A MAJORIZATION CONDITION FOR MULTI-INPUT NETWORKED STABILIZATION VIA CODING/CONTROL CO-DESIGN*

WEI CHEN†, SONGBAI WANG†, AND LI QIU†

Abstract. In this paper, we investigate the stabilization of a multi-input continuous-time linear system over a communication network modeled as the cascade of an encoder, a set of parallel channels, and a decoder. The channel capacities or signal-to-noise ratios are assumed to be fixed a priori. The encoder/decoder pair can be designed, leading to a stabilization problem via coding/control co-design. From a demand/supply balance point of view, the design of the encoder/decoder pair has the effect of reshaping the demands for communication resource from different control inputs to match the given supplies. We study the problem for the case of AWGN channels and fading channels, respectively. In both cases, we arrive at a unified necessary and sufficient condition under which the coding/control co-design problem is solvable. The condition is given by a majorization type relation. As we go along, a systematic procedure is also put forward to carry out the coding/control co-design. A numerical example is presented to illustrate our results.

Key words. networked control systems, networked stabilization, majorization, coding/control co-design, topological entropy

AMS subject classifications. 93B52, 93C35, 93D15

1. Introduction. A networked control system (NCS) is a feedback system in which the feedback loop is closed over a communication network. It has been well recognized that in networked control, whether stabilization can be achieved or not critically depends on the information constraints in the communication network. As such, a primary concern of networked stabilization is to find a fundamental limitation on the information constraints so as to render stabilization possible. For a single-input system, the networked stabilization problem has been extensively studied under various information constraints. See [1, 37, 38, 54] for data rate constraint, [3] for signal-to-noise ratio constraint, [14, 19] for quantization, [13] for fading effect, and [56] for combined effect of quantization and packet drop, etc. Interestingly, all these studies converge to a unified fundamental limitation on the information constraints required for stabilization given by the topological entropy of the open-loop plant, i.e., the logarithm of the absolute product of unstable poles for a discrete-time plant, or the sum of the unstable poles for a continuous-time plant.

There have been results reported for the stabilization of multi-input NCSs as well [19, 21, 22, 57, 58, 45, 46, 8, 62]. The idea of channel resource allocation has been proposed in [46] and followed by several other works such as [8, 15, 61, 62], etc. It was assumed therein that the channel capacities can be allocated among different input channels subject to a total capacity constraint. This gives rise to a channel/controller co-design problem that is shown to be solvable, if and only if the total channel capacity is greater than the topological entropy of the open-loop plant. Similar ideas, although not stated explicitly, were used in [30, 49] to study multi-input networked stabilization

*This work was partially supported by three grants from the Research Grants Council of Hong Kong Special Administrative Region, China (HKUST617912, HKUST16208414, and Theme-Based Research Scheme T23-701/14-N). Earlier preliminary versions of this paper containing part of the results have appeared in the proceedings of 2015 American Control Conference and the 34th Chinese Control Conference.

†Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China (email: wchenust@gmail.com, swangas@connect.ust.hk, eeqiu@ust.hk).

over parallel Gaussian channels. Therein a constraint on total transmission power is imposed while the power in each individual channel remains flexible.

Another approach exploiting diagonal encoding and decoding has been developed in pioneering works [57, 58] to study stabilization of discrete-time multi-input multi-output (MIMO) systems over parallel communication channels with individual signal-to-noise ratio constraints. Therein the authors established a necessary and sufficient condition for the networked stabilizability given in terms of the values and directions of unstable poles of the plant. Recently, a necessary and sufficient condition of similar spirit has been obtained in [45] for the stabilizability of discrete-time MIMO systems over stochastic multiplicative noise channels.

Inspired by the existing works, we study in this paper the networked stabilization of multi-input NCSs in which the communication network is modeled as a cascade of an encoder matrix, a set of individually constrained channels, and a decoder matrix. Both AWGN channels and fading channels are considered. The encoder matrix and decoder matrix are to be designed, leading to a coding/control co-design stabilization problem. A necessary and sufficient condition is obtained for networked stabilizability given explicitly in terms of a majorization type relation. Partial results in this paper have appeared in earlier conference versions [9, 10].

As mentioned, similar problem formulations have been adopted in [57, 58]. The main differences between those works and this paper are two folds. Firstly, the encoder matrix and decoder matrix were assumed to be diagonal in [57, 58]. In the current paper, they are allowed to be general full matrices and not necessarily square. When diagonal coding is used, each channel transmits a scaled version of a particular control input. In contrast, when full matrices are used, both scaling and rotation are enabled through coding and each channel transmits a linear combination of all control inputs. Secondly, it was assumed in [57, 58] the number of channels are the same as that of the control inputs. In our work, we allow the possibility to use more channels than the number of control inputs. Both the general encoder/decoder pair and the redundancy in the number of channels provide us more flexibility in the design, and thus, enables further relaxation of the existing stabilizability conditions.

We wish to mention that majorization is a powerful mathematical tool [35] which has been studied in mathematics by giants like Hardy, Littlewood, and Pólya in their masterpiece [25] and has been widely used in statistics in the past 100 years. Its engineering applications also appear in a broad range of areas, including but not limited to wireless communication [42], information theory [11], operations research [6], and power systems [40, 24]. The application of majorization in system and control theory can be traced back to Rosenbrock's structure theorem on invariant factor assignment which was first reported in [47] and later revisited and extended in [63, 64, 65]. Recently, majorization has also been utilized to investigate the remote state estimation with communication costs [32, 39].

The rest of this paper is organized as follows. Section 2 formulates the problem. Section 3 presents some preliminary knowledge. Section 4 and 5 give the main results. A numerical example is worked out in Section 6. Some conclusion remarks follow in Section 7. The notation in this paper is more or less standard and will be made clear as we proceed.

2. Problem Formulation. Consider the NCS depicted in Fig. 2.1. The plant $[A|B]$ is a continuous-time linear time-invariant system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that $[A|B]$ is unstable but stabilizable. Let the state $x(t)$ be available for feedback. If the communication network between the controller and the plant is ideal, i.e., $u(t) = v(t)$, one can easily design a state feedback controller $v(t) = Fx(t)$ so that the closed-loop system is stable. However, such state feedback design faces challenges when the communication network is not ideal, i.e., $u(t)$ is a distorted version of $v(t)$, since the achievability of stabilization will depend on the transmission accuracy of the communication network. We confine our attention to LTI controllers in this paper. A general concern of networked stabilization is to find a fundamental limitation on the quality of the communication network such that the NCSs can be stabilized.

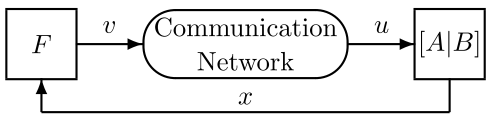

FIG. 2.1. *State feedback via communication network.*

The idea of channel resource allocation as in [46, 8, 15, 61, 62] has been successful in revealing the minimum total channel capacity rendering stabilization possible for multi-input NCSs. This is based on a crucial assumption that the channel capacities can be allocated among different input channels. What if the channel capacities are individually constrained and thus not allocatable? In that case, the works in [57, 58] suggested the use of a diagonal encoding and decoding scheme as an additional design freedom to facilitate the networked stabilization. Driven by curiosity, we wish to ask: What if full matrices, not necessarily diagonal, are exploited to implement the coding scheme? Will that bring in new advantages? Also, is it possible and beneficial to use more channels than the number of control inputs?

To answer these questions, we study the stabilization of a continuous-time multi-input linear system over a communication network modeled as the cascade of a full matrix encoder T, a set of l parallel channels, and a full matrix decoder R, as shown in Fig. 2.2. The channels are now fixed a priori. Assume $l \geq m$, i.e., the number of channels can be greater than or equal to the number of control inputs. Later as we proceed, it will become clear that $l < m$ may also be valid in some cases. The encoder matrix $T \in \mathbb{R}^{l\times m}$ and decoder matrix $R \in \mathbb{R}^{m\times l}$ are to be designed subject to certain mild constraint. The specific form of the constraint will depend on the model of the channels.

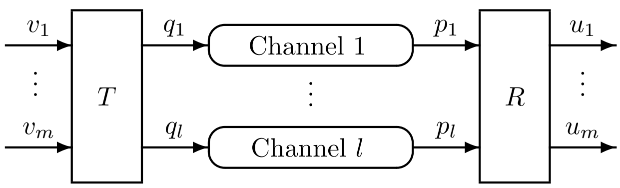

FIG. 2.2. *Communication Channels with Encoding and Decoding.*

Note that with T and R being full matrices, each channel now transmits a linear combination of all the control inputs. This imposes a combined effect of scaling and rotation on the control signals, which is different from the scenario in [57, 58], wherein each channel transmits a scaled version of a particular control input.

