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Unitary dilation approach to contractive matrix completion☆

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Abstract

In this paper a new method to construct all contractive matrix completion to a block triangular matrix is given. This new method uses only elementary matrix operations, and has the advantages of better numerical reliability and ease in implementation on computers.

Keywords: Triangular matrices; Nest algebra; Matrix dilation; Matrix completion; Contractive matrices; Unitary matrices

1. Introduction

In this paper we address the following problem: Given a block matrix

$$M=\begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1l}\\ M_{21} & M_{22} & \cdots & M_{2l}\\ \vdots & \vdots & \ddots & \vdots\\ M_{l1} & M_{l2} & \cdots & M_{ll}\end{bmatrix}, \tag{1}$$

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characterize all block lower triangular matrices of the form

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ T_{21} & T_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_{l1} & T_{l2} & \cdots & T_{ll} \end{bmatrix} \tag{2}$$

satisfying

$$\|M + T\| < 1.$$

Here the norm used is the spectral norm, namely, the largest singular value.

This problem, in addition to being interesting mathematically, arises in the $\mathscr{H}_\infty$ control of periodic/multirate systems [4,9,10]. The common approach to controller design involving periodic/multirate systems is to apply a technique called lifting [8] to convert the design problem into an equivalent linear, time-invariant one; however, the resultant linear, time-invariant problem has to satisfy a new design condition, the so-called causality constraint [4,7]. Roughly speaking, the causality constraint requires that the direct feedthrough terms in the lifted controllers be block lower triangular under certain coordinate transformations [4,7]. The study of the set of all $\mathscr{H}_\infty$ suboptimal controllers in the periodic/multirate framework intrinsically relates to the matrix completion problem just described, see [9,10] for the connection.

Let us introduce some notation related to the matrices in (1) and (2). Let the size of M_{ij} be $m_i \times n_j$. The ordered sets of integers $\{m_1, \ldots, m_l\}$ and $\{n_1, \ldots, n_l\}$ are denoted by $\tilde{m}$ and $\tilde{n}$, respectively. The set of matrices of the form in (1) is denoted by $\mathscr{M}(\tilde{m}, \tilde{n})$. The set of all block lower triangular matrices of the form in (2) is denoted by $\mathscr{T}(\tilde{m}, \tilde{n})$. The set of all (block) strictly lower triangular matrices, namely, matrices of the form in (2) with

$$T_{ii} = 0, \quad i = 1, \ldots, l,$$

is denoted by $\mathscr{T}_s(\tilde{m}, \tilde{n})$.

The following theorem is known in various forms [1–3,6,12].

Theorem 1. *Let $M \in \mathscr{M}(\tilde{m}, \tilde{n})$. The following statements are equivalent*:

(a) *There exists $T \in \mathscr{T}(\tilde{m}, \tilde{n})$ such that $\|M + T\| < 1$.*

(b)

$$\max_{1 \leqslant i \leqslant l} \left\| \begin{bmatrix} M_{1(i+1)} & \cdots & M_{1l} \\ \vdots & & \vdots \\ M_{i(i+1)} & \cdots & M_{il} \end{bmatrix} \right\| < 1.$$

(c) *There exists*

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

with $W_{11} \in \mathscr{T}(\tilde{m}, \tilde{m})$, $W_{12} \in \mathscr{T}(\tilde{m}, \tilde{n})$, $W_{21} \in \mathscr{T}_s(\tilde{n}, \tilde{m})$, *and* $W_{22} \in \mathscr{T}(\tilde{n}, \tilde{n})$ *such that*

$$W^* J W = G^* J G,$$

where

$$G = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

(d) *There exists*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

with $P_{11} \in \mathscr{T}(\tilde{m}, \tilde{n})$, $P_{12} \in \mathscr{T}(\tilde{m}, \tilde{m})$, $P_{21} \in \mathscr{T}(\tilde{n}, \tilde{n})$, $P_{22} \in \mathscr{T}_s(\tilde{n}, \tilde{m})$, *and* P_{12}, P_{21} *both invertible such that*

$$\begin{bmatrix} M + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary.

If W or P is obtained as in (c) or (d) in Theorem 1, the set of all $T \in \mathscr{T}(\tilde{m}, \tilde{n})$ such that $\|M + T\| < 1$ is characterized by results in either of the following two theorems.

