and dynamics, if represented by transfer functions of finite McMillan degrees. The modeling error between the mathematical model and its corresponding physical process is inevitable due to nonlinearities, infinite-dimensionality, and time-varying nature of the physical system. In order to effectively cope with unmodeled dynamics and parameters variations, new approaches and methods are indispensable.

Nevertheless we are motivated by the developments of adaptive control, and $\mathcal{H}_\infty$ control in the past two decades. Both are capable of dealing with model uncertainties. Roughly speaking adaptive control is effective in tackling time-varying systems or systems with uncertain parameters, while $\mathcal{H}_\infty$ control is more powerful in coping with unmodeled dynamics. Moreover adaptive control is a time-domain approach, whereas $\mathcal{H}_\infty$ control involves optimization in frequency-domain. Although the differences between the two prevent them from unifying together meaningfully, we propose a novel approach in this paper to tackle robust adaptive control. Our goal is to maximize stability margins and to optimize the performance index for adaptive control systems via $\mathcal{H}_\infty$-based robust control. The foundation of our proposed approach is the *uncertainty equivalence principle*, contrast to the conventional *certainty equivalence principle*. In general, the modeling error in adaptive estimation can not be quantified at each time instant in terms of $\mathcal{H}_\infty$-norm based on real time data, which represents only one time sample path. Hence an equivalent measure of the modeling error will be proposed. Even though the $\mathcal{H}_\infty$-norm of the error system can not be quantified, the output signal of the error system is guaranteed to satisfy the same energy amplification constraint as the $\mathcal{H}_\infty$-norm, thereby providing an equivalent description of the dynamics uncertainty, and enabling applications of $\mathcal{H}_\infty$ optimization in adaptive control systems to achieve equivalent stability margin and performance comparable to those

*The research is supported in part by AFOSR and by the Research Grants Council of Hong Kong.

§ Department of Electrical and Computer Engineering, Louisiana State University, Baton Rouge, LA 70803-5901.
† Department of Electrical and Electronic Engineering, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.

0-7803-7516-5/02/$17.00 ©2002 IEEE

achievable in $\mathcal{H}_\infty$ control. Moreover we will propose a specific adaptive feedback control system focusing on stable plants in this paper, and prove its stability, and performance under under the condition that the adaptively estimated plant model converges. It is interesting to note that persistent excitation of the exogenous signals is not required for our proposed adaptive control system.

2 Uncertainty Equivalence Principle

In $\mathcal{H}_\infty$-based robust control, the unmodeled dynamics are assumed to be stable, and quantified by the $\mathcal{H}_\infty$-norm. However by definition, $\mathcal{H}_\infty$-norm of a stable dynamic system is the worst-case gain amplification over energy bounded signals, while the time-domain signal in practical operation is unlikely to be the worst-case one. Hence for system identification with time-domain signals, it is impractical to estimate the modeling error in $\mathcal{H}_\infty$ norm that is especially true for adaptive estimation. This seems to be a disadvantage for time-domain modeling techniques. However if we examine it further, this disadvantage is actually an advantage for adaptive feedback control systems. The pivot is the *uncertainty equivalence principle.*

Denote ℓ_+^2 as the collection of all the causal signals (which can be vector-valued for each time instance t) having bounded energy. Then for any $s(t) \in \ell_+^2$, its ℓ_2-norm is defined by

$$\|s\|_2 := \sqrt{\sum_{t=0}^{\infty} \|s(t)\|^2} = \sqrt{\sum_{t=0}^{\infty} s'(t)s(t)} < \infty.$$

Assume that the uncertainty represented by its transfer function $\Delta(z)$ is ℓ^2-BIBO stable. Then its $\mathcal{H}_\infty$-norm $\|\Delta\|_\infty$ is bounded, determined by its frequency response. Let $\mu = \{\mu(t)\}_{t=0}^{\infty} \in \ell_+^2$ be the input. Then the output $\nu = \{\nu(t)\}_{t=0}^{\infty} \in \ell_+^2$. Moreover

$$\delta = \|\Delta\|_\infty := \sup_{\omega \in \mathbf{R}} \overline{\sigma}\left(\Delta(e^{j\omega})\right) = \sup_{\mu \in \ell_+^2} \frac{\|\nu\|_2}{\|\mu\|_2} \quad (1)$$

with $\overline{\sigma}(\cdot)$ the maximum singular value. That is, $\mathcal{H}_\infty$-norm is the worst-case energy amplification, or ℓ^2-gain. The collection of all stable rational transfer functions is denoted by $\mathbf{R}\mathcal{H}_\infty$, and its closure is denoted by $\mathcal{H}_\infty$.

