Characterizing the Positive Semidefiniteness of Signed Laplacians via Effective Resistances

Wei Chen, Ji Liu, Yongxin Chen, Sei Zhen Khong, Dan Wang, Tamer Başar, Li Qiu, and Karl H. Johansson

***Abstract*— A symmetric signed Laplacian matrix uniquely defines a resistive electrical circuit, where the negative weights correspond to negative resistances. The positive semidefiniteness of signed Laplacian matrices is studied in this paper using the concept of effective resistance. We show that a signed Laplacian matrix is positive semidefinite with a simple zero eigenvalue if, and only if, the underlying graph is connected, and a suitably defined effective resistance matrix is positive definite.**

I. Introduction

Over the past few decades, there has been considerable attention paid to developing algorithms for information distribution and computation among a group of interactive agents via local interactions [1]–[4]. Distributed computation and control problems of various types [5]–[9] arise naturally in large-scale networks due to their fault tolerance and cost saving features, amongst others. The Laplacian matrices play a salient role in both the design and analysis of distributed algorithms, such as those in consensus [10] and distributed optimization [11]. The convergence of such algorithms relies on the nice property that the Laplacian matrix of a positively weighted graph is positive semidefinite, and has a simple zero eigenvalue if, and only if, the graph is connected [12].

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W. Chen is with the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, Berkeley, CA 94720, USA. wchenust@gmail.com

W. Chen was with the ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, Sweden, and the Department of Electronic & Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China.

J. Liu and T. Başar are with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. jiliu@illinois.edu, basar1@illinois.edu

Y. Chen is with the Department of Mechanical Engineering, University of Minnesota, Minneapolis, MN 55455, USA. chen2468@umn.edu

S.Z. Khong is with the Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN 55455, USA. szkhong@umn.edu

D. Wang and L. Qiu are with the Department of Electronic & Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China. dwangah@connect.ust.hk, eeqiu@ust.hk

K. H. Johansson is with the ACCESS Linnaeus Center, School of Electrical Engineering, KTH Royal Institute of Technology, Stockholm, Sweden. kallej@kth.se

In this paper, signed Laplacian matrices associated with signed weighted graphs having both positive and negative edge weights are investigated. In a realistic network, negative weights may arise from faulty processes occurring in distributed computation or communication among agents. For example, sign errors may be present in some communication channels. In this case, the actual weights used in the updates of the distributed algorithms can be negative, yielding signed Laplacians with negative weights. Another possible occurrence of signed Laplacians comes from adversarial attacks on a network. For example, in a continuous-time linear consensus network [10], an external attacker may intentionally hack the communication link between some pairs of neighboring agents by switching the signs of the values transmitted via the link, with the purpose of preventing the agents from reaching a consensus.

In both cases described above, negative weights appear in the associated Laplacian matrices. Recently, increasing interest has been drawn to networks described by signed weighted graphs [13], in which the signed Laplacians naturally play a role. It has been demonstrated in [14] that negative weights can somehow help accelerate the convergence rates of distributed averaging algorithms.

It is worth mentioning that distributed computation and control is not the only area in which signed Laplacians occur. Recently, it has been reported that signed Laplacians can help provide a graph-theoretical perspective in power system stability analysis [15], [16] and biological networks [17]. All in all, there is ample motivation to study the properties of signed Laplacians with negative weights.

As a starting point, we examine the positive semidefiniteness of signed Laplacians in this paper. Specifically, we are interested in the condition under which signed Laplacians with negative weights are positive semidefinite and have a simple zero eigenvalue. This is interesting because the latter gives a necessary and sufficient condition for the corresponding linear consensus process to reach a consensus [18].

In general, signed Laplacians may exhibit negative eigenvalues and/or multiple zero eigenvalues, even when the underlying signed graphs are connected. In a recent paper [19], signed Laplacians with only one negative weight have been investigated. It has been shown in [19] that such a signed Laplacian is positive semidefinite if, and only if, the effective resistance over the negatively weighted edge is nonnegative. The result has been extended therein to signed Laplacians with multiple negative weights, but with the restriction that the negatively weighted edges are isolated in different cycles in the graphs. Later, the same results were reestablished in [20] using geometrical and passivity-based approaches, leading to a significant simplification of the proof and more transparent physical interpretations in terms of circuit theory. Notwithstanding this, necessary and sufficient conditions for general signed Laplacians with multiple negative weights to be positive semidefinite are still lacking.

The main contribution of this paper lies in the establishment of a necessary and sufficient condition under which a signed Laplacian, without any restrictions on the negatively weighted edges in the corresponding graph, is positive semidefinite and has a simple zero eigenvalue. The condition can be well interpreted by checking the passivity of a suitably defined electrical circuit via effective resistance matrices.