We model the channels in two different ways. Firstly, the AWGN channels as in Fig. 2.3 are considered. The channel input signal q_i is a stationary process with a certain predetermined admissible power level P_i, namely,

$$\boldsymbol{E}[q_i^2] < P_i. \tag{2.1}$$

The signal transmission is corrupted by a zero-mean white Gaussian noise d_i with power spectral density N_i. The ratio $\boldsymbol{E}[q_i^2]/N_i$ is called the signal-to-noise ratio of the channel. We consider the idealized case of infinite-bandwidth AWGN channels. Such idealization simplifies the technicality, yet fulfills the purpose to bring out the essential information limitation on networked stabilization. The capacity of such an infinite-bandwidth AWGN channel with input power constraint P_i is given by [12]

$$\mathfrak{C}_i = \frac{1}{2}\frac{P_i}{N_i}. \tag{2.2}$$

The total channel capacity is given by $\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots + \mathfrak{C}_l$.

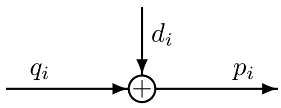

FIG. 2.3. *An AWGN channel.*

We do not assume monotonicity among the channel capacities $\mathfrak{C}_i, i = 1, 2, \ldots, l$. Due to the predetermined admissible power levels, the capacities are fixed a priori and thus, cannot be allocated as in [46, 8, 61, 62]. Nevertheless, the encoder/decoder pair now provides a substitute design freedom in place of the channel resource allocation. Specifically, the encoder matrix T and the decoder matrix R are to be designed subject to the following constraint:

$$RT = I. \tag{2.3}$$

The problem becomes to jointly design the controller and the encoder/decoder pair so as to stabilize the system subject to the input power constraints (2.1). This gives rise to a stabilization problem through coding/control co-design. We are interested in obtaining a fundamental limitation on the channel capacities such that the resulting coding/control co-design problem is solvable.

Other than the AWGN channels, we also consider the fading channels as shown in Fig. 2.4. Here, the channel input signal q_i is subject to a stochastic multiplicative noise $\kappa_i(t)$, where $\kappa_i(t)$ is a continuous-time white noise process with mean $\boldsymbol{E}[\kappa_i(t)] = \mu_i$ and autocovariance $\boldsymbol{E}[(\kappa_i(t) - \mu_i)(\kappa_i(t+\tau) - \mu_i)] = \sigma_i^2\delta(\tau)$. The signal-to-noise ratio of such a fading channel is $\mathrm{SNR}_i = \frac{\mu_i^2}{\sigma_i^2}$. For future use, denote $M = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_l\}$, and $\Sigma^2 = \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_l^2\}$.

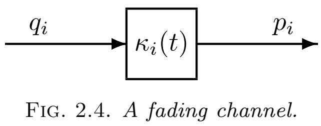

FIG. 2.4. *A fading channel.*

As before, we do not assume any kind of monotonicity among $\mathrm{SNR}_i, i = 1, 2, \ldots, l$. Also, the signal-to-noise ratios of the channels are given a priori and fixed. The lack of channel resource allocation can be compensated by the substitute design freedom given by the coding mechanism. A slight difference from the case of AWGN channels is that now the encoder/decoder pair is to be designed subject to the following constraint:

$$RMT = I, \tag{2.4}$$

which simply says that the output signal u has the same mean as the input signal v. The controller designer is to jointly design the controller and encoder/decoder pair so as to stabilize the system, leading to another stabilization problem via coding/control co-design. We are interested in finding a fundamental limitation on the signal-to-noise ratios such that the resulting coding/control co-design problem is solvable.

While Shannon capacity is used for the infinite-bandwidth AWGN channels, the Shannon capacity of a fading channel when the distribution of the multiplicative noise is unknown cannot be explicitly given [23]. It has been recognized that the Shannon capacity is in general not sufficient to characterize the channels in a feedback loop due to the causality constraint. Several attempts have been made in the literature to define a suitable capacity notion for channels in a feedback system from an information-theoretic point of view, for instance, [48, 36, 44].

This paper concentrates on continuous-time systems. A discrete-time counterpart is also under our investigation, for which some preliminary results have been reported in [59]. Although nowadays communication is mainly done digitally, most physical systems to be controlled are in continuous-time. The controllers are often designed in a continuous-time setting which can be implemented via discrete-time approximation. As such, the capacity requirement for networked stabilization obtained for continuous-time systems is of fundamental importance and is approximately equal to that derived for the discretised systems due to the growing sampling rate. We also wish to mention that much effort has been devoted to the performance limitation of networked control systems on top of the networked stabilization, for instance, [27, 18, 31, 51, 53], etc.

Note that the communication network considered here can be regarded as a special case of the MIMO transceiver recently developed in communication theory and widely applied in wireless communication [42, 55]. As depicted in Fig. 2.5, a typical MIMO transceiver consists of an encoder T, a MIMO channel characterized by a channel matrix Z and additive white noise d, together with a decoder R. It is often built in such a way that the dimensions of q and p are higher than the dimension of v and u so as to exploit the spacial diversity to increase the data transmission capacity [42, 55]. Clearly, the communication model in Fig. 2.2 with AWGN channels can be seen as a simple MIMO transceiver with $Z = I$. In this regard, this work serves as a starting point in studying the networked stabilization over a general MIMO transceiver.

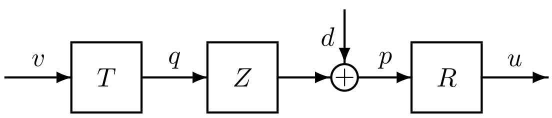

Fig. 2.5. *A MIMO transceiver.*

Before proceeding, we recall an important notion called topological entropy. Given a continuous-time linear system $\dot{x}(t) = Ax(t)$, its topological entropy is defined as the quantity $H(A) = \sum_{\Re(\lambda_i)>0} \lambda_i$, where λ_i are the eigenvalues of A. This quantity appeared frequently in recent studies of networked control [1, 3, 7, 8, 38, 49, 61] as a measure of instability of a linear system based on the minimum resource required for stabilizing the system. It also appeared in early control system literature, mostly in the performance limitation of feedback control [16, 17]. A notion of topological feedback entropy has recently been proposed in [38] which is shown to coincide with the topological entropy in the case of an LTI system.

3. Preliminary.

3.1. Cyclic decomposition. Let A be an $n \times n$ real matrix. The minimal polynomial of A is the monic polynomial $\alpha(\lambda)$ of least degree such that $\alpha(A) = 0$. The minimal polynomial of a matrix is unique. We say that A is cyclic if its minimal polynomial has degree n, or equivalently, its minimal polynomial coincides with its characteristic polynomial. The following lemma gives the cyclic decomposition of a linear system, which plays an essential role in later developments.

LEMMA 3.1. *Given a stabilizable linear system $[A|B]$ with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, there exist nonsingular matrices P and Q such that*

$$[P^{-1}AP|P^{-1}BQ] = \left[\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix} \middle| \begin{bmatrix} b_1 & * & \cdots & * & * \\ 0 & b_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & * & * \\ 0 & \cdots & 0 & b_k & * \end{bmatrix}\right], \tag{3.1}$$

where $A_i, i = 1, 2, \ldots, k$, are cyclic with minimal polynomials $\alpha_i(\lambda)$, such that $\alpha_1(\lambda) = \alpha(\lambda)$ and $\alpha_{i+1}(\lambda)$ divides $\alpha_i(\lambda)$ for $i = 1, 2, \ldots, k-1$. Moreover, the cyclic subsystems $[A_i|b_i], i = 1, 2, \ldots, k$, are stabilizable.

The symbol $*$ in (3.1) represents the upper-right off-diagonal blocks of $P^{-1}BQ$ that are nonzero in general but not much relevant in the developments to follow. The number k as in Lemma 3.1 is referred to as the cyclic index of A and is unique. In fact, it is given by the largest geometric multiplicity of the eigenvalues of A. The minimal polynomials $\alpha_i(\lambda), i = 1, 2, \ldots, k$, are also unique. Since $\alpha_{i+1}(\lambda)$ divides $\alpha_i(\lambda)$, it follows that the spectrum of A_{i+1} is contained in the spectrum of A_i. Consequently, there naturally holds $H(A_1) \geq H(A_2) \geq \cdots \geq H(A_k)$. For more details on the cyclic decomposition of a matrix and cyclic decomposition of a linear system, one can refer to [20, 60].