For any signal $s = \{s(t)\}_{t=0}^{\infty} \in \ell_+^2$, we define $\pi_T, T \geq 0$, as the projection operator satisfying

$$\pi_T\left[\{s(t)\}_{t=0}^{\infty}\right] = \{s(t)\}_{t=0}^{T}, \quad \|s\|_{[2,T]} = \|\pi_T[s]\|_2.$$

By slight abuse of notation, ℓ^2-gain can also be defined over the finite time horizon by

$$\delta_T = \|\Delta\|_{[\infty,T]} := \sup_{\mu \in \ell_+^2} \frac{\|\nu\|_{[2,T]}}{\|\mu\|_{[2,T]}}.$$

For any input/output pair (μ, ν), there hold $\|\nu\|_{[2,T]} \leq \delta_T \|\mu\|_{[2,T]}$ for $T \geq 0$, and $\delta = \lim_{T\to\infty} \delta_T$. However it is difficult to determine δ_T for each $T \geq 0$ using only the time-domain measurement data, which requires the presence of the worst-case input/output signals. Indeed for adaptive feedback control, only one time sample-path is available. If $\Delta(z)$ represents the unmodeled dynamics at time $t = T$, then it is in general infeasible to estimate δ_T, based on the time-domain measurement data, due to the possible absence of the worst-case input/output signals. Moreover the modeling error in adaptive control varies with respect to time t, and depends on the input and output of the true system, due to the use of the adaptive estimation algorithm. Hence Δ represents a time-varying nonlinear system. For this reason it is more suitable to be denoted by a nonlinear operator $\Delta_{\mathcal{N}}(\cdot)$, even though the true plant is linear and time-invariant, and the nominal plant model is parameterized by fixed order linear time-invariant systems. Such a time-varying nonlinear system is said to be ℓ^2-BIBO stable, if

$$\|\Delta_{\mathcal{N}}\|_{[\infty,T]} := \sup_{\mu \in \ell_+^2} \frac{\|\Delta_{\mathcal{N}}(\mu)\|_{[2,T]}}{\|\mu\|_{[2,T]}} < \infty \;\forall\; T \geq 0. \quad (2)$$

The transition from $\Delta(z)$ to $\Delta_{\mathcal{N}}(\cdot)$ is important. Although the frozen model uncertainty at each time t is linear, and may be represented by $\Delta(z)$, its $\mathcal{H}_\infty$-norm $\|\Delta\|_\infty$ can not be quantified based on input/output pair $(\{\mu(t)\}_{t=0}^{T}, \{\nu(t)\}_{t=0}^{T})$ in general. But if an adaptive estimation algorithm can ensure $\|\Delta_{\mathcal{N}}(\mu)\|_{[2,T]} \leq \epsilon\|\mu\|_{[2,T]}$ for some $\epsilon > 0$, and any $\{\mu(t)\}_{t=0}^{T}$, and $T \geq 0$, then ϵ can serve as an upper bound for $\|\Delta_{\mathcal{N}}(\mu)\|_{[2,T]}$. Moreover the following result can be obtained, which illustrates our proposed concept of *uncertainty equivalence principle.* See also [15].

Proposition 2.1 *Consider the uncertain feedback system in Figure 1 where both $H_{\mathcal{N}}(\cdot)$, with* 2×1 block, *and $\Delta_{\mathcal{N}}$ are ℓ^2-BIBO stable. Suppose that for any pair $(\{\mu(t)\}_{t=0}^{T}, \{\nu(t)\}_{t=0}^{T})$, $\|\nu\|_{[2,T]} \leq \epsilon\|\mu\|_{[2,T]}$, and $\|H_{\mathcal{N}}\|_{[\infty,T]} \leq \gamma$ for some $\epsilon > 0$, $\gamma > 0$, and all $T \geq 0$. Then the nonlinear feedback system in Figure 1 is stable, and satisfies*

$$\|\rho\|_{[2,T]} \leq \frac{\gamma}{1-\epsilon\gamma}\|r\|_{[2,T]} \quad \forall\; T \geq 0, \quad (3)$$

provided that $\epsilon\gamma < 1$. The upper bound $\gamma/(1-\epsilon\gamma)$ will be referred to as equivalent $\mathcal{H}_\infty$ performance.