We note that the problem considered here is also related to the literature on the problem of bounding the number of negative and zero eigenvalues of signed Laplacians [21].

Notation: We write x' to denote the transpose of a vector x and A' for a matrix A. The range and kernel of A are denoted, respectively, by $\operatorname{ran}(A)$ and $\ker(A)$. The spectral radius of a square matrix A is denoted by $\rho(A)$. We use $\mathbf{1}$ to denote the vector with all entries equal to 1, while the size of the vector is to be understood from the context. Denote by u_i the vector with the ith entry equal to 1 and other entries equal to 0. We define $u_{ij} = u_i - u_j$. For a symmetric matrix S, we write $S \geq 0$ if S is positive semidefinite, and $S > 0$ if S is positive definite.

The rest of the paper is organized as follows. In Section II, signed Laplacians are introduced and some existing results are reviewed. Some preliminary knowledge on effective resistance matrices is presented in Section III. The main result of the paper is given in Section IV together with a proper physical interpretation. The analysis and proofs of the results are given in Section V. A simulation example is provided in Section VI to validate the results. The paper ends with some concluding remarks in Section VII.

II. Signed Laplacians

Consider an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ which consists of a set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$ and a set of edges $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$. We use (i, j) to denote the edge connecting node i and node j, and associate with each edge $(i, j) \in \mathcal{E}$ a nonzero real-valued weight a_{ij} that can be either positive or negative. If there is no edge connecting node i and node j, a_{ij} is understood to be zero. Such a graph is called a *signed weighted graph*. For brevity, hereinafter the signed weighted graphs are also referred to as signed graphs.

Denote by $\mathcal{E}_+$ ($\mathcal{E}_-$, respectively) the subset of $\mathcal{E}$ containing all the edges with positive weights (negative weights, respectively). Denote by $\mathbb{G}_+ = (\mathcal{V}, \mathcal{E}_+)$ ($\mathbb{G}_- = (\mathcal{V}, \mathcal{E}_-)$, respectively) the spanning subgraph[1] of $\mathbb{G}$ whose edge set is given by $\mathcal{E}_+$ ($\mathcal{E}_-$, respectively).

A spanning tree $\mathbb{T}$ of an undirected graph $\mathbb{G}$ is a spanning subgraph that is a tree. A spanning tree exists if, and only if, the underlying graph is connected. If the graph is not connected, a spanning forest $\mathbb{F}$ is considered instead, which is a spanning subgraph containing a spanning tree in each connected component of the graph. A spanning tree can be regarded as a special case of a spanning forest. Therefore, hereinafter we shall use $\mathbb{F}$ to represent a spanning tree or a spanning forest of a graph $\mathbb{G}$.

For a signed graph introduced above, the associated *signed Laplacian matrix* $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ is defined by

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j=1, j\neq i}^{n} a_{ij}, & i = j. \end{cases} \tag{1}$$

Clearly, L is symmetric, and thus has real eigenvalues. Also, L has a zero eigenvalue with a corresponding eigenvector being $\mathbf{1} \in \mathbb{R}^n$.

When all the edges have positive weights, L reduces to the conventional Laplacian matrix, to which a substantial literature is dedicated [22]. However, due to the presence of negative weights, a signed Laplacian L has some significant differences from the conventional Laplacians. Firstly, L is no longer an M-matrix[2] when negative weights are present, for which many well-studied properties of M-matrices do not hold. Secondly, L is not necessarily positive semidefinite as opposed to the conventional Laplacians. Thirdly, while the multiplicity of the zero eigenvalue of a conventional Laplacian is equal to the number of connected components in the underlying graph, this is in general not true for a signed Laplacian. All these differences necessitate the development of a theory for signed graphs and the associated signed Laplacian matrices.

We study the spectral properties of signed Laplacians with negative weights. Specifically, positive semidefiniteness is examined in this paper. It is well known that a conventional Laplacian matrix is always positive semidefinite, and it has a simple zero eigenvalue if, and only if, the underlying graph is connected [12]. This is no longer the case for general signed Laplacians with negative weights. The following simple example demonstrates that a signed Laplacian may have negative eigenvalues and multiple zero eigenvalues even when the graph is connected. Consider a complete graph with three nodes. Let $a_{12} = -1$, $a_{13} = 2$, and $a_{23} = 2$. It follows from (1) that L has a zero eigenvalue of multiplicity two. Furthermore, when $a_{12} < -1$, L has one negative, one zero, and one positive eigenvalue.

We wish to understand under what conditions a signed Laplacian with both positive and negative weights is positive semidefinite and has a simple zero eigenvalue. This is of wide interest to many applications, for instance, the behavior of a linear consensus process under adversarial attacks [23].