REMARK 1. *The subsystems $[A_i|b_i], i = 1, 2, \ldots, k$, are referred to as the cyclic subsystems of $[A|B]$. The role of nonsingular matrices P and Q can be considered as linear transformations in the state space and input space, respectively. The following implication can be inferred from Lemma 3.1. In the cyclic decomposition (3.1), A_1 contains the greatest number of unstable poles of $[A|B]$ that can be stabilized by a single control input up to linear transformations in the input space; likewise, A_1 together with A_2 contains the greatest number of unstable poles of $[A|B]$ that can be stabilized by two control inputs up to linear transformations in the input space; and so on so forth.*

3.2. Optimal complementary sensitivity. For the NCS considered in this paper, assume temporarily that the channels are ideal and the encoder/decoder pair is simply trivial, i.e., $l = m$ and $T = R = I$. Then, the complementary sensitivity function at the plant input is given by $\boldsymbol{T}(s) = F(sI - A - BF)^{-1}B$. As shown in many existing works [3, 13, 49, 61] as well as later developments in this work, the feedback stabilization in the presence of either additive noise or multiplicative noise is closely related to the $\mathcal{H}_2$ optimal $\boldsymbol{T}(s)$. For preparation, the following lemma gives a solution to $\mathcal{H}_2$ optimal $\boldsymbol{T}(s)$.

LEMMA 3.2 ([7]). *There holds*

$$\inf_{F:A+BF \text{ is stable}} \|\boldsymbol{T}(s)\|_2 = [2H(A)]^{\frac{1}{2}}.$$

When A has no eigenvalues on the imaginary axis, the infimum can be achieved by the optimal state feedback gain $F = -B'X$, where X is the stabilizing solution to the algebraic Riccati equation

$$A'X + XA - XBB'X = 0.$$

3.3. Two mixed norms for systems. We define two mixed norms for an $m \times m$ stable transfer function $\boldsymbol{G}(s)$ as follows:

$$\|\boldsymbol{G}(s)\|_{2,1} = \left(\max_{1\le j\le m} \sum_{i=1}^{m} \|\boldsymbol{G}(s)_{ij}\|_2^2\right)^{\frac{1}{2}}, \quad \|\boldsymbol{G}(s)\|_{2,\infty} = \left(\max_{1\le i\le m} \sum_{j=1}^{m} \|\boldsymbol{G}(s)_{ij}\|_2^2\right)^{\frac{1}{2}},$$

where $\|\boldsymbol{G}(s)_{ij}\|_2$ is the $\mathcal{H}_2$ norm of the (i,j)th entry of $\boldsymbol{G}(s)$.

A useful lemma is given below. The proof is simply a specialization of Theorem 2 in [50] to the matrix $\left[\|\boldsymbol{G}(s)_{ij}\|_2^2\right]$ and is thus omitted here for brevity.

LEMMA 3.3. *There holds*

$$\rho\left(\left[\|\boldsymbol{G}(s)_{ij}\|_2^2\right]\right) = \inf_{D\in\mathcal{D}} \|D^{-1}\boldsymbol{G}(s)D\|_{2,1}^2 = \inf_{D\in\mathcal{D}} \|D^{-1}\boldsymbol{G}(s)D\|_{2,\infty}^2,$$

where $\mathcal{D}$ is the set of all $m \times m$ diagonal matrices with positive diagonal entries.

3.4. Majorization. As will be seen later, the main results in this paper are given in terms of majorization type conditions. Here, we briefly review some basic concepts and properties in majorization theory. For an extensive treatment of majorization and its applications, one can refer to [35].

For $x, y \in \mathbb{R}^n$, we denote by $x^\uparrow$ and $y^\uparrow$ the rearranged versions of x and y so that their elements are arranged in a non-decreasing order. We say that x is majorized by y, denoted by $x \preccurlyeq y$, if

$$\sum_{i=1}^{k} x_i^\uparrow \ge \sum_{i=1}^{k} y_i^\uparrow, \text{ for } k = 1, \ldots, n-1, \text{ and } \sum_{i=1}^{n} x_i^\uparrow = \sum_{i=1}^{n} y_i^\uparrow. \tag{3.2}$$

Majorization orders the level of fluctuations when the averages are the same. In other words, $x \preccurlyeq y$ says that the elements of x are more even or, less spread out, than the elements of y.

Now, if the last equality in (3.2) is changed to an inequality $\ge$, x is said to be weakly majorized by y from above, denoted by $x \preccurlyeq^w y$. Further, if all the inequalities $\ge$ in (3.2), including the last equality, are changed to strict inequalities $>$, then x is said to be strictly weakly majorized by y from above, denoted by $x \prec^w y$.

Note that the majorization $\preccurlyeq$ and the weak majorization $\preccurlyeq^w$ are pre-orders in $\mathbb{R}^n$. Also note that when two vectors are compared via majorization or weak majorization, the order of elements in the vectors is irrelevant.

The following lemma characterizes the relation between majorization and weak majorization.

LEMMA 3.4 ([35]). *$x \preccurlyeq^w y$ ($x \prec^w y$, respectively), if and only if there exists z such that $x \geq z$ ($x > z$, respectively), and $z \preccurlyeq y$.*

Another useful lemma is given below.

LEMMA 3.5 ([35]). *There exists a real symmetric matrix X with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and diagonal elements $d_1, d_2, \ldots, d_n$, if and only if*

$$\begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix}' \preccurlyeq \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}'.$$

When the condition in Lemma 3.5 is satisfied, efficient algorithms for finding the desired symmetric matrix X have also been developed in the literature [5, 28].

4. Stabilizability over AWGN Channels with Linear Coding. In the case when AWGN channels are considered, the closed-loop NCS is depicted in Fig. 4.1, where T and R are to be designed subject to the constraint (2.3).

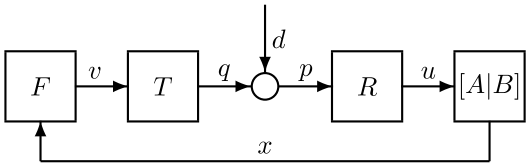

FIG. 4.1. *NCS over AWGN channels with linear coding.*

We confine our attention to the scenario when the closed-loop system is internally stable such that the distribution of all signals converges exponentially to a stationary distribution. Without loss of generality, we assume that the closed-loop system has reached its steady state and all the signals are wide sense stationary. According to our setup, the total noise $d = \begin{bmatrix} d_1 & d_2 & \cdots & d_l \end{bmatrix}'$ is a vector white Gaussian noise with power spectral density $N = \operatorname{diag}\{N_1, N_2, \ldots, N_l\}$. The complementary sensitivity function, i.e., the closed-loop transfer function from d to q, is given by

$$\boldsymbol{T}(s) = TF(sI - A - BRTF)^{-1}BR = TF(sI - A - BF)^{-1}BR.$$

Then, the power spectrum density of q_i has the expression $\{\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*\}_{ii}$ and consequently, the power of q_i is given by $\boldsymbol{E}[q_i^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*\}_{ii}d\omega$, where $\{\cdot\}_{ii}$ stands for the ith diagonal element of a matrix. It follows that the input power constraint (2.1) can be rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*\}_{ii}d\omega < P_i.$$

In view of (2.2), such constraint can be further translated into

$$\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{N^{-\frac{1}{2}}\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*N^{-\frac{1}{2}}\right\}_{ii}d\omega < \mathfrak{C}_i. \tag{4.1}$$

The objective is to find the requirement on the channel capacities $\mathfrak{C}_i, i = 1, 2, \ldots, l$, such that the networked stabilization can be accomplished subject to the constraints (4.1) via a judicious coding/control co-design. It is also our hope to come up with a systematic procedure to perform the coding/control co-design.

In order to stabilize the NCS, each control input requires certain communication resource for the transmission purpose. As such, the control inputs can be considered as the demand side for the communication resource, while the channels are considered as the supply side. The supply capabilities of the channels are characterized by their capacities. The challenge lies in that the channel capacities are given a priori and thus, the demand and supply may not match in general. To resolve such imbalance of demand and supply, it is crucial to understand that the encoder matrix T has an effect of mixing the demands from different control inputs. With that in mind, an interesting idea would be to exploit the coding mechanism judiciously so that after the mixing, the demands will be reshaped properly so as to match the supplies. For comparison, the channel resource allocation adopted in [46, 8, 15, 61] does the exact opposite, i.e., tailoring the supplies to match the demands. Note that demand shaping is a quite general principle in economics. It has led to many successful stories in engineering fields as well such as power systems [40, 52], transportation [41], and data networks [33], etc.

The idea of demand shaping turns out to work perfectly. We arrive at a necessary and sufficient condition for the solvability of the coding/control co-design problem given in terms of a weak majorization relation, as shown in the following theorem. The proof is simply making the above understanding precise in a formal way.