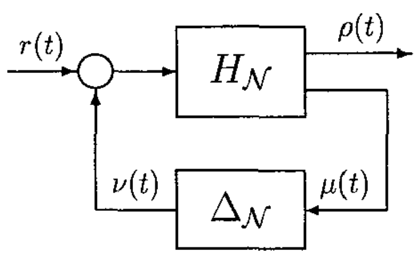

Figure 1: Uncertain Feedback System

Proposition 2.1 seems to be a very simple result. Its significance is far reaching for adaptive model estimation and adaptive controller design, because $\mathcal{H}_\infty$-norm of the uncertainty in the form of $\Delta(z)$ at each frozen time $T \geq 0$ remains unknown which may be greater than ϵ, yet stability and performance of the equivalent nonlinear feedback system are nevertheless ensured. Indeed for adaptive estimation there is no guarantee for the $\mathcal{H}_\infty$-norm of the frozen model uncertainty $\Delta(z)$ at each time instant due to the lack of the worst-case signal. But if the upper bound ϵ under which $\|\nu\|_{[2,T]} \leq \epsilon\|\mu\|_{[2,T]}$ can be estimated and ensured for each pair $(\{\mu(t)\}_{t=0}^T, \{\nu(t)\}_{t=0}^T)$, and each $T \geq 0$, then the underlying adaptive feedback control system can be converted into an equivalent nonlinear feedback system as in Figure 1 where $H_\mathcal{N}$ is a function (often a linear fractional transform) of the adaptive feedback controller, which is again nonlinear and time-varying due to its dependence on the input/output signals, even though it is often parameterized by fixed order linear systems. Hence if the adaptive feedback controller is designed appropriately such that each frozen transfer function matrix $H(z) \in \mathbf{R}\mathcal{H}_\infty$ with upper bound γ minimized, and $\gamma\epsilon << 1$ at each time instant $t = T \geq 0$, then both stability and performance of the adaptive feedback control system can be ensured. Moreover the structure of the feedback system in Figure 1 is quite general, including the so called "two-block" problem in $\mathcal{H}_\infty$-based robust control. Many meaningful and practical feedback control systems admit the two-block structure. Proposition 2.1 illustrates the essence of *uncertainty equivalence principle*, and is the foundation of our proposed $\mathcal{H}_\infty$-based robust adaptive control.

Remark 2.2 There are two key issues for the robust adaptive control problem to be studied in this paper in order for the *uncertainty equivalence principle* to be applicable. The first is associated with adaptive estimation: for a given form of model uncertainty, how to minimize the upper bound ϵ under which $\|\nu\|_{[2,T]} \leq \epsilon\|\mu\|_{[2,T]}$ holds for each $T \geq 0$. The upper bound ϵ will be called *equivalent uncertainty bound*, which is indeed an upper bound for $\|\Delta_\mathcal{N}\|_{[\infty,T]}$, when the modeling error is viewed as a nonlinear system rather than the linear one due to its dependence on the input, output, and the estimation algorithm. The second is associated with adaptive control: how to design the fixed order linear feedback controller such that the equivalent nominal feedback system $H_\mathcal{N}(\cdot)$ as in Figure 1 satisfies $\|H_\mathcal{N}\|_{[\infty,T]} \leq \gamma$ for each $T \geq 0$ where $\gamma < \epsilon^{-1}$. We would like to point out that the least-squares type algorithm is a natural candidate to approach the first issue, while $\mathcal{H}_\infty$ control design methodology is a natural candidate to the second. However both need be studied before used for $\mathcal{H}_\infty$-based robust adaptive control. ∎

In the following section we propose a new adaptive control algorithm, to not only estimate the nominal plant model, but also quantify the equivalent uncertainty bound, and design robust adaptive feedback controllers. Only stable plants are considered.

3 Robust Adaptive Control

In this section we consider adaptive feedback control for stable plants, which admit continuous frequency response. The feedback system has the standard form as in Figure 2 where the plant $P(z)$ has m-input and p-output, and is stable. Our design objective is to synthesize the feedback controller $K(z)$ such that the closed-loop system is stable, and

$$\|M^{-1}(I - K\hat{P})^{-1}\|_\infty \leq 1 \quad (4)$$

where $\hat{P}(z)$ is the approximate plant model, and $M(z)$ represents the ideal sensitivity function.

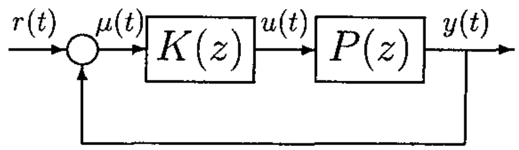

Figure 2: Feedback System

If (4) holds, then the nominal sensitivity $\left(I - K(z)\hat{P}(z)\right)^{-1}$ will be no worse then the ideal sensitivity function $M(z)$. A simple way to obtain $M(z)$ is through using some first order or second order proto-type systems. We are now led to consider adaptive estimation algorithms to identify the approximate model $\hat{P}(z)$ such that $\|\Delta M\|_\infty$ is minimized, where $\Delta(z) = P(z) - \hat{P}(z)$ is the additive uncertainty. By the small gain theorem, the feedback system in Figure 1 is stable, if $K(z)$ stabilizes $\hat{P}(z)$, and

$$\|M^{-1}K(I - \hat{P}K)^{-1}\|_\infty < \epsilon^{-1}. \quad (5)$$

The nominal performance plus robust stability condition yields the following performance index:

$$J = \left\|M^{-1}(I - K\hat{P})^{-1}\left[\begin{array}{cc} \alpha I & K \end{array}\right]\right\|_\infty \quad (6)$$

with $\alpha > 0$ a trade-off parameter. A large value of α (greater than 1) gives more weighting on the performance, while a small value of α (smaller than 1) gives more weighting to the stability margin. It is noted that $J < \epsilon^{-1}$ is required for the stability insurance in the worst-case by (5).