Before proceeding, let us introduce a useful factorization of the signed Laplacians. Let $W = \operatorname{diag}\{w_1, w_2, \ldots, w_m\}$ denote an $m \times m$ diagonal matrix with diagonal elements

[1]A spanning subgraph of $\mathbb{G}$ is a graph which contains the same set of nodes as $\mathbb{G}$ and whose edge set is a subset of that of $\mathbb{G}$.

[2]A square matrix M is said to be an M-matrix if it can be expressed as $M = sI - B$, where I is the identity matrix, B is nonnegative, and $s \geq \rho(B)$.

given by the edge weights, i.e.,

$$w_k = a_{ij}, \text{ for } (i,j) = e_k.$$

Also, assign an (arbitrary) orientation to each edge of the graph, i.e., for each edge $e_k \in \mathcal{E}$, denote one endpoint as the head and the other as the tail. Then, the oriented incidence matrix $D = [d_{ik}] \in \mathbb{R}^{n\times m}$ is defined as:

$$d_{ik} = \begin{cases} 1, & \text{if } i \text{ is the head of } e_k, \\ -1, & \text{if } i \text{ is the tail of } e_k, \\ 0, & \text{otherwise.} \end{cases}$$

An important property of the incidence matrix is $D'\mathbf{1} = 0$. Now, with the weight matrix W and the incidence matrix D defined above, notice that the signed Laplacian matrix L can be factorized as

$$L = DWD'.$$

It is worth noting that while the incidence matrix D depends on the choice of orientations, the signed Laplacian L does not. To see this, suppose that the orientation of edge e_k is changed and the orientations of other edges remain the same. Denote the resultant incidence matrix by $\tilde{D}$. Then, $\tilde{D} = DS$, where S is a diagonal matrix whose kth diagonal entry is -1 and other diagonal entries are 1. Therefore, $\tilde{D}W\tilde{D}' = DSWSD' = DWS^2D' = DWD'$.

III. Preliminaries on Effective Resistance Matrix

Consider an undirected connected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$. Associate with each edge a resistor of (possibly negative) resistance $r_k = 1/w_k$, where w_k is the weight on edge e_k. In other words, the weight w_k is the conductance of the corresponding resistor. Define $R = W^{-1} = \text{diag}\{r_1, r_2, \ldots, r_m\}$.

Let $\boldsymbol{c} \in \mathbb{R}^n$ be a vector whose entries denote the amount of current injected to each node by external independent sources. Assume that the sum of the entries of $\boldsymbol{c}$ is zero, i.e., $\boldsymbol{c}'\mathbf{1} = 0$, meaning that there is no current accumulation in the electrical network. Denote by $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{i} \in \mathbb{R}^m$ the vector of voltages at all nodes and the vector of currents through all edges, respectively. Then, Kirchhoff's current law [24] asserts that the difference between the outgoing current and the incoming current through the edges adjacent to a given node equals to the external current injection at that node, i.e.,

$$D\boldsymbol{i} = \boldsymbol{c}.$$

On the other hand, Ohm's law [24] asserts that the current across each edge is given by the voltage difference divided by the resistance, i,e.,

$$WD'\boldsymbol{v} = \boldsymbol{i}. \tag{2}$$

Combining the above two equalities, we have

$$DWD'\boldsymbol{v} = L\boldsymbol{v} = \boldsymbol{c}. \tag{3}$$

When the Laplacian L has a simple zero eigenvalue, we can solve the above equation to yield

$$\boldsymbol{v} = L^\dagger \boldsymbol{c} + \alpha\mathbf{1}, \tag{4}$$

where $L^\dagger$ is the Moore-Penrose pseudoinverse of L and α is an arbitrary real number. The electric power of the network is given by $\boldsymbol{v}'\boldsymbol{c}$. The electrical network is said to be passive [25] if $\boldsymbol{v}'\boldsymbol{c} \geq 0$, and strictly passive if $\boldsymbol{v}'\boldsymbol{c} > 0$.

Let $\boldsymbol{c} = u_{ij}$. This means that a unit of current is injected into node i and extracted from node j. In light of the voltage formula (4), the voltage difference between these two nodes is given by $u_{ij}'L^\dagger u_{ij}$. This quantity is called the *effective resistance* across the pair (i,j), and we denote it by

$$r_{\text{eff}}(i,j) = u_{ij}'L^\dagger u_{ij}.$$

When all the edge weights are positive, it has been shown that the effective resistance serves as a distance function in the node set of a weighted graph [26].