THEOREM 4.1. *$[A|B]$ is stabilizable over AWGN channels through coding/control co-design, if and only if*

$$\begin{bmatrix}\mathfrak{C}_1 & \mathfrak{C}_2 & \cdots & \mathfrak{C}_l\end{bmatrix}' \prec^w \begin{bmatrix}H(A_1) & H(A_2) & \cdots & H(A_k) & 0 & \cdots & 0\end{bmatrix}', \tag{4.2}$$

where $H(A_i), i = 1, 2, \ldots, k$, are the topological entropies of the cyclic subsystems $[A_i|b_i]$ as in (3.1).

Proof. For brevity, assume that all the eigenvalues of A lie in the open right half complex plane. This assumption can be removed following the same arguments as in [3, 8, 46, 61].

We first show the necessity. Note that the channels can always be reordered so that their capacities are arranged in a non-increasing order. Therefore, without loss of generality, assume that $\mathfrak{C}_1 \geq \mathfrak{C}_2 \geq \cdots \geq \mathfrak{C}_l$. Suppose there exists a state feedback gain F together with an encoder matrix T and a decoder matrix R such that the NCS is stabilized and the constraints (4.1) are satisfied. It suffices to show the inequality

$$\sum_{i=j}^{l} \mathfrak{C}_i > \sum_{i=j}^{k} H(A_i) \tag{4.3}$$

holds for $j = 1, 2, \ldots, k$. The case when $j = 1$ is quite straightforward since

$$\begin{aligned}\sum_{i=1}^{l} \mathfrak{C}_i >& \sum_{i=1}^{l} \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{N^{-\frac{1}{2}}\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^* N^{-\frac{1}{2}}\right\}_{ii} d\omega \\ =& \frac{1}{2}\|N^{-\frac{1}{2}}\boldsymbol{T}(s)N^{\frac{1}{2}}\|_2^2 \geq H(A) = \sum_{i=1}^{k} H(A_i),\end{aligned}$$

where the second inequality follows from Lemma 3.2. We proceed to show the case when $j = 2$. Let us carry out the controllable-uncontrollable decomposition to the system $[A|BR]$ with respect to the first column of BR, i.e., find a state space transformation $x(t) = Py(t)$ such that the system $[A|BR]$ is transformed to

$$\begin{bmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ 0 & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} + \begin{bmatrix}\tilde{B}_{11} & \tilde{B}_{12}\\ 0 & \tilde{B}_{22}\end{bmatrix}\begin{bmatrix}p_1(t)\\ \tilde{p}(t)\end{bmatrix}, \tag{4.4}$$

where

$$\begin{aligned} p_1(t) &= q_1(t) + d_1(t), \\ \tilde{p}(t) &= \tilde{q}(t) + \tilde{d}(t), \\ \tilde{q}(t) &= \begin{bmatrix} q_2(t) & q_3(t) & \cdots & q_l(t) \end{bmatrix}', \\ \tilde{d}(t) &= \begin{bmatrix} d_2(t) & d_3(t) & \cdots & d_l(t) \end{bmatrix}'. \end{aligned}$$

Set $\tilde{F} = TFP$ and partition $\tilde{F}$ compatibly as $\tilde{F} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix}$. Consider the subsystem $[\tilde{A}_{22}|\tilde{B}_{22}]$ with closed-loop dynamics:

$$\begin{aligned} \dot{y}_2(t) &= \tilde{A}_{22} y_2(t) + \tilde{B}_{22}\tilde{q}(t) + \tilde{B}_{22}\tilde{d}(t), \\ \tilde{q}(t) &= \tilde{F}_{22} y_2(t) + \tilde{F}_{21} y_1(t). \end{aligned}$$

Applying the Laplace transform to both sides of the above equations yields

$$\mathcal{L}(\tilde{q}(t)) = \begin{bmatrix} \tilde{\boldsymbol{T}}_{21}(s) & \tilde{\boldsymbol{T}}_{22}(s) \end{bmatrix} \begin{bmatrix} \mathcal{L}(y_1(t)) \\ \mathcal{L}(\tilde{d}(t)) \end{bmatrix},$$

where

$$\begin{aligned} \tilde{\boldsymbol{T}}_{21}(s) &= \tilde{F}_{21} + \tilde{F}_{22}(sI - \tilde{A}_{22} - \tilde{B}_{22}\tilde{F}_{22})^{-1}\tilde{B}_{22}\tilde{F}_{21}, \\ \tilde{\boldsymbol{T}}_{22}(s) &= \tilde{F}_{22}(sI - \tilde{A}_{22} - \tilde{B}_{22}\tilde{F}_{22})^{-1}\tilde{B}_{22}. \end{aligned}$$

Since $y_1(t)$ is independent of $\tilde{d}(t)$, we have

$$\boldsymbol{E}[q_{i+1}^2] \geq \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\tilde{\boldsymbol{T}}_{22}(j\omega)\tilde{N}\tilde{\boldsymbol{T}}_{22}(j\omega)^*\}_{ii} d\omega,$$

for $i = 1, 2, \ldots, l-1$, where $\tilde{N} = \operatorname{diag}\{N_2, N_3, \ldots, N_l\}$. Consequently, there holds

$$\sum_{i=2}^{l} \mathfrak{C}_i > \sum_{i=2}^{l} \frac{1}{2} \frac{\boldsymbol{E}[q_i^2]}{N_i} \geq \frac{1}{2} \|\tilde{N}^{-\frac{1}{2}} \tilde{\boldsymbol{T}}_{22}(s) \tilde{N}^{\frac{1}{2}}\|_2^2 \geq H(\tilde{A}_{22}).$$

Meanwhile, it can be inferred from Remark 1 that $H(\tilde{A}_{22}) \geq \sum_{i=2}^{k} H(A_i)$ and, thus, $\sum_{i=2}^{l} \mathfrak{C}_i > \sum_{i=2}^{k} H(A_i)$. In analogy to the above procedure, we can verify the validity of inequality (4.3) for $j = 3, \ldots, k$ as well, which completes the necessity proof.

To show the sufficiency, we will seek a state feedback gain F together with an encoder matrix T and a decoder matrix R such that the NCS is stabilized and the constraints (4.1) are satisfied. Without loss of generality, assume that $[A|B]$ is in the cyclic decomposition form (3.1), where each cyclic subsystem $[A_i|b_i], i = 1, 2, \ldots, k$, is stabilizable with state dimension n_i. For each $[A_i|b_i]$, we can design a stabilizing state feedback gain f_i such that $\|\boldsymbol{T}_i(s)\|_2^2 = 2H(A_i)$, where

$$\boldsymbol{T}_i(s) = f_i(sI - A_i - b_i f_i)^{-1} b_i. \tag{4.5}$$

The existence of such f_i is due to Lemma 3.2. Let $f = \operatorname{diag}\{f_1, f_2, \ldots, f_k\}$ and design

$$F = \begin{bmatrix} f \\ 0_{(m-k)\times n} \end{bmatrix}. \tag{4.6}$$

It is straightforward to verify that F is stabilizing, i.e., $A + BF$ is stable. Regarding the encoder/decoder pair, let

$$T = N^{\frac{1}{2}} U D^{-1} \text{ and } R = D U' N^{-\frac{1}{2}}, \tag{4.7}$$

where $U \in \mathbb{R}^{l \times m}$ is an isometry to be designed and $D = \operatorname{diag}\{1, \epsilon, \ldots, \epsilon^{m-1}\}$ with ϵ being a small positive number. Also set $S = \operatorname{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{k-1} I_{n_k}\}$. Then

$$\boldsymbol{T}(s) = TF(sI - A - BF)^{-1} BR = N^{\frac{1}{2}} U \bar{F} (sI - \bar{A} - \bar{B}\bar{F})^{-1} \bar{B} U' N^{-\frac{1}{2}},$$

where

$$\bar{F} = D^{-1} F S = F, \tag{4.8}$$

$$\bar{A} = S^{-1} A S = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix}, \tag{4.9}$$

$$\bar{B} = S^{-1} B D = \begin{bmatrix} b_1 & o(\epsilon) & \cdots & o(\epsilon) & o(\epsilon) \\ 0 & b_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) & o(\epsilon) \\ 0 & \cdots & 0 & b_k & o(\epsilon) \end{bmatrix}, \tag{4.10}$$

and $\frac{o(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon \to 0$. It follows that

$$\begin{aligned} &\frac{1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} N^{-\frac{1}{2}} \boldsymbol{T}(j\omega) N \boldsymbol{T}(j\omega)^* N^{-\frac{1}{2}} d\omega \\ &= U \left(\operatorname{diag}\left\{ \frac{\|\boldsymbol{T}_1(s)\|_2^2}{2}, \ldots, \frac{\|\boldsymbol{T}_k(s)\|_2^2}{2}, 0, \ldots, 0 \right\} + o(\epsilon) \right) U' \\ &= U \left(\operatorname{diag}\{H(A_1), \ldots, H(A_k), 0, \ldots, 0\} + o(\epsilon) \right) U'. \end{aligned} \tag{4.11}$$

