Continuous-time Indefinite Linear Quadratic Optimal Control with Random Input Gains

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***Abstract*— In this paper, the indefinite linear quadratic (LQ) optimal control of continuous-time linear time-invariant (LTI) systems with random input gains is studied. One main novelty of this work is the use of channel/controller co-design framework which bridges and integrates the design of the channels and controller. The co-design is carried out by the twist of channel resource allocation, i.e., the channel capacities can be allocated among the input channels by the control designer subject to an overall capacity constraint. With the channel/controller co-design, necessary and sufficient conditions for the well-posedness as well as the attainability of the indefinite LQ problem concerned are obtained. The optimal control law is given by a linear state feedback associated with the mean-square stabilizing solution of a modified algebraic Riccati equation.**

I. Introduction

Stochastic systems are attracting more and more attention due to wide applications in different areas, such as networked control systems (NCSs), financial engineering, etc. Parallel to the control theory for deterministic systems, the stabilization as well as the optimal control of the stochastic systems have been investigated widely. A general study of the stochastic control systems can be found in [5], [6], [8] and the references therein.

The LQ optimal control is one of the most important classes of optimal control problems, in both theory and application. It adopts a quadratic cost function of the plant state and control input with a symmetric weighting matrix. In general, the weighting matrix can be indefinite.

The definite stochastic LQ optimal control is much related to the stochastic $\mathcal{H}_2$ control and has been studied extensively in the literature. In particular, much research has been done recently in the context of NCSs [7], [11], [12], [19], most of which treats the definite LQ optimal control as part of the Linear Quadratic