In many cases, it is also interesting to consider the voltage difference across a node pair (i,j) when a unit of current is injected and extracted from another node pair (k,l). Such a quantity is called the *mutual effective resistance* between the two node pairs:

$$r_{\text{mut}}((i,j),(k,l)) = u_{ij}'L^\dagger u_{kl}.$$

Since $L^\dagger$ is symmetric, we have

$$r_{\text{mut}}((i,j),(k,l)) = r_{\text{mut}}((k,l),(i,j)).$$

If we confine our attention to the adjacent node pairs, both the effective resistance and mutual effective resistance can be captured by an *effective resistance matrix* $\Gamma = [\gamma_{kl}] \in \mathbb{R}^{m\times m}$ defined as

$$\Gamma = D'L^\dagger D.$$

Clearly, Γ is a symmetric matrix. The diagonal entries of Γ correspond to the effective resistances and the off-diagonal entries correspond to the mutual effective resistances.

IV. Main Results

Consider a signed graph $\mathbb{G}$ equipped with the signed Laplacian matrix L. Let $\mathbb{G}_+ = (\mathcal{V}, \mathcal{E}_+)$ and $\mathbb{G}_- = (\mathcal{V}, \mathcal{E}_-)$ be defined as before. We express the graph $\mathbb{G}$ as the union of three subgraphs:

$$\mathbb{G} = \mathbb{F}_- \cup \mathbb{C}_- \cup \mathbb{G}_+,$$

where $\mathbb{F}_- = (\mathcal{V}, \mathcal{E}_{\mathbb{F}_-})$ is a spanning forest of $\mathbb{G}_-$ and $\mathbb{C}_-$ is a spanning subgraph of $\mathbb{G}_-$ containing the remaining edges of $\mathbb{G}_-$. With a proper labeling of the edges, the incidence matrix D and the weight matrix W can be written without loss of generality in the following form:

$$\begin{aligned} D &= \begin{bmatrix} D_{\mathbb{F}_-} & D_{\mathbb{C}_-} & D_{\mathbb{G}_+} \end{bmatrix}, \\ W &= \text{diag}\left\{W_{\mathbb{F}_-}, W_{\mathbb{C}_-}, W_{\mathbb{G}_+}\right\}. \end{aligned}$$

Consequently, the effective resistance matrix admits the form

$$\Gamma = D'L^\dagger D = \begin{bmatrix} D_{\mathbb{F}_-}'L^\dagger D_{\mathbb{F}_-} & D_{\mathbb{F}_-}'L^\dagger D_{\mathbb{C}_-} & D_{\mathbb{F}_-}'L^\dagger D_{\mathbb{G}_+} \\ D_{\mathbb{C}_-}'L^\dagger D_{\mathbb{F}_-} & D_{\mathbb{C}_-}'L^\dagger D_{\mathbb{C}_-} & D_{\mathbb{C}_-}'L^\dagger D_{\mathbb{G}_+} \\ D_{\mathbb{G}_+}'L^\dagger D_{\mathbb{F}_-} & D_{\mathbb{G}_+}'L^\dagger D_{\mathbb{C}_-} & D_{\mathbb{G}_+}'L^\dagger D_{\mathbb{G}_+} \end{bmatrix}.$$

Our main result in this paper relies critically on the submatrix

$$\Gamma_{\mathbb{F}_-} = D_{\mathbb{F}_-}'L^\dagger D_{\mathbb{F}_-}. \tag{5}$$

Denote the dimensions of $D_{\mathbb{F}_-}$ by $n \times m_1$. Since $\mathbb{F}_-$ is a spanning forest, it follows that $D_{\mathbb{F}_-}$ has full column rank (see Theorem 2.5 in [27]).

The main result of the paper now follows.

Theorem 1: A signed Laplacian L is positive semidefinite with a simple zero eigenvalue if, and only if, the underlying signed graph $\mathbb{G}$ is connected, and $\Gamma_{\mathbb{F}_-} > 0$.

It should be clear that the choice of a spanning forest $\mathbb{F}_-$ in $\mathbb{G}_-$ is not unique. Nevertheless, Theorem 1 holds for any choice of $\mathbb{F}_-$. The theorem is proved in Section V via a passivity-based approach. An alternative geometrical proof is also presented in the Appendix.

This theorem builds a bridge linking the spectral properties of the signed Laplacians to their underlying graph-theoretic meanings. As in Section III, one can associate a signed graph $\mathbb{G}$ with a resistive electrical network, wherein the negative weights correspond to negative resistances. Then, Theorem 1 can be physically interpreted as follows. See the next section for detailed reasoning.

A signed Laplacian L is positive semidefinite with a simple zero eigenvalue if, and only if, $\mathbb{G}$ is connected and the associated electrical network is strictly passive.

From Theorem 1, one can deduce the following corollary, which has also been shown in [19, Theorem III.3] and [20, Theorem 3.2]. The proof is omitted here due to page limit.