Let the feedback controller $K(z) = V^{-1}(z)U(z)$, and the nominal plant model $\hat{P}(z) = \hat{N}(z)M^{-1}(z)$ be their respective coprime factorizations such that the Bezout identity

$$V(z)M(z) - U(z)\hat{N}(z) = \mathcal{K}(z)G(z) = I \quad (7)$$

is satisfied, where $G(z) = \left[\begin{array}{c} \alpha^{-1}M(z) \\ \hat{N}(z) \end{array}\right]$, and

$$\mathcal{K}(z) = \left[\begin{array}{cc} \alpha V(z) & -U(z) \end{array}\right]. \quad (8)$$

Then both $\mathcal{K}(z)$, and $G(z)$ are stable, and the Bezout identity (7) is equivalent to that $\mathcal{K}(z)$ is a stable left inverse of $G(z)$. In this case the performance index has another expression

$$J = \|[\begin{array}{cc} \alpha V & -U \end{array}]\|_\infty = \|\mathcal{K}\|_\infty. \quad (9)$$

That is, our design goal is to search for a stable left inverse $\mathcal{K}(z)$ to $G(z)$ such that its $\mathcal{H}_\infty$-norm is minimized. But the difficult part is the use of the adaptive algorithms to estimate the desired approximate model $\hat{P}(z)$, and to synthesize the required feedback controller $K(z)$ at each time instant $t \geq 0$.

3.1 Adaptive Model Estimation via RLS

For the purpose of adaptive estimation, we write $K(z) = M(z)C_M(z)$. Since $M(z)$ can be chosen such that both $M(z)$ and $M^{-1}(z)$ are causal and stable transfer functions, $C_M(z) = M^{-1}(z)K(z)$ is well defined. Now the feedback system in Figure 2 is equivalent to the one in Figure 3.

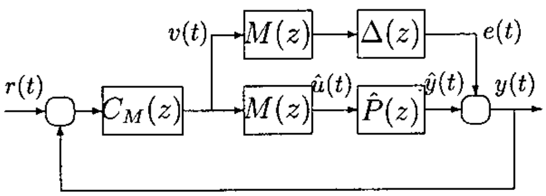

Figure 3: Equivalent Feedback System

The model estimation objective of minimizing $\|\Delta M\|_\infty$ implies that the RLS algorithm can be used to identify $\hat{P}(z)$, based on the measurement data $(\{v(t)\}, \{y(t)\})$, in light of the *uncertainty equivalence principle.* Because of the optimality of the least-squares algorithm, the equivalent uncertainty bound associated with modeling $\hat{N}(z) = \hat{P}(z)M(z)$ is the smallest possible. Moreover since $\{v(t)\}$ is the internal signal of the feedback controller $K(z)$, it can be assumed to be noise-free in its measurement.

Let $\mathcal{H}_2$ be the $\mathcal{Z}$-transform of ℓ_+^2. It follows that $\mathcal{H}_2$ is also a Hilbert space. Let $\{\psi_i(z)\}_{i=1}^{\infty}$ be a complete orthonormal basis for $\mathcal{H}_2$. Then stability and smoothness of $P(z)$ and $M(z)$ imply that there exist $\{\theta_i\}_{i=1}^{\infty}$ such that

$$N(z) = P(z)M(z) = \sum_{i=1}^{\infty} \theta_i \psi_i(z).$$

Our RLS estimation algorithm uses real-time data $\{v(t)\}, \{y(t)\})$ to identify the optimal approximate model parameterized by

$$\begin{aligned} \hat{N}(z) &= \hat{P}(z)M(z) = \sum_{i=1}^{n} \hat{\theta}_i \psi_i(z) = \hat{\Theta}\Psi(z), \\ \Psi(z) &= \left[\ \psi_1'(z) \quad \psi_2'(z) \quad \cdots \quad \psi_\kappa'(z)\ \right]', \quad (10) \\ \hat{\Theta} &= \left[\ \hat{\theta}_1 \quad \hat{\theta}_2 \quad \cdots \quad \hat{\theta}_\kappa\ \right], \end{aligned}$$