When the relation (4.2) holds, by Lemma 3.4, there exists a vector $\begin{bmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_l \end{bmatrix}'$ such that

$$\begin{bmatrix} \mathfrak{C}_1 & \mathfrak{C}_2 & \ldots & \mathfrak{C}_l \end{bmatrix}' > \begin{bmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_l \end{bmatrix}', \tag{4.12}$$

and

$$\begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_l \end{bmatrix}' \preccurlyeq \begin{bmatrix} H(A_1) & H(A_2) & \cdots & H(A_k) & 0 & \cdots & 0 \end{bmatrix}'. \tag{4.13}$$

Further, in view of (4.13) and Lemma 3.5, an isometry U can be constructed such that

$$\{U \left(\operatorname{diag}\{H(A_1), \ldots, H(A_k), 0, \ldots, 0\} \right) U'\}_{ii} = \gamma_i, \tag{4.14}$$

for $i = 1, 2, \ldots, l$. Finally, putting (4.11), (4.12), and (4.14) together, we can verify that the constraints (4.1) are satisfied when ϵ is sufficiently small. This completes the proof. □

We wish to mention that the design of encoder/decoder pair and controller is not unique in general. The constructive proof of Theorem 4.1 gives one way, among others, for such a coding/control co-design. The state feedback gain F is designed as in (4.6) by solving $\mathcal{H}_2$ optimal complementary sensitivity for each cyclic subsystem $[A_i|b_i]$. The encoder/decoder pair is designed as in (4.7), where D is a suitably chosen scaling matrix, and U is an isometry which can be constructed by exploiting the algorithm proposed in [5].

Let us now revisit the intuition of demand shaping to better digest this result. By the knowledge of topological entropy and cyclic decomposition, the vector on the right-hand side of (4.2) can be considered as the intrinsic demands for the communication resource. Due to the coding mechanism, such demands will be mixed, leading to a set of reshaped demands that are more even, or less spread out. The total demand will remain the same after the mixing. Combining these observations yields that in order to enable the reshaped demands to match the given supplies, the supplies must meet the following two requirements:

1. The total supply is greater than the total demand, i.e., $\mathfrak{C} > H(A)$.
2. The supplies from the communication channels are less spread out than the most uneven demands.

The above two requirements are exactly what the condition (4.2) says in light of the physical interpretation of weak majorization.

From the majorization condition (4.2), we can also infer that in some cases, it may even be possible to stabilize the NCS with fewer channels than the number of control inputs. In fact, the minimum number of channels needed for stabilization equals the number of unstable cyclic subsystems $[A_i|b_i]$ yielded from the cyclic decomposition (3.1). This is consistent with earlier studies [26, 60] in the literature that investigate the minimum number of control inputs required to stabilize a linear system.

One can further deduce the following two corollaries from Theorem 4.1.

COROLLARY 4.2. *If the cyclic decomposition of* $[A|B]$ *has only one unstable cyclic subsystem, then* $[A|B]$ *is stabilizable over AWGN channels through coding/control co-design, if and only if* $\mathfrak{C} > H(A)$.

Proof. When $[A|B]$ has only one unstable cyclic subsystem, the weak majorization condition (4.2) becomes

$$\begin{bmatrix}\mathfrak{C}_1 & \mathfrak{C}_2 & \cdots & \mathfrak{C}_l\end{bmatrix}' \prec^w \begin{bmatrix}H(A) & 0 & \cdots & 0\end{bmatrix}',$$

which holds, if and only if $\mathfrak{C} > H(A)$. □

Corollary 4.2 is consistent with the result in [49]. Also, it includes the single-input system as a special case since a stabilizable single-input system only has one unstable cyclic subsystem. Hence, this corollary suggests that to stabilize a single-input system via coding/control co-design, we only require the total capacity be greater than the topological entropy of the open-loop plant. How the individual channel capacities are distributed is not relevant in this case.

COROLLARY 4.3. *If* $\mathfrak{C}_1 = \mathfrak{C}_2 = \cdots = \mathfrak{C}_l$*, then* $[A|B]$ *is stabilizable over AWGN channels through coding/control co-design, if and only if* $\mathfrak{C} > H(A)$.

Proof. When $\mathfrak{C}_1 = \mathfrak{C}_2 = \cdots = \mathfrak{C}_l$, the condition (4.2) becomes

$$\frac{1}{l}\begin{bmatrix}\mathfrak{C} & \mathfrak{C} & \cdots & \mathfrak{C}\end{bmatrix}' \prec^w \begin{bmatrix}H(A_1) & H(A_2) & \cdots & H(A_k) & 0 & \cdots & 0\end{bmatrix}',$$

which holds, if and only if $\mathfrak{C} > H(A)$. □

Corollary 4.3 suggests that identical communication channels can help each other best in transmitting the signals.

Note that we use full matrices T and R to perform encoding and decoding, which allows both scaling and rotation of control signals. If only scaling is allowed in some practical scenarios, the stabilizability condition has been investigated in [57, 58]. In the present setup, through the combined role of scaling and rotation, one can in effect change the output directions of the unstable poles, and thus obtain a more relaxed condition in terms of the majorization relation (4.2) due to the additional freedom provided by rotation.

5. Stabilizability over Fading Channels with Linear Coding. The same idea extends to the networked stabilization over fading channels with linear coding. In this case, the closed-loop system has the form as shown in Fig. 5.1, where $\kappa(t) = \operatorname{diag}\{\kappa_1(t), \kappa_2(t), \ldots, \kappa_l(t)\}$ stands for the multiplicative noise in the fading channels. The signal-to-noise ratios of these fading channels are fixed. As before, the encoder matrix T and decoder matrix R are to be designed subject to a mild constraint as in (2.4), leading to a stabilization problem via coding/control co-design.

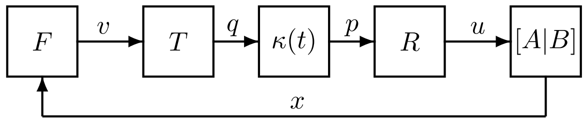

FIG. 5.1. *NCS over fading channels with linear coding.*

Now, the dynamics of closed-loop system is given by $\dot{x}(t) = [A + BR\kappa(t)TF]x(t)$. Different from the case of AWGN channels, here we do not assume that the closed-loop system has reached its steady state. Denote by $X(t) = \boldsymbol{E}[x(t)x(t)']$ the state covariance. By Itô's formula [29], the evolution of $X(t)$ is governed by the following matrix differential equation:

$$\dot{X}(t) = (A + BF)X(t) + X(t)(A + BF)' + BR[\Sigma^2 \odot (TFX(t)F'T')]R'B',$$

where $\odot$ means Hadamard product. We say that $[A|B]$ is mean-square stabilizable if there exist a state feedback gain F and an encoder/decoder pair such that the closed-loop system in Fig. 5.1 is mean-square stable, i.e., $\lim_{t\to\infty} X(t) = 0$. The aim is to find the requirement on $\mathrm{SNR}_i, i = 1, 2, \ldots, l$, such that the networked stabilization can be accomplished via coding/control co-design.

The next lemma provides several equivalent criteria in verifying the mean-square stabilizability. The equivalence between the implications (a), (b), and (c) can be shown by specializing the well-established results in stochastic control [2]. The equivalence between the implications (a) and (d) is simply the continuous-time counterpart of Theorem 6.4 in [13]. The details of the proof are omitted here for brevity.

LEMMA 5.1. *The following statements are equivalent:*

(a) $[A|B]$ *is mean-square stabilizable.*

(b) There exist a state feedback gain and an encoder/decoder pair such that the matrix inequality

$$(A + BF)X + X(A + BF)' + BR[\Sigma^2 \odot (TFXF'T')]R'B' < 0$$

has a solution $X > 0$.

(c) There exists an encoder/decoder pair such that the matrix inequality

$$A'X + XA - XBRM[\Sigma^2 \odot (R'B'XBR)]^{-1}MR'B'X < 0 \qquad (5.1)$$

has a solution $X > 0$.

(d) There exist a state feedback gain and an encoder/decoder pair such that

$$\inf_{D\in\mathcal{D}} \|D^{-1}\boldsymbol{T}(s)\Phi D\|_{2,1} < 1, \tag{5.2}$$

where $\mathcal{D}$ is the set of $m\times m$ diagonal matrices with positive diagonal entries, $\Phi = M^{-1}\Sigma$, and $\boldsymbol{T}(s) = TF(sI-A-BF)^{-1}BRM$.