Corrollary 1: Let L be the Laplacian of a signed graph $\mathbb{G}$. If there is no cycle in $\mathbb{G}$ containing two negatively weighted edges, then L is positive semidefinite and has a simple zero eigenvalue if, and only if, $\mathbb{G}$ is connected and $r_{\text{eff}}(i,j) > 0$ for all $(i,j) \in \mathcal{E}_-$.

The computational complexity of effective resistance is an issue worth special attention. Although the expression of $\Gamma_{\mathbb{F}_-}$ as in (5) involves $L^\dagger$, it does not mean that one needs to compute $L^\dagger$, which is of high complexity. In fact, it has been shown in [26] that the effective resistance across a pair of nodes solely depends on all the paths inbetween. It is also not difficult to show that the mutual effective resistance between two pairs of nodes solely depends on all the cycles containing them. In this sense, computing the effective resistance matrix can be done locally. This partially explains why checking the semidefiniteness of a signed Laplacian via Theorem 1 is more advantageous than directly computing the eigenvalues of the signed Laplacian. How to compute the effective resistance in a distributed way is also of great interest, and is under our current investigation.

Remark 1: When $\mathbb{G}$ has no negatively weighted edges, i.e., $\mathcal{E}_- = \emptyset$, L reduces to a conventional Laplacian matrix which is always positive semidefinite. In such a special case, $\Gamma_{\mathbb{F}_-}$ becomes irrelevant and, thus, Theorem 1 indicates that L has a simple zero eigenvalue if, and only if, $\mathbb{G}$ is connected. In this regard, Theorem 1 is also a generalization of the well-known positive semidefiniteness result for the conventional Lapacians.

V. ANALYSIS

In this section, we provide proofs for the main result stated in Section IV. The necessity proof is mainly algebraic reasoning, while the sufficiency proof is based on the passivity of an electrical circuit.

For preparation, we first introduce two useful lemmas. The proofs are omitted here due to page limit.

Lemma 1: If a signed Laplacian matrix L has a simple zero eigenvalue, the underlying signed graph $\mathbb{G}$ is connected.

Lemma 2: The spectrum of the signed Laplacian L is monotonically increasing with respect to each edge weight of the underlying signed graph $\mathbb{G}$.

Now we present the proof of Theorem 1.

Proof of Theorem 1: We first prove the necessity. Suppose therefore that L is positive semidefinite and has a simple zero eigenvalue. From Lemma 1, we know that $\mathbb{G}$ is connected. It remains to show that $\Gamma_{\mathbb{F}_-} > 0$. Note that since L is positive semidefinite with a simple zero eigenvalue, so is $L^\dagger$. This follows from the singular value decomposition expression of the pseudoinverse [28]. Then, from formula (5), there holds $\Gamma_{\mathbb{F}_-} \geq 0$ whenever $L \geq 0$. Furthermore, the range of $D_{\mathbb{F}_-}$ is orthogonal to the kernel of $L^\dagger$, yielding that $\Gamma_{\mathbb{F}_-}$ has no zero eigenvalues. This completes the proof of necessity.

We now turn to the proof of sufficiency. Suppose therefore that $\mathbb{G}$ is connected, and $\Gamma_{\mathbb{F}_-} > 0$.

It suffices to show that $\boldsymbol{v}' L \boldsymbol{v} > 0$ for all $\boldsymbol{v} \perp \mathbf{1}$. To this end, we shall appeal to the notion of passivity of an electrical network. Let $\boldsymbol{v}$ be the voltage induced by an external current $\boldsymbol{c}$. Then, in view of (3), $\boldsymbol{v}' L \boldsymbol{v} > 0$ for all $\boldsymbol{v} \perp \mathbf{1}$ is equivalent to $\boldsymbol{v}' \boldsymbol{c} > 0$ for all $\boldsymbol{c} \perp \mathbf{1}$. Physically, it means that we need to show that the associated electrical network is strictly passive.

Let $\boldsymbol{c}$ be an arbitrary external current that induces voltage $\boldsymbol{v}$ and current $\boldsymbol{i}$ in the network. We claim that there exists another external current of the form $\tilde{\boldsymbol{c}} = D_{\mathbb{F}_-}\xi$ that induces voltage $\tilde{\boldsymbol{v}}$ and current $\tilde{\boldsymbol{i}}$ such that $\tilde{\boldsymbol{i}}_k = \boldsymbol{i}_k$ for all $e_k \in \mathcal{E}_-$, where $\xi \in \mathbb{R}^{m_1}$. To see this, combining (2) and (3) yields

$$WD'L^\dagger \tilde{\boldsymbol{c}} = WD'L^\dagger D_{\mathbb{F}_-}\xi = \tilde{\boldsymbol{i}}.$$