Theorem 2. *Let* $M \in \mathscr{M}(\tilde{m}, \tilde{n})$ *and assume condition* (c) *in Theorem* 1 *is satisfied. Then the set of all* $T \in \mathscr{T}(\tilde{m}, \tilde{n})$ *such that* $\|M + T\| < 1$ *is given by*

$$\left\{ T = Q_1 Q_2^{-1} \colon \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = W^{-1} \begin{bmatrix} U \\ I \end{bmatrix},\ U \in \mathscr{T}(\tilde{m}, \tilde{n}),\ \text{and}\ \|U\| < 1 \right\}. \tag{3}$$

The proof of Theorem 2 can be done following the developments in [2,3], which are rather heavy. A more elementary proof is given in [10].

Theorem 3. *Let* $M \in \mathscr{M}(\tilde{m}, \tilde{n})$ *and assume condition* (d) *in Theorem* 1 *is satisfied. Then the set of all* $T \in \mathscr{T}(\tilde{m}, \tilde{n})$ *such that* $\|M + T\| < 1$ *is given by*

$$\left\{ T = P_{11} + P_{12} U (I - P_{22} U)^{-1} P_{21} \colon U \in \mathscr{T}(\tilde{m}, \tilde{n})\ \text{and}\ \|U\| < 1 \right\}. \tag{4}$$

Proof. Since the matrix

$$\begin{bmatrix} M + P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is unitary and P_{12}, P_{21} are invertible, it follows from [11] that the map

$$U \mapsto = M + P_{11} + P_{12} U (I - P_{22} U)^{-1} P_{21}$$

is a bijection from the open unit ball of $\mathscr{M}(\mathbb{F}^{m \times n})$ onto itself. What is left to show is that $T = P_{11} + P_{12} U (I - P_{22} U)^{-1} P_{21} \in \mathscr{T}(\tilde{m}, \tilde{n})$ iff $U \in \mathscr{T}(\tilde{m}, \tilde{n})$. The "if" part follows from simple matrix manipulation. For the “only if” part, assume $T = P_{11} + P_{12}U(I - P_{22}U)^{-1}P_{21} \in \mathscr{T}(\tilde{m}, \tilde{n})$ for some $U \in \mathscr{M}(\tilde{m}, \tilde{n})$; we need to show that U too belongs to $\mathscr{T}(\tilde{m}, \tilde{n})$. Simple algebra gives

$$P_{12}^{-1}(T - P_{11})P_{21}^{-1} = \left[I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22}\right]U. \tag{5}$$

Since

$$\begin{aligned} I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22} &= I + P_{12}^{-1}P_{12}U(I - P_{22}U)^{-1}P_{21}P_{21}^{-1}P_{22} \\ &= I + U(I - P_{22}U)^{-1}P_{22} \\ &= (I - UP_{22})^{-1}, \end{aligned}$$

it follows that $I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22}$ is invertible. Hence from (5)

$$U = \left[I + P_{12}^{-1}(T - P_{11})P_{21}^{-1}P_{22}\right]^{-1}P_{12}^{-1}(T - P_{11})P_{21}^{-1}.$$

Therefore U belongs to $\mathscr{T}(\tilde{m}, \tilde{n})$. □

For a contractive matrix A, define the entropy of A by

$$\mathscr{I}(A) = -\ln\det[I - A^*A].$$

The characterizations in Theorems 2 and 3 also give easy expression to the T which minimizes $\mathscr{I}(M + T)$.

Theorem 4. *Let $M \in \mathscr{M}(\tilde{m}, \tilde{n})$ and assume condition* (c) *or* (d) *in Theorem* 1 *is satisfied. Then the unique T satisfying $\|M + T\| < 1$ which minimizes $\mathscr{I}(M + T)$ is given by $T = P_{11}$ or $T = -W_{11}^{-1}W_{12}$.*

Establishing and applying these results rely on a constructive proof of Theorem 1. The equivalence of (a) and (b) in Theorem 1 follows from the Arveson’s distance formula [5]. There have been several proofs [2,3,10,12] for the equivalence of (b), (c), (d) in the literature, all taking the route (b) ⇒ (c) ⇒ (d) ⇒ (b). The key step is (b) ⇒ (c) in which a method to construct W needs to be given. The methods in [2,3,10] are based on J-unitary matrices or J-spectral factorizations whereas the method in [12] may be considered as a finite-dimensional analogue of Schur’s algorithm. These constructions are all quite involved. Once W is given, P in Theorem 1(d) can be computed easily as

$$P = \begin{bmatrix} -W_{11}^{-1}W_{12} & W_{11}^{-1} \\ W_{22} - W_{21}W_{11}^{-1}W_{12} & W_{21}W_{11}^{-1} \end{bmatrix}.$$