with $\kappa > 0$ an integer in the sense that $\|e\|_{[2,T]}$ is minimized for each $T \geq 0$, where $\{e(t)\}$ is the error signal in Figure 3. The use of more general basis functions can incorporate the *a priori* information about the plant model, such as the locations of its dominate poles, to achieve smaller modeling errors. Denote

$$\phi_i(t) = \psi_i(q)v(t), \quad i = 1, 2, \cdots, \kappa, \qquad (11)$$

with q the unit advance operator. Then the output of the approximate model $\hat{P}(z)$ has the form

$$\hat{y}(t) = \hat{\Theta}\phi(t), \quad \phi(t) = \left[\ \phi_1'(t) \quad \cdots \quad \phi_\kappa'(t)\ \right]'. \qquad (12)$$

Let $\Phi_T = [\ \phi(0) \quad \phi(1) \quad \cdots, \quad \phi(T)\ ]$. Then

$$\hat{Y}_T = [\ \hat{y}(0) \quad \hat{y}(1) \quad \cdots \quad \hat{y}(T)\ ] = \hat{\Theta}\Phi_T. \qquad (13)$$

The RLS algorithm computes recursively $\hat{\Theta} = \hat{\Theta}_T$, the optimal solution, to achieve

$$\inf_{\hat{\Theta}} \|Y_T - \hat{\Theta}'\Phi_T\|_2 \qquad (14)$$

where $Y_T = [\ y(0) \quad \cdots \quad y(T)\ ]$. It is noted that the conventional RLS algorithm needs be adapted to our case due to the multivariable nature, which yields the following for computing the optimal solution to (14):

$$\begin{aligned} \hat{\Theta}_T &= \hat{\Theta}_{T-1} + (y_T - \hat{\Theta}_{T-1}\phi_T)\Psi_T\phi_T' X_{T-1}, \\ X_T &= X_{T-1} - X_{T-1}\phi_T\Psi_T\phi_T' X_{T-1} \qquad (15) \end{aligned}$$

for $T = 1, 2, \cdots$ where $\Psi_T = (I + \phi_T' X_{T-1}\phi_T)^{-1}$. It is noted that the first a few $\hat{\Theta}_T$ (for T close to 0) need be computed from other methods, because $X_T = (\Phi_T\Phi_T')^{-1}$ does not exist for small T. But one may also employ $X_0 = rI$ with large r value to begin the recursive computation, which does not affect the asymptotic performance of the RLS algorithm.

Theorem 3.1 *Consider the feedback system in Figure 3. Suppose that the measurement output $y(t) = w(t) + \eta(t)$ with $w(t)$ the true output of $P(z)$, and $\eta(t)$ the additive noise satisfying $\|\eta\|_{[2,T]} \leq \epsilon_T\|v\|_{[2,T]}$ for some $\epsilon_T > 0$. Let $\{\psi_i(z)\}_{i=1}^{\infty}$ be the given complete orthonormal basis for $\mathcal{H}_2$. Define*

$$\delta_\kappa^* := \inf_{\hat{\Theta}} \|P - \hat{\Theta}\Psi\|_\infty$$

with $\hat{\Theta}$ and $\Psi(z)$ as in (10). If the feedback system in Figure 3 is at rest at time $t = 0$, then the use of the RLS algorithm as in (15) for estimation of $\hat{P}(z)$ ensures that

$$\|w - \hat{y}\|_{[2,T]} \leq (\delta_\kappa^* + 2\epsilon_T)\|v\|_{[2,T]} \quad \forall\ T \geq 0. \qquad (16)$$

As mentioned earlier, $v(t)$ is assumed to be noise-free because it is an internal signal of the feedback controller. The assumption on the additive noise $\eta(t)$ in measurement of the output data $y(t)$ implies that ϵ_T^{-1} is the signal-to-noise ratio in the time interval $[0, T]$. The inequality (16) shows that the equivalent uncertainty bound for the weighted modeling error $\Delta(z)M(z)$ is no more than $\varepsilon(T) = \delta_\kappa^* + 2\epsilon_T$ with $\Delta(z)$ the additive modeling error at the frozen time $t = T$. Now the problem for adaptive control is the synthesis of the robust feedback controller based on each estimated approximate model.