Note that if the identity $\tilde{\boldsymbol{i}}_k = \boldsymbol{i}_k$ holds for all $e_k \in \mathcal{E}_{\mathbb{F}_-}$, then it holds for all $e_k \in \mathcal{E}_-$. This follows from the fact that $\mathbb{F}_- = (\mathcal{V}, \mathcal{E}_{\mathbb{F}_-})$ is a spanning forest of $\mathbb{G}_- = (\mathcal{V}, \mathcal{E}_-)$, whereby the voltage across every edge in $\mathcal{E}_-$ is fixed whenever the current through every $e_k \in \mathcal{E}_{\mathbb{F}_-}$ is decided. Therefore, it boils down to solving the equation

$$W_{\mathbb{F}_-} D'_{\mathbb{F}_-} L^\dagger D_{\mathbb{F}_-}\xi = W_{\mathbb{F}_-}\Gamma_{\mathbb{F}_-}\xi = \boldsymbol{i}_{\mathbb{F}_-},$$

where $\boldsymbol{i}_{\mathbb{F}_-}$ denotes the internal current in $\mathbb{F}_-$. Clearly, the above equation has a unique solution since $\Gamma_{\mathbb{F}_-} > 0$ and $W_{\mathbb{F}_-}$ is nonsingular. Now let $\boldsymbol{p} = \boldsymbol{i} - \tilde{\boldsymbol{i}}$. We have

$$\begin{aligned} \boldsymbol{v}'\boldsymbol{c} &= \sum_{l=1}^{m} \boldsymbol{i}_l^2 r_l = \sum_{l=1}^{m} (\tilde{\boldsymbol{i}}_l + \boldsymbol{p}_l)^2 r_l \\ &= \sum_{l=1}^{m} \tilde{\boldsymbol{i}}_l^2 r_l + 2\sum_{l=1}^{m} \tilde{\boldsymbol{i}}_l r_l \boldsymbol{p}_l + \sum_{l=1}^{m} \boldsymbol{p}_l^2 r_l \\ &= \xi'\Gamma_{\mathbb{F}_-}\xi + 2\sum_{l=1}^{m} \tilde{\boldsymbol{i}}_l r_l \boldsymbol{p}_l + \sum_{l: e_l \in \mathcal{E}_+} \boldsymbol{p}_l^2 r_l, \end{aligned} \tag{6}$$

where the first equality follows from the fact that any power injected into the electrical network is equal to the sum of power dissipated through all positive resistances subtracted by the sum of power generated by all negative resistances. Note that the first term and the third term in (6) are both nonnegative and they cannot be zero at the same time since $\mathbb{G}$ is connected. Therefore, it remains to examine the second term. As a matter of fact,

$$\begin{aligned}\sum_{l=1}^{m} \tilde{\boldsymbol{i}}_l r_l \boldsymbol{p}_l = \tilde{\boldsymbol{i}}' R \boldsymbol{p} &= (W D' L^\dagger D_{\mathbb{F}_-} \xi)' R W D' L^\dagger (\boldsymbol{c} - \tilde{\boldsymbol{c}}) \\ &= \xi' D'_{\mathbb{F}_-} L^\dagger D W D' L^\dagger (\boldsymbol{c} - \tilde{\boldsymbol{c}}) \\ &= \xi' D'_{\mathbb{F}_-} L^\dagger L L^\dagger (\boldsymbol{c} - \tilde{\boldsymbol{c}}) \\ &= \xi' D'_{\mathbb{F}_-} L^\dagger (\boldsymbol{c} - \tilde{\boldsymbol{c}}) \\ &= \xi' D'_{\mathbb{F}_-} (\boldsymbol{v} - \tilde{\boldsymbol{v}}) \\ &= 0.\end{aligned}$$

This yields that $\boldsymbol{v}'\boldsymbol{c} > 0$ for all $\boldsymbol{c} \perp \mathbf{1}$, which completes the proof. ∎

Remark 2: According to Lemma 1, the connectedness of the signed graph $\mathbb{G}$ can be guaranteed by L having a simple zero eigenvalue. In fact, when L is positive semidefinite with a simple zero eigenvalue, one can conclude that $\mathbb{G}_+$ is connected, which is stronger than $\mathbb{G}$ being connected. To see this, assume $\mathbb{G}_+$ is disconnected. Then, the Laplacian matrix associated with $\mathbb{G}_+$ has multiple zero eigenvalues. In view of Lemma 2, the signed Laplacian matrix associated with $\mathbb{G}$ either has multiple zero eigenvalues or negative eigenvalues or both. This contradicts the hypothesis that L is positive semidefinite and has a simple zero eigenvalue. Hence, $\mathbb{G}_+$ has to be connected.

VI. An Example

In this section, we provide a simulation example to validate the main result presented in Section IV.