However, in many applications the characterization in Theorem 3 is preferred because it connects better to the results on $\mathscr{H}_\infty$ control in the literature. Hence it is desirable to have a way to compute P directly and more efficiently. In this paper we will give a direct, simple constructive proof of (b) ⇒ (d) in Theorem 1 based on elementary matrix operations. This, together with an easy expression of W in terms of P,

$$W = \begin{bmatrix} P_{12}^{-1} & -P_{12}^{-1}P_{11} \\ P_{22}P_{12}^{-1} & P_{21} - P_{22}P_{12}^{-1}P_{11} \end{bmatrix},$$

facilitates a different proof for the equivalence of (b), (c), and (d). Numerical experience shows that the new construction indeed is easier to implement on computers and requires less computation time.

2. Construction of P

In the following, we assume the condition in Theorem 1(b) is satisfied and present the new construction of P given in Theorem 1(d).

Lemma 1. *Assume the matrices E, F, and H, of appropriate dimensions, satisfy the conditions*:

$$[E \quad F]\begin{bmatrix} E^* \\ F^* \end{bmatrix} = I, \qquad \left\| \begin{bmatrix} F \\ H \end{bmatrix} \right\| < 1.$$

Then there exists a matrix G satisfying

$$\left\| \begin{bmatrix} E & F \\ G & H \end{bmatrix} \right\| \leqslant 1,$$

$$[G \quad H]\begin{bmatrix} E^* \\ F^* \end{bmatrix} = 0,$$

$$\|[G \quad H]\| < 1.$$

An explicit formula for this matrix is

$$G = -HF^*(EE^*)^{-1}E.$$

Proof. It follows from [6] that there exists a matrix G such that

$$\left\| \begin{bmatrix} E & F \\ G & H \end{bmatrix} \right\| \leqslant 1.$$

Among all such G characterized in [6] in terms of a free contractive matrix, the "central" one obtained by setting the free contractive matrix to zero is

$$G = -HF^*(I - FF^*)^{-1}E = -HF^*(EE^*)^{-1}E.$$

Using this G, we have

$$[G \quad H]\begin{bmatrix} E^* \\ F^* \end{bmatrix} = -HF^*(EE^*)^{-1}EE^* + HF^* = 0.$$

and

$$[G \quad H]\begin{bmatrix} G^* \\ H^* \end{bmatrix} = HF^*(EE^*)^{-1}FH^* + HH^* = H(I - F^*F)^{-1}H^* < I.$$

The last inequality follows from $\left\| \begin{bmatrix} F \\ H \end{bmatrix} \right\| < 1.$ □

To avoid awkward notation, we redefine

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

Then

$$\begin{bmatrix} M+A & B \\ C & D \end{bmatrix}$$
$$= \begin{bmatrix} M_{11}+A_{11} & M_{12} & \cdots & M_{1l} & B_{11} & 0 & \cdots & 0 \\ M_{21}+A_{21} & M_{22}+A_{22} & \cdots & M_{2l} & B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{l1}+A_{l1} & M_{l2}+A_{l2} & \cdots & M_{ll}+A_{ll} & B_{l1} & B_{l2} & \cdots & B_{ll} \\ C_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ C_{21} & C_{22} & \cdots & 0 & D_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{l1} & C_{l2} & \cdots & C_{ll} & D_{l1} & D_{l2} & \cdots & 0 \end{bmatrix}.$$

We need to choose A_{ij}, B_{ij}, C_{ij}, for $i \geqslant j$, and D_{ij} for $i > j$. This will be done in the following order: In the ith step, determine those blocks in the $(l+i)$th row and the ith row:

Step 1: Set $C_{11} = I$, $M_{11} + A_{11} = 0$, and choose B_{11} so that

$$\begin{bmatrix} M_{12} & \cdots & M_{1l} & B_{11} \end{bmatrix}$$

is a co-isometry. Theorem 1(b) implies that any B_{11} chosen in this way is nonsingular.