3.2 Controller Design via $\mathcal{H}_\infty$ Filtering

The RLS algorithm yields a time-varying approximate model $\hat{P}_t(q) = \hat{N}_t(q)M^{-1}(q)$ with q the unit advance operator. We associate a state-space realization with the generalized plant model:

$$G_t(q) = \begin{bmatrix} M(q) \\ \alpha^{-1}\hat{N}_t(q) \end{bmatrix} = \left[\begin{array}{c|c} A_t & B_t \\ \hline C_t & D_t \end{array}\right], \qquad (17)$$

for $t \geq 0$ with the fixed order n. The assumption on $M(z)$ implies that D_t has full column rank, and $G_t(q)$ is strictly minimum phase, satisfying for each frozen time $t \geq 0$,

$$\text{rank}\begin{bmatrix} A_t - zI & B_t \\ C_t & D_t \end{bmatrix} = n + m \quad \forall\, |z| \geq 1. \quad (18)$$

As D_t of size $(p+m) \times m$ has full column rank, there exist D_t^+ of size $m \times (p+m)$, $D_{t\perp}$ of size $(p+m) \times p$, and $D_{t\perp}^+$ of size $p \times (p+m)$ such that

$$\begin{bmatrix} D_t^+ \\ D_{t\perp}^+ \end{bmatrix} \begin{bmatrix} D_t & D_{t\perp} \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}. \quad (19)$$

There are many choices for D_t^+, $D_{t\perp}$, and $D_{t\perp}^+$. A simplest one is the following:

$$D_t^+ = (D_t'D_t)^{-1}D_t', \;\; D_{t\perp}D_{t\perp}' = I - D_t(D_t'D_t)^{-1}D_t',$$

and $D_{t\perp}^+ = D_{t\perp}'$. It follows that at each frozen time t, there exists a stable left inverse $G_t^+(q)$ to $G_t(q)$ such that $G_t^+(q)G_t(q) = I$. It is noted that

$$G_t(q) = \left[I + C_t(qI - A_t)^{-1}(B_tD_t^+ - L_tD_{t\perp}')\right] D_t$$

for any constant matrix L_t, in form of state estimation gain matrix. Hence

$$\begin{aligned} G_t^+(q) &= D_t^+\left[I + C_t(qI - A_t)^{-1}B_{tL}\right]^{-1} \quad (20) \\ &= D_t^+\left[I - C_t(qI - A_{tL})^{-1}B_{tL}\right], \end{aligned}$$

with $B_{tL} := B_tD^+ - L_tD_{t\perp}'$, and $A_{tL} := A_t - B_{tL}C_t$. The strict minimum phase condition (18) is equivalent to the existence of L_t such that A_{tL} is stable.

Denote $D_{ta} = \begin{bmatrix} D_t & D_{t\perp} \end{bmatrix}$, and

$$G_{ta}(q) = \begin{bmatrix} G_t(q) & G_{t\perp}(q) \end{bmatrix} := \left[\begin{array}{c|c} A_t & B_{tL} \\ \hline C_t & I \end{array}\right] D_{ta}.$$

Then the frozen time system $G_{ta}(z)$ is square, and strictly minimum phase, provided that $G_t(z)$ is strictly minimum phase, and L_t is stabilizing at the frozen time t. The above results in

$$G_{ta}^{-1}(q) = \begin{bmatrix} G_t^+(q) \\ G_{t\perp}^+(q) \end{bmatrix} = \begin{bmatrix} D_t^+ \\ D_{t\perp}' \end{bmatrix} \left[\begin{array}{c|c} A_{tL} & B_{tL} \\ \hline -C_t & I \end{array}\right]$$

which is stable as well at each frozen time t. We therefore have a dynamic version to (19):

$$\begin{bmatrix} G_t^+(q) \\ G_{t\perp}^+(q) \end{bmatrix} \begin{bmatrix} G_t(q) & G_{t\perp}(q) \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}. \quad (21)$$

The next result characterizes all stable left inverses of $G_t(q)$, which is in essence the Youla parameterization. Thus the proof is omitted.

Proposition 3.2 *Let $G_t(q)$ as in (17) satisfy (18) with D_t full rank for all $t \geq 0$, and $G_t^+(q)$ and $G_{t\perp}^+(q)$ be defined as in (21). Then all stable left inverses of $G_t(q)$ at each frozen time t are parameterized by*

$$G_t^{\text{inv}}(q) = G_t^+(q) + Q_t(q)G_{t\perp}^+(q) = \begin{bmatrix} I & Q_t(q) \end{bmatrix} G_{at}^{-1}(q)$$

where $Q_t(q)$ is stable, and arbitrary.