Consider an undirected signed graph $\mathbb{G}$, as shown in Fig. 1, which consists of eight vertices, eight positive edges, and six negative edges. Then, its corresponding signed Laplacian L can be easily written. We construct a spanning forest of $\mathbb{G}_-$, as shown in Fig. 2. Then, the effective resistance matrix $\Gamma_{\mathbb{F}_-}$ can be readily computed via (5). Simulation shows that the spectrum of $\Gamma_{\mathbb{F}_-}$ is $\{1.4093, 0.6309, 0.0922, 0.0457\}$, which implies that $\Gamma_{\mathbb{F}_-}$ is positive definite, and the spectrum of L is $\{0, 1.1259, 3.4600, 6.3543, 10.1882, 20.6872, 27.1570, 45.2274\}$, which implies that L is positive semidefinite and has a simple zero eigenvalue. Therefore, the simulation is consistent with Theorem 1.

VII. Conclusion

In this paper, we have studied the positive semidefiniteness of a signed Laplacian matrix associated with a signed graph. We have shown that a signed Laplacian matrix is positive semidefinite and has a simple zero eigenvalue if, and only if, the underlying graph is connected, and a suitably defined effective resistance matrix is positive definite. This result bridges the spectral properties of signed Laplacians with their graph-theoretical meanings. It can be physically interpreted via the passivity of an associated resistive electrical network.

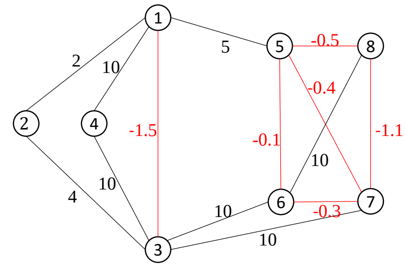

Fig. 1. A signed graph $\mathbb{G}$

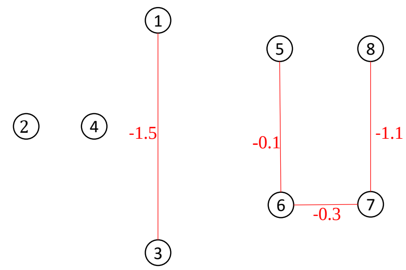

Fig. 2. A spanning forest of $\mathbb{G}_-$

The results in this paper significantly generalize the existing ones in [19] and [20].

Our future work aims at extending the results in this paper to directed signed graphs. Such an extension, however, appears to be challenging. While signed Laplacians have been introduced for directed graphs in the literature [29], how to suitably define effective resistances and physically interpret them in terms of electrical circuits remains to be investigated.

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Appendix

A geometrical proof of Theorem 1: We start from the necessity. If the signed Laplacian L is positive semidefinite and has a simple zero eigenvalue, then

$$x'Lx = x'DWD'x > 0 \tag{7}$$

for all nonzero $x \perp \mathbf{1}$. Let $\hat{D} = D\sqrt{|W|}$ and $\hat{W} = \operatorname{sign}(W)$ where $|W|$, $\sqrt{|W|}$, and the signum function $\operatorname{sign}(W)$ are applied entrywise (and $\operatorname{sign}(0) := 0$). Then, $L = \hat{D}\hat{W}\hat{D}'$ and therefore (7) is equivalent to

$$x'Lx = x'\hat{D}\hat{W}\hat{D}'x > 0,\ \forall x \perp \mathbf{1}. \tag{8}$$

It follows that $\ker\left(\hat{D}'\right) = \ker(D')$ is the one-dimensional subspace containing $\mathbf{1}$, which implies that $\operatorname{rank}(D') = n-1$ and, thus, $\mathbb{G}$ is connected. The above equation (8) can be rewritten as

$$y'\hat{W}y > 0$$

for any nonzero vector y in the range of $\hat{D}'$. That is,

$$\sum_{i:e_i\in\mathcal{E}_+} y_i^2 > \sum_{j:e_j\in\mathcal{E}_-} y_j^2,\ \forall y \in \operatorname{ran}\left(\hat{D}'\right). \tag{9}$$

Let $\mathcal{A}$ be the span of $\{u_i \in \mathbb{R}^m \mid e_i \in \mathcal{E}_-\}$ and $\mathcal{B}$ be the range of $\hat{D}'$, then (9) is equivalent to

$$\|y - P_{\mathcal{A}}y\|_2^2 > \|P_{\mathcal{A}}y\|_2^2,\ \forall y \in \mathcal{B},$$

where $P_{\mathcal{A}}y$ means the projection of y onto $\mathcal{A}$. The above inequality says that for any nonzero element in $\mathcal{B}$, its distance to the space $\mathcal{A}$ is greater than the length of its projection onto $\mathcal{A}$. From a geometric point of view, this is equivalent to saying there is no nonzero vectors $y \in \mathcal{B}$ and $z \in \mathcal{A}$ such that the angle between them is less than or equal to $\pi/4$. This angle condition holds if, and only if