Applying Lemma 5.1 together with the demand shaping idea as in the previous section, we manage to show the following result.

THEOREM 5.2. *$[A|B]$ is mean-square stabilizable over fading channels through coding/control co-design, if and only if*

$$\frac{1}{2}\begin{bmatrix}\mathrm{SNR}_1 & \mathrm{SNR}_2 & \cdots & \mathrm{SNR}_l\end{bmatrix}' \prec^w \begin{bmatrix}H(A_1) & H(A_2) & \cdots & H(A_k) & 0 & \cdots & 0\end{bmatrix}', \tag{5.3}$$

where $H(A_i), i = 1,2,\ldots,k$, are the topological entropies of the cyclic subsystems $[A_i|b_i]$ as in (3.1).

Proof. As before, assume that all the eigenvalues of A lie in the open right half complex plane for brevity.

We first show the necessity. Suppose $[A|B]$ is mean-square stabilizable, we shall show the relation (5.3) holds. Without loss of generality, assume that $\mathrm{SNR}_1 \geq \mathrm{SNR}_2 \geq \cdots \geq \mathrm{SNR}_l$. It suffices to show

$$\frac{1}{2}\sum_{i=j}^{l}\mathrm{SNR}_i > \sum_{i=j}^{k} H(A_i) \tag{5.4}$$

holds for $j = 1,2,\ldots,k$. We start from the case when $j = 1$. By Lemma 5.1, there exists an encoder/decoder pair such that the matrix inequality (5.1) has a solution $X > 0$. Pre-multiplying and post-multiplying $X^{-\frac{1}{2}}$ on both sides of (5.1), and taking trace for both sides of the resulted inequality yields

$$\begin{aligned}\mathrm{tr}(X^{-\frac{1}{2}}A'X^{\frac{1}{2}}) + \mathrm{tr}(X^{\frac{1}{2}}AX^{-\frac{1}{2}}) - \mathrm{tr}\big\{X^{\frac{1}{2}}BRM[\Sigma^2\odot(R'B'XBR)]^{-1}MR'B'X^{\frac{1}{2}}\big\}\\ = 2H(A) - \sum_{i=1}^{l}\mathrm{SNR}_i < 0,\end{aligned}$$

that validates (5.4) when $j = 1$. We now proceed to the case when $j = 2$. Let us carry out the controllable-uncontrollable decomposition to the system $[A|BR]$ with respect to the first column of BR such that $[A|BR]$ is transformed to the form (4.4), where

$$\begin{aligned}p_1(t) &= \kappa_1(t)q_1(t),\\ \tilde{p}(t) &= \tilde{\kappa}(t)\tilde{q}(t),\\ \tilde{q}(t) &= \begin{bmatrix}q_2(t) & q_3(t) & \cdots & q_l(t)\end{bmatrix}',\\ \tilde{\kappa}(t) &= \begin{bmatrix}\kappa_2(t) & \kappa_3(t) & \cdots & \kappa_l(t)\end{bmatrix}'.\end{aligned}$$

Set $\tilde{F} = TFP$ and partition $\tilde{F}$ compatibly as $\tilde{F} = \begin{bmatrix}\tilde{F}_{11} & \tilde{F}_{12}\\ \tilde{F}_{21} & \tilde{F}_{22}\end{bmatrix}$. Consider the subsystem $[\tilde{A}_{22}|\tilde{B}_{22}]$ with closed-loop dynamics

$$\dot{y}_2(t) = [\tilde{A}_{22} + \tilde{B}_{22}\tilde{\kappa}(t)\tilde{F}_{22}]y_2(t) + \tilde{B}_{22}\tilde{\kappa}(t)\tilde{F}_{21}y_1(t),$$

where the term $\tilde{F}_{21}y_1(t)$ can be treated as a noise, the power of which goes to zero as time goes to infinity. Then, by mimicking the proof of the case when $j = 1$, one can validate the inequality (5.4) when $j = 2$. In the same spirit, one can also validate the inequality (5.4) when $j = 3, \ldots, k$, which completes the necessity proof.

For the sufficiency, we will seek a state feedback gain F together with an encoder matrix T and a decoder matrix R such that the inequality (5.2) is satisfied. Without loss of generality, assume that $[A|B]$ is in the cyclic decomposition form (3.1), where each cyclic subsystem $[A_i|b_i], i = 1, 2, \ldots, k$, is stabilizable with state dimension n_i. For each $[A_i|b_i]$, we can design a stabilizing state feedback gain f_i such that $\|\boldsymbol{T}_i(s)\|_2^2 = 2H(A_i)$, where $\boldsymbol{T}_i(s)$ is given by (4.5). The existence of such f_i is guaranteed by Lemma 3.2. Let $f = \mathrm{diag}\{f_1, f_2, \ldots, f_k\}$ and design F as in (4.6). Regarding the encoder/decoder pair, let

$$T = M^{-\frac{1}{2}}UD^{-1} \text{ and } R = DU'M^{-\frac{1}{2}}, \tag{5.5}$$

where $U \in \mathbb{R}^{l\times m}$ is an isometry to be designed and $D = \mathrm{diag}\{1, \epsilon, \ldots, \epsilon^{m-1}\}$ with ϵ being a small positive number. Also set $S = \mathrm{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{k-1}I_{n_k}\}$. Then

$$\boldsymbol{T}(s) = TF(sI - A - BF)^{-1}BRM = M^{-\frac{1}{2}}U\bar{F}(sI - \bar{A} - \bar{B}\bar{F})^{-1}\bar{B}U'M^{\frac{1}{2}},$$

where $\bar{F}$, $\bar{A}$, and $\bar{B}$ are as in (4.8), (4.9), and (4.10), respectively. It follows that

$$\begin{aligned}\|M^{\frac{1}{2}}\boldsymbol{T}(s)\Phi M^{-\frac{1}{2}}\|_{2,1}^2 &= \max_{1\le j\le l} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\Phi M^{-\frac{1}{2}}\boldsymbol{T}^*(j\omega)M\boldsymbol{T}(j\omega)M^{-\frac{1}{2}}\Phi\right\}_{jj} d\omega \\ &= \max_{1\le j\le l}\left\{\Phi U(\mathrm{diag}\{\|\boldsymbol{T}_1(s)\|_2^2, \ldots, \|\boldsymbol{T}_k(s)\|_2^2, 0, \ldots, 0\} + o(\epsilon))U'\Phi\right\}_{jj} \\ &= \max_{1\le j\le l}\left\{\Phi U(\mathrm{diag}\{2H(A_1), \ldots, 2H(A_k), 0, \ldots, 0\} + o(\epsilon))U'\Phi\right\}_{jj}.\end{aligned} \tag{5.6}$$

When the relation (5.3) holds, by Lemma 3.4, there exists a vector $\begin{bmatrix}\gamma_1 & \gamma_2 & \ldots & \gamma_l\end{bmatrix}'$ such that the inequalities

$$\frac{1}{2}\begin{bmatrix}\mathrm{SNR}_1 & \mathrm{SNR}_2 & \ldots & \mathrm{SNR}_l\end{bmatrix}' > \begin{bmatrix}\gamma_1 & \gamma_2 & \ldots & \gamma_l\end{bmatrix}'$$

and (4.13) are satisfied. Further, by Lemma 3.5, an isometry U can be constructed such that the equality (4.14) holds for $i = 1, 2, \ldots, l$. Now, substituting (4.14) into (5.6) yields $\|M^{\frac{1}{2}}\boldsymbol{T}(s)\Phi M^{-\frac{1}{2}}\|_{2,1}^2 = \max_{1\le j\le l}\frac{2\gamma_j}{\mathrm{SNR}_j} + o(\epsilon)$. It is clear that when ϵ is sufficiently small, we have $\|M^{\frac{1}{2}}\boldsymbol{T}(s)\Phi M^{-\frac{1}{2}}\|_{2,1} < 1$, which completes the proof. □

All the discussions following Theorem 4.1 apply here as well. Also, Corollary 4.2 and Corollary 4.3 can be adapted straightforwardly to the case of fading channels. We omit the detailed statements for brevity.

6. Example. Consider the following unstable system $[A|B]$:

$$A = \begin{bmatrix}4 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad B = \begin{bmatrix}1 & 1\\ 1 & 1\\ 1 & 1\\ 0 & 1\end{bmatrix},$$

with initial condition $x_0 = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}'$. Clearly, $[A|B]$ is stabilizable. Moreover, it is already in the cyclic decomposition form (3.1) with cyclic subsystems

$$[A_1|b_1] = \left[\begin{bmatrix}4 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\middle|\begin{bmatrix}1\\1\\1\end{bmatrix}\right] \text{ and } [A_2|b_2] = [1|1].$$

It follows that $H(A_1) = 4+2+1 = 7$, and $H(A_2) = 1$.