By the expression of $G_t^+(q)$ and $G_{t\perp}^+(q)$ as in (21), and an abuse of notation, the parameterized stable left inverse $G_t^{\text{inv}}(q)$ has the form

$$\left[\begin{array}{c|c} A_t - B_tD_t^+C_t + L_tD_{t\perp}'C_t & -B_tD_t^+ + L_tD_{t\perp}' \\ \hline (D_t^+ + Q_t(q)D_{t\perp}')C_t & (D_t^+ + Q_t(q)D_{t\perp}') \end{array}\right]. \quad (22)$$

In light of the discussion earlier, our controller design objective is to synthesize a stable left inverse $\mathcal{K}_t(q) = G_t^{\text{inv}}(q)$, such that $\|G_t^{\text{inv}}\|_\infty$ is minimized for each frozen time $t \geq 0$ over all possible state estimation gains L_t, and stable dynamic systems $Q_t(q)$. While this is feasible and amounts to the $\mathcal{H}_\infty$ filtering problem in light of the expression of the left inverse in (22), it requires to solve the stabilizing solution to some discrete-time algebraic Riccati equation which can be numerically demanding. As an alternative, we view $\mathcal{K}_t(q) = G_t^{\text{inv}}(q)$ as a time-varying system, and seek L_t and $Q_t(q)$ such that $\|\mathcal{K}_t\|_{[\infty,T]} = \|G_t^{\text{inv}}\|_{[\infty,T]} < \gamma$ whenever feasible. We have the following result based on [3].

Theorem 3.3 *Consider the time-varying left inverses $G_t^{\text{inv}}(q)$ in (22) where $0 \leq t \leq T$. Then there exist $\{L_t\}_{t=0}^T$, and $\{Q_t(q)\}_{t=0}^T$ such that $\|G_t^{\text{inv}}\|_{[\infty,T]} < \gamma$, if and only if the following difference Riccati equation (DRE)*

$$\begin{aligned} Y_{t+1} &= A_0Y_tA_0' + B_t(D_t'D_t)^{-1}B_t' \quad (23) \\ &\quad -S_t(I - \gamma^2D_tD_t' + C_tY_tC_t')^{-1}S_t' \end{aligned}$$

with $Y_0 = 0$, $A_0 = A_t - B_tD_t^+C_t$ and $S_t = (A_0Y_tC' - BD^+)$ satisfies $Y_{t+1} \geq 0$ and

$$(D_t'D_t)^{-1} + D_t^+C_tY_tC_t'\Gamma_t(D_t^+)' < \gamma^2 I \quad (24)$$

for $t = 0, 1, \cdots, T$ with $\Gamma_t = (I + D_{t\perp}D_{t\perp}'C_tY_tC_t')^{-1}$. In this case, a left inverse in the form of (22) satisfying $\|G_t^{\text{inv}}\|_{[\infty,T]} < \gamma$ is specified by

$$\begin{bmatrix} L_t \\ Q_t \end{bmatrix} = -\begin{bmatrix} A_t - B_tD_t^+C_t \\ D_t^+C_t \end{bmatrix} Y_tC_t'D_{t\perp}\Gamma_t. \quad (25)$$

That is, $Q_t(q)$ can be chosen as a time-dependent constant matrix.

It is now clear that the design of $\mathcal{K}_t(q) = G_t^{\text{inv}}(q)$ requires to compute recursively (23) given each new estimated model. Since each iteration in DRE (23) requires only $\mathcal{O}(n^2)$ with n the order of the estimated model, the computational complexity for controller design is comparable to that of the RLS algorithm for model estimation.

3.3 Asymptotic Stability and Performance

The RLS algorithm for adaptive estimation, and $\mathcal{H}_\infty$ filtering algorithm for controller design are all based on the approximate model. The real issue is the stability, and performance of the feedback system with the true plant as in Figure 2. Recall that the feedback controller is now time-varying, thus denoted by $K_t(q)$, and is given by $K_t(q) = V_t^{-1}(q)U_t(q)$ with $V_t(q)$ and $U_t(q)$ the same as in the partition of $\mathcal{K}_t(q)$, as (8) which is the left inverse of $G_t(q)$. However the feedback controller $K_t(q)$ is implemented as in Figure 3 with $C_{tM}(q) = M^{-1}(q)K_t(q)$. We have the following result concerning asymptotic stability and performance of our proposed adaptive control system in this section.

Theorem 3.4 *Consider the proposed adaptive control system with RLS estimation algorithm as in (15), and $\mathcal{H}_\infty$ filtering algorithm as in Theorem 3.3. Let the feedback system be as in Figure 3 where the measurement data $y(t) = w(t) + \eta(t)$ with $\eta(t)$ the observation noise satisfying $\|\eta\|_{[2,T]} \leq \epsilon_T \|v\|_{[2,T]}$ for some $\epsilon_T > 0$, and each $T > 0$. Suppose that the true plant is linear, time-invariant, and stable, and $X_T \to 0$ as $T \to \infty$, with X_T as defined in (15). Assume further that the DRE (23) admits the solution $Y_{t+1} \geq 0$ such that (24) holds for $0 \leq t \leq T$ and some $\gamma > 0$. If in addition $Y_T \to Y \geq 0$ as $T \to \infty$ for which*

$$\lim_{t \to \infty} \left[(D_t'D_t)^{-1} + D_t^+ C_t Y_t C_t' \Gamma_t (D_t^+)'\right] < \gamma^2 I,$$

then the proposed adaptive feedback control system is stable asymptotically, and admits the performance bound equivalent to $\|M^{-1}(I-KP)^{-1}\|_\infty < \gamma/(1-\gamma\varepsilon)$, provided that $\varepsilon\gamma < 1$, where $\varepsilon = \lim_{T\to\infty} \delta_n^* + 2\epsilon_T$.