$$\|z\|_2^2 > 2\|P_{\mathcal{B}}z\|_2^2,\ \forall z \in \mathcal{A}. \tag{10}$$

Let $|\mathcal{E}_-| = m_2$. With some relabeling of the edges, one can make $\mathcal{E}_- = \{e_1, e_2, \ldots, e_{m_2}\}$. Then, any nonzero $z \in \mathcal{A}$ has the form

$$z = \begin{bmatrix} u_1 & u_2 & \ldots & u_{m_2} \end{bmatrix}\zeta,$$

where $\zeta = \begin{bmatrix} \zeta_1 & \zeta_2 & \ldots & \zeta_{m_2} \end{bmatrix}' \neq 0$. The condition (10) can thus be rewritten as

$$\|\zeta\|_2^2 > 2\left\|\begin{bmatrix} P_{\mathcal{B}}u_1 & P_{\mathcal{B}}u_2 & \ldots & P_{\mathcal{B}}u_{m_2} \end{bmatrix}\zeta\right\|_2^2$$

for any nonzero ζ, or equivalently,

$$\begin{bmatrix} 1-2u_1'P_{\mathcal{B}}u_1 & -2u_1'P_{\mathcal{B}}u_2 & \cdots & -2u_1'P_{\mathcal{B}}u_{m_2} \\ -2u_2'P_{\mathcal{B}}u_1 & 1-2u_2'P_{\mathcal{B}}u_2 & \cdots & -2u_2'P_{\mathcal{B}}u_{m_2} \\ \vdots & \vdots & \ddots & \vdots \\ -2u_{m_2}'P_{\mathcal{B}}u_1 & -2u_{m_2}'P_{\mathcal{B}}u_2 & \cdots & 1-2u_{m_2}'P_{\mathcal{B}}u_{m_2} \end{bmatrix} > 0. \tag{11}$$

Since

$$P_{\mathcal{B}}z = \hat{D}'(\hat{D}\hat{D}')^\dagger\hat{D}z = \sqrt{|W|}D'(D|W|D')^\dagger D\sqrt{|W|}z,$$

the inequality (11) can be rewritten as

$$-2D_{\mathbb{G}_-}'\tilde{L}^\dagger D_{\mathbb{G}_-} + |W_{\mathbb{G}_-}|^{-1} > 0, \tag{12}$$

where $\tilde{L} = D|W|D'$, $D_{\mathbb{G}_-} = \begin{bmatrix} D_{\mathbb{F}_-} & D_{\mathbb{C}_-} \end{bmatrix}$, and $W_{\mathbb{G}_-} = \operatorname{diag}\{W_{\mathbb{F}_-}, W_{\mathbb{C}_-}\}$. Since $\mathbb{F}_-$ is a spanning forest of $\mathbb{G}_-$, there exists a matrix T such that $D_{\mathbb{G}_-} = D_{\mathbb{F}_-}T$. Combining this and the fact that $\mathbb{G}$ is connected, whereby $D_{\mathbb{F}_-}'\tilde{L}^\dagger D_{\mathbb{F}_-}$ is invertible, it follows that (12) is equivalent, via Schur complement, to

$$\begin{bmatrix} |W_{\mathbb{G}_-}|^{-1} & T' \\ T & \frac{1}{2}(D_{\mathbb{F}_-}'\tilde{L}^\dagger D_{\mathbb{F}_-})^{-1} \end{bmatrix} > 0.$$

Using Schur complement again yields

$$(D_{\mathbb{F}_-}'\tilde{L}^\dagger D_{\mathbb{F}_-})^{-1} - 2T|W_{\mathbb{G}_-}|T' > 0.$$

Now, noting $\tilde{L} = L + 2D_{\mathbb{G}_-}|W_{\mathbb{G}_-}|D_{\mathbb{G}_-}'$ and applying Matrix Inversion Lemma to L, we have

$$L^\dagger = \tilde{L}^\dagger + 2\tilde{L}^\dagger D_{\mathbb{G}_-}(|W_{\mathbb{G}_-}|^{-1} - 2D_{\mathbb{G}_-}'\tilde{L}^\dagger D_{\mathbb{G}_-})^{-1}D_{\mathbb{G}_-}'\tilde{L}^\dagger.$$

Applying Matrix Inversion Lemma again to $\Gamma_{\mathbb{F}_-}$ yields

$$\Gamma_{\mathbb{F}_-}^{-1} = (D_{\mathbb{F}_-}'\tilde{L}^\dagger D_{\mathbb{F}_-})^{-1} - 2T|W_{\mathbb{G}_-}|T' > 0.$$

All the above steps can go in the converse direction to show the sufficiency. The proof is thus completed.