We first consider the stabilization over three AWGN channels. For simplicity, let $N = I$. The admissible transmission power of these channels are given by $P_1 = 9.1$, $P_2 = 3.1$, and $P_3 = 4.1$. In view of (2.2), the channel capacities are $\mathfrak{C}_1 = 4.55$, $\mathfrak{C}_2 = 1.55$, and $\mathfrak{C}_3 = 2.05$. One can easily verify that the strictly weak majorization relation (4.2) holds. By Theorem 4.1, the networked stabilization can be accomplished via certain coding/control co-design. One such co-design is carried out as below.

For the controller design, we solve the $\mathcal{H}_2$ optimal $\boldsymbol{T}_i(s)$ as in (4.5) for each cyclic subsystem $[A_i|b_i], i = 1, 2$, yielding the optimal feedback gains $f_1 = \begin{bmatrix}-40 & 36 & -10\end{bmatrix}$ and $f_2 = -2$, respectively. Let

$$F = \mathrm{diag}\{f_1, f_2\} = \begin{bmatrix}-40 & 36 & -10 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}. \tag{6.1}$$

As for the coding design, let the encoder T and decoder R be as in (4.7) with

$$U = \begin{bmatrix}0.7817 & 0.4714\\ 0.4629 & 0\\ -0.4179 & 0.8819\end{bmatrix} \text{ and } D = \begin{bmatrix}1 & 0\\ 0 & 0.1\end{bmatrix}.$$

With this co-design, we observe that the closed-loop poles are exactly the mirror images of the open-loop poles with respect to the imaginary axis. This validates the stability of the closed-loop system. Moreover, further computation yields

$$\boldsymbol{E}[q_1^2] = 9.0848 < P_1, \quad \boldsymbol{E}[q_2^2] = 3.0299 < P_2, \quad \boldsymbol{E}[q_3^2] = 4.0249 < P_3,$$

i.e., the input power constraints (2.1) are satisfied. Combining these observations, we see that the networked stabilization is accomplished via this coding/control co-design.

Next, we consider the case of three fading channels. The mean and covariance of the multiplicative noise $\kappa(t)$ are given by

$$M = \begin{bmatrix}2 & 0 & 0\\ 0 & 0.6 & 0\\ 0 & 0 & 0.9\end{bmatrix}, \quad \Sigma^2 = \begin{bmatrix}0.35 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & 0.25\end{bmatrix},$$

respectively. The signal-to-noise ratios of these fading channels are $\mathrm{SNR}_1 = 11.4286$, $\mathrm{SNR}_2 = 1.8$, and $\mathrm{SNR}_3 = 3.24$. Again, we can verify that the strictly weak majorization relation (5.3) is satisfied and, thus, the mean-square stabilization can be accomplished via coding/control co-design. One such co-design is carried out as below.

For the controller design, we use the same state feedback gain as in (6.1). For the coding design, let the encoder T and decoder R be as in (5.5) with

$$U = \begin{bmatrix}0.8952 & -0.1993\\ 0 & 0.8944\\ 0.4456 & 0.4004\end{bmatrix} \text{ and } D = \begin{bmatrix}1 & 0\\ 0 & 0.1\end{bmatrix}.$$

Under this coding/control co-design, the closed-loop system is mean-square stable. As in Fig. 6.1, the Frobenius norm of the state covariance goes to zero asymptotically.

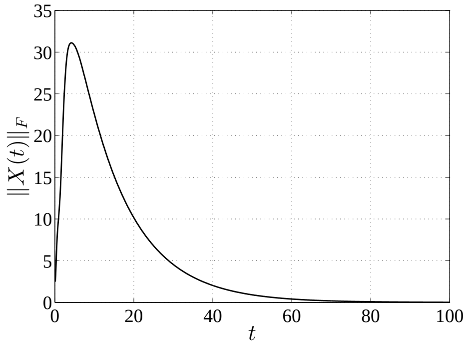

FIG. 6.1. *Closed-loop evolution of* $\|X(t)\|_F$.

7. Conclusion. In this paper, we study the stabilization of a continuous-time linear system over a communication network modeled as the cascade of an encoder, a set of parallel channels, and a decoder. The channel capacities or signal-to-noise ratios are fixed while the encoder/decoder pair can be designed. We first consider the AWGN channels and then the fading channels. In both cases, the resulting coding/control co-design problem is shown to be solvable if and only if a majorization type condition is satisfied.

We conclude by echoing the motivating questions raised before. Both the general full matrix encoder/decoder pair and the redundancy in the communication channels bring in new advantages in stabilizing NCSs.

1) The use of full matrix encoder/decoder pair enables further relaxation of the stabilizability condition compared to that obtained in [57, 58], where diagonal encoder/decoder pair was exploited.
2) Using more channels than the number of control inputs has the advantage of reducing the capacity requirement for individual channels. For example, one can use a number of channels with small capacities to stabilize a single-input system as long as the total capacity is greater than the topological entropy. Similar ideas of introducing redundancy can be found in many other areas as well such as error-correcting codes [43], over-sampled filter banks [4], and signal compression [34], etc.
3) By virtue of the coding mechanism, in some cases, it may even be possible to stabilize an NCS with fewer channels than the number of control inputs. The minimum number of channels needed equals the number of unstable cyclic subsystems yielded from the cyclic decomposition.

Acknowledgment. The authors would like to thank Professor Weizhou Su of South China University of Technology, Professor Bo Bernhardsson of Lund University, Professor Lidong He of Nanjing University of Science and Technology, and Dr. Wei Jiang of Hong Kong University of Science and Technology for helpful discussions.

REFERENCES

[1] J. Baillieul, "Feedback coding for information based control: Operating near the data rate limit", in *Proc. 41th IEEE Conf. Decision Contr.*, pp. 3229-3236, Las Vegas, NV, 2002.

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[3] J. H. Braslavsky, R. H. Middleton, and J. S. Freudenberg, "Feedback stabilization over signal-

to-noise ratio constrained channels", *IEEE Trans. Automat. Contr.*, vol. 52, pp. 1391-1403, Aug. 2007.

[4] L. Chai, J. Zhang, C. Zhang, and E. Mosca, "Frame-theory-based analysis and design of oversampled filter banks: direct computational method", *IEEE Trans. Signal Process.*, vol. 55, pp. 507-519, Feb. 2007.

[5] N. N. Chan and K.-H. Li, "Diagonal elements and eigenvalues of a real symmetric matrix", *J. Math. Anal. Appl.*, vol. 91, pp. 562-566, 1983.

[6] C.-S. Chang and D. D. Yao, "Rearrangement, majorization and stochastic scheduling", *Math. Oper. Res.*, vol. 18, pp. 658-684, 1993.

[7] W. Chen, "Topological entropy of continuous-time linear systems", MPhil Thesis, Hong Kong University of Science and Technology, 2010.

[8] W. Chen and L. Qiu, "Stabilization of networked control systems with multirate sampling", *Automatica*, vol. 49, no. 6, pp. 1528-1537, 2013.

[9] W. Chen, S. Wang, and L. Qiu, "MIMO control using MIMO communication: a majorization condition for networked stabilizability", in *Proc. 2015 Amer. Control Conf.*, pp. 2205-2210, Chicago, USA, 2015.

[10] W. Chen, S. Wang, and L. Qiu, "A majorization condition for MIMO stabilizability via MIMO transceivers with pure fading subchannels", in *Proc. 34th Chinese Contr. Conf.*, pp. 6800-6804, Hangzhou, China, 2015.

[11] F. Cicalese and U. Vaccaro, "Supermodularity and subadditivity properties of the entropy on the majorization lattice", *IEEE Trans. Inf. Theory*, vol. 48, pp. 933-938, Apr. 2002.

[12] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, 1991.

[13] N. Elia, "Remote stabilization over fading channels", *Syst. Control Lett.*, vol. 54, no. 3, pp. 237-249, 2005.

[14] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information", *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1384-1400, Sept. 2001.

[15] Y. Feng, X. Chen, and G. Gu, "Quantized state feedback control for multiple-input systems subject to signal-to-noise ratio constraints", in *Proc. 52th IEEE Conf. Decision Contr.*, pp. 7229-7234, Florence, Italy, 2013.

[16] J. S. Freudenberg and D. Looze, "Right half plane poles and zeros and design tradeoffs in feedback systems", *IEEE Trans. Automat. Contr.*, vol. 30, pp. 555-565, Jun. 1985.

[17] J. S. Freudenberg and D. Looze, " A sensitivity tradeoff for plants with time delay", *IEEE Trans. Automat. Contr.*, vol. 32, pp. 99-104, Feb. 1987.