Theorem 3.4 shows the asymptotic robust stability and equivalent performance for our proposed adaptive control system. It should be mentioned that the parameter α can be made time-varying, which should be taken a small value (< 1) at the beginning stage, and a relatively large value (> 1) at the later stage of the adaptive control. This will help ensure the existence of the required solution to DRE (23) and satisfaction of the condition (24). Similarly the ideal sensitivity $M(z)$ can also be made time-varying with lower requirements on its performance at the beginning stage, and be gradually strengthened at the later stage. For the same reason, γ can be made time-varying as well with γ close to, but smaller than $1/\varepsilon$ at the beginning stage of the adaptive control. Gradually it can move to a value of $1/(\beta\varepsilon)$ with $\beta \geq 2$ in order to have better stability margin and performance.

4 Conclusion

Robust adaptive control has been tackled in this paper. The key ingredient is the new concept of *uncertainty equivalence principle*, which motivated quantification of the equivalent uncertainty bound for adaptive estimation of the plant model, and helped introduce $\mathcal{H}_\infty$ control into the adaptive controller design. A new adaptive control system is proposed with robust stability and equivalent performance established under the convergence assumption for the estimation algorithms. Although only stable plants are considered in this paper, unstable plants can also be tackled which will be reported elsewhere. It is seen that the successful unification of adaptive control and $\mathcal{H}_\infty$-based robust control empowers robust adaptive control, enabling the proposed adaptive feedback control systems to achieve robust stability and performance comparable to those achievable by $\mathcal{H}_\infty$ control. The results in this paper shed some light to new direction for robust adaptive control.

References

[1] D.W. Clark, E. Mosca and R. Scattolini, "Robustness of an adaptive predictive controller," *IEEE Trans. Automat. Contr.*, vol. 39, pp1052, 1994.

[2] A. Datta and P.A. Ioannou, "Robust adaptive controller: a unified approach," *Proceedings of IEEE*, vol. 79, pp1736, 1991.

[3] M. Green and D.J.N. Limebeer, *Linear Robust Control*, Prentice Hall, 1995.

[4] L. Guo, "Self-convergence of weighted least-squares with applications to stochastic adaptive control," *IEEE Trans. Automat. Contr.*, vol. 41, pp79, 1996.

[5] P.A. Ionanou and J. Sun, *Robust Adaptive Control*, Prentice-Hall, New Jersey, 1996.

[6] P.A. Ioannou and K.S. Tsakalis, "A robust direct adaptive controller," *IEEE Trans. Automat. Contr.*, vol. 31, pp1033, 1986.

[7] P.P. Khargonekar and R. Ortega, "Comments on the robust stability analysis of adaptive controllers using normalization," *IEEE Trans. Automat. Contr.*, vol. 34, pp478, 1989.

[8] R.H. Middleton, G.C. Goodwin, D.J. Hill, and D.Q. Mayne, "Design issues in adaptive control," *IEEE Trans. Automat. Contr.*, vol. 33, pp50, 1988.

[9] S.M. Naik, P.R. Kumar and B.E. Ydsite, "Robust continuous-time adaptive control by parameter projection," *IEEE Trans. Automat. Contr.*, vol. 37, pp182, 1992.

[10] A. Pinkus, *n-Widths in Approximation Theory*, pp250, Springer-Verlag, Berlin, 1985.

[11] C.E. Rohrs, L. Valavani, M. Athans, and G. Stein, "Robustness of continuous-time adaptive control algorithms in the presence of unmodeled dynamics," *IEEE Trans. Automat. Contr.*, vol. 30, pp881, 1985.

[12] S. Sastry and M. Modson, *Adaptive Control – Stability, Convergence, and Robustness*, Prentice-Hall, New Jersey, 1989.

[13] G. Tao and P. Ioannou, "Robust adaptive control – a modified scheme," *Int J. Contr.*, vol. 54, pp241-256, 1992.

[14] C. Wen and D.J. Hill, "Global boundedness of discrete-time adaptive control just using estimator projection," *Automatica*, vol. 28, pp1143, 1992.

[15] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems, Part I," *IEEE Trans. on Auto. Contr.*, pp228, 1966.

[16] K. Zhou, J. Doyle and K. Glover, *Robust and Optimal Control*, Englewood Cliffs, NJ: Printice-Hall 1987.