Step i, $i = 2, \ldots, l-1$: Set $C_{i1} = 0$ and choose the rest of the $(l+i)$th row so that it is a co-isometry and is orthogonal to all of the previously determined rows. This requires

$$\begin{bmatrix} C_{i2} & \cdots & C_{ii} & D_{i1} & \cdots & D_{i(i-1)} \end{bmatrix}^*$$

to be an isometry onto the kernel of

$$\begin{bmatrix} M_{12} & \cdots & M_{1i} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ M_{(i-1)2}+A_{(i-1)2} & \cdots & M_{(i-1)i} & B_{(i-1)1} & \cdots & B_{(i-1)(i-1)} \\ C_{22} & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{(i-1)2} & \cdots & 0 & D_{(i-1)1} & \cdots & 0 \end{bmatrix}.$$

Then set $M_{i1} + A_{i1} = 0$ and choose

$$\begin{bmatrix} M_{i2} + A_{i2} & \cdots & M_{ii} + A_{ii} & B_{i1} & \cdots & B_{i(i-1)} \end{bmatrix}$$

in such a way so that

$$\begin{bmatrix} M_{12} & \cdots & M_{1i} & M_{1(i+1)} & \cdots & M_{1l} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ M_{(i-1)2} + A_{(i-1)2} & \cdots & M_{(i-1)i} & M_{(i-1)(i+1)} & \cdots & M_{(i-1)l} & B_{(i-1)1} & \cdots & B_{(i-1)(i-1)} \\ M_{i2} + A_{i2} & \cdots & M_{ii} + A_{ii} & M_{i(i+1)} & \cdots & M_{il} & B_{i1} & \cdots & B_{i(i-1)} \\ C_{22} & \cdots & 0 & 0 & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{i2} & \cdots & C_{ii} & 0 & \cdots & 0 & D_{i1} & \cdots & D_{i(i-1)} \end{bmatrix}$$

is a contraction and its ith block row is orthogonal to all other block rows. This is possible following Lemma 1, condition (b), and the fact that

$$\begin{bmatrix} M_{12} & \cdots & M_{1i} & M_{1(i+1)} & \cdots & M_{1l} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ M_{(i-1)2} + A_{(i-1)2} & \cdots & M_{(i-1)i} & M_{(i-1)(i+1)} & \cdots & M_{(i-1)l} & B_{(i-1)1} & \cdots & B_{(i-1)(i-1)} \\ C_{22} & \cdots & 0 & 0 & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{i2} & \cdots & C_{ii} & 0 & \cdots & 0 & D_{i1} & \cdots & D_{i(i-1)} \end{bmatrix}$$

is a co-isometry. Finally determine B_{ii} so that

$$\begin{bmatrix} M_{i2} + A_{i2} & \cdots & M_{ii} + A_{ii} & M_{i(i+1)} & \cdots & M_{il} & B_{i1} & \cdots & B_{ii} \end{bmatrix}$$

is a co-isometry. By Lemma 1, any B_{ii} chosen in such a way is nonsingular.

Step l: Set $C_{l1} = 0$ and choose the rest of the $2l$th row so that it is orthogonal to all the previously determined rows. This requires

$$\begin{bmatrix} C_{l2} & \cdots & C_{ll} & D_{l1} & \cdots & D_{l(l-1)} \end{bmatrix}^*$$

to be an isometry onto the kernel of

$$\begin{bmatrix} M_{12} & \cdots & M_{1l} & B_{11} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ M_{(l-1)2} + A_{(l-1)2} & \cdots & M_{(l-1)l} & B_{(l-1)1} & \cdots & B_{(l-1)(l-1)} \\ C_{22} & \cdots & 0 & D_{21} & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{(l-1)2} & \cdots & 0 & D_{(l-1)1} & \cdots & 0 \end{bmatrix}.$$

Finally set

$$\begin{bmatrix} M_{l1} + A_{l1} & \cdots & M_{ll} + A_{ll} & B_{l1} & \cdots & B_{l(l-1)} \end{bmatrix} = 0$$

and $B_{ll} = I$.

The above construction guarantees that the matrix

$$\begin{bmatrix} M + A & B \\ C & D \end{bmatrix} \tag{6}$$

is unitary, B is invertible, and $D \in \mathscr{T}_s(\tilde{n}, \tilde{m})$. The invertibility of C follows from that of B and the fact that the matrix in (6) is unitary.

3. Concluding remarks

The new procedure reported in this paper has the advantage of simplicity and computational efficiency. More specifically, if we compare this new procedure with the method based on J-spectral factorizations, we see that the new computation does not involve squaring matrix data in that there is no need to compute matrices of the form A^*A. The procedure can be implemented using little more than the singular value decomposition. Squaring matrices is undesirable from a numerical point of view since it reduces the number of significant digits by half. The computation of G in Lemma 1 appears to require EE^*, but actually $(EE^*)^{-1}E$ can be computed using the singular value decomposition of E without explicitly forming EE^*.

In conclusion, the new procedure is numerically more efficient and reliable.

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