[18] J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky, "Minimum variance control over a Gaussian communication channel", *IEEE Trans. Automat. Contr.*, vol. 56, pp. 1751-1765, Aug. 2011.

[19] M. Fu and L. Xie, "The sector bound approach to quantized feedback control", *IEEE Trans. Automat. Contr.*, vol. 50, pp. 1698-1711, Nov. 2005.

[20] F. R. Gantmacher, *The Theory of Matrices*, vol. 1, Chelsea, New York, 1959.

[21] H. Gao and T. Chen, "A new approach to quantized feedback control systems", *Automatica*, vol. 44, no. 2, pp. 534-542, 2008.

[22] E. Garone, B. Sinopoli, A. Goldsmith, and A. Casavola, "LQG control for MIMO systems over multiple erasure channels with perfect acknowledgment", *IEEE Trans. Automat. Contr.*, vol. 57, pp. 450-456, Feb. 2012.

[23] A. Goldsmith, *Wireless Communications*, Cambridge University Press, 2005.

[24] H. Hao and W. Chen, "Characterizing flexibility of an aggregation of deferrable loads", in *Proc. 53rd IEEE Conf. Decision Contr.*, pp. 4059-4064, Los Angeles, USA, 2014.

[25] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, 1952.

[26] M. Heymann, "On the input and output reducibility of multivariable linear systems", *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 563-569, Oct. 1970.

[27] O. C. Imer, S. Yüksel, and T. Basar, "Optimal control of LTI systems over unreliable communication links", *Automatica*, vol. 42, no. 9, pp. 1429-1439, 2006.

[28] E. L. Isaacson, "Solution to similarity and the diagonal of a matrix", *Amer. Math. Monthly*, vol. 87, pp. 62-63, 1980.

[29] F. C. Klebaner, *Introduction to Stochastic Calculus with Applications*, 3rd ed., Imperial College Press, London, 2012.

[30] Y. Li, E. Tuncel, J. Chen, and W. Su, "Optimal tracking performance of discrete-time systems over an additive white noise channel", in *Proc. 48th IEEE Conf. Decision Contr.*, pp. 2070-2075, Shanghai, P.R.China, 2009.

[31] D. Li and N. Hovakimyan, "Bode-like integral for continuous-time closed-loop systems in the presence of limited information", *IEEE Trans. Automat. Contr.*, vol. 58, pp. 1457-1469,

Jun. 2013.
[32] G. M. Lipsa and N. C. Martins, "Remote state estimation with communication costs for first-order LTI systems", *IEEE Trans. Automat. Contr.*, vol. 56, pp. 2013-2025, Sept. 2011.
[33] P. Loiseau, G. Schwartz, J. Musacchio, S. Amin, and S. S. Sastry, "Incentive mechanisms for internet congestion management: fixed-budget rebate versus time-of-day pricing", *IEEE/ACM Trans. Netw.*, vol. 22, pp. 647-661, 2014.
[34] S. Mallat and S. Zhong, "Characterization of signals from multiscale edges", *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 14, pp. 710-732, Jul. 1992.
[35] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theorey of Majorization and Its Applications*, 2nd ed., Springer, New York, 2011.
[36] G. N. Nair, "A nonstochastic information theory for communication and state estimation", *IEEE Trans. Automat. Contr.*, vol. 58, pp. 1497-1510, Jun. 2013.
[37] G. N. Nair and R. J. Evans, "Exponential stabilisability of finite-dimensional linear systems with limited data rates", *Automatica*, vol. 39, no. 4, pp. 585-593, 2003.
[38] G. N. Nair, R. J. Evans, I. M. Y. Mareels, and W. Moran, "Topological feedback entropy and nonlinear stabilization", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1585-1597, Sept. 2004.
[39] A. Nayyar, T. Basar, D. Teneketzis, and V. V. Veeravalli, "Optimal strategies for communication and remote estimation with an energy harvesting sensor", *IEEE Trans. Automat. Contr.*, vol. 58, pp. 2246-2260, Sept. 2013.
[40] A. Nayyar, M. N.-Pincetic, K. Poolla and P. Varaiya, "Duration-differentiated energy services with a continuum of loads", *IEEE Trans. Control Network Syst.*, vol.3, pp. 182-191, Jun. 2016.
[41] Y. Nie and Y. Yin, "Managing rush hour travel choices with tradable credit scheme", *Transport. Res. B-Meth.*, vol. 50, pp. 1-19, 2013.
[42] D. P. Palomar and Y. Jiang, *MIMO Transceiver Design via Majorization Theory*, Now Publishers Inc., Hanover, MA, USA, 2007.
[43] W. W. Peterson and E. J. Weldon, Jr., *Error-Correcting Codes*, 2nd ed., MIT Press: Cambridge, Mass., 1972.
[44] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Feedback in the non-asymptotic regime", *IEEE Trans. Inf. Theory*, vol. 57, pp. 4903-4925, Aug. 2011.
[45] T. Qi, J. Chen, W. Su, and M. Fu, "Control under stochastic multiplicative uncertainties: Part I, fundamental conditions of stabilizability", *IEEE Trans. Automat. Contr.*, vol. 62, pp. 1269-1284, Mar. 2017.
[46] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation", *IEEE Trans. Automat. Contr.*, vol. 58, pp. 554-568, Mar. 2013.
[47] H. H. Rosenbrock, *State-Space and Multivariable Theory*, Thomas Nelson, London, 1970.
[48] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link-part I: scalar systems", *IEEE Trans. Inf. Theory*, vol. 52, pp. 3369-3395, Aug. 2006.
[49] Z. Shu and R. H. Middleton, "Stabilization over power-constrained parallel Gaussian channels", *IEEE Trans. Automat. Contr.*, vol. 56, pp. 1718-1724, July. 2011.
[50] J. Stoer and C. Witzgall, "Transformations by diagonal matrices in a normed space", *Numer. Math.*, vol. 4, pp. 158-171, 1962.
[51] W. Su, J. Chen, M. Fu, and T. Qi, "Control under stochastic multiplicative uncertainties: Part II, optimal design for performance", *IEEE Trans. Automat. Contr.*, vol. 62, pp. 1285-1300, Mar. 2017.
[52] C.-W. Tan and P. Varaiya, "Interruptible electric power service contracts", *J. Econ. Dynam. Control*, vol. 17, pp. 495-517, May 1993.
[53] T. Tanaka, K. K. Kim, P. A. Parrilo, and S. K. Mitter, "Semidefinite programming approach to Gaussian sequential rate-distortion trade-offs", *IEEE Trans. Automat. Contr.*, vol. 62, pp. 1896-1910, Apr. 2017.
[54] S. Tatikonda and S. Mitter, "Control under communication constraints", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1056-1068, July. 2004.
[55] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.
[56] K. Tsumura, H. Ishii, and H. Hoshina, "Tradeoffs between quantization and packet loss in networked control of linear systems", *Automatica*, vol. 45, no. 12, pp. 2963-2970, 2009.
[57] F. J. Vargas, E. I. Silva, and J. Chen, "Stabilization of two-input two-output systems over SNR-constrained channels", *Automatica*, vol. 49, no. 10, pp. 3133-3140, 2013.
[58] F. J. Vargas, J. Chen, and E. I. Silva, "On stabilizability of MIMO systems over parallel noisy channels", in *Proc. 53rd IEEE Conf. Decision Contr.*, pp. 6074-6079, Los Angeles, USA, 2014.

[59] S. Wang, W. Chen, and L. Qiu, "Feasible channel capacity region for MIMO stabilization via MIMO communication", in *Proc. 54th IEEE Conf. Decision Contr.*, pp. 2483-2488, Osaka, Japan, 2015.

[60] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed., Springer-Verlag, 1985.

[61] N. Xiao and L. Xie, "Feedback stabilization over stochastic multiplicative input channels: continuous-time case", in *Proc. Int. Conf. Contr., Automation, Robotics and Vision*, pp. 543-548, Singapore, 2010.

[62] N. Xiao, L. Xie, and L. Qiu, "Feedback stabilization of discrete-time networked systems over fading channels", *IEEE Trans. Automat. Contr.*, vol. 57, pp. 2176-2189, Sept. 2012.

[63] I. Zaballa, "Interlacing and majorization in invariant factor assignment problems", *Linear Algebra Appl.*, vol. 121, pp. 409-421, 1989.

[64] I. Baragaña, V. Fernández, and I. Zaballa, "Linear systems with prescribed similarity structural invariants", *SIAM J. Control Optim.*, vol. 38, no. 4, pp. 1033-1049, 2000.

[65] P. Zagalak and V. Kučera, "The general problem of pole assignment: a polynomial equation approach", *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 286-289, 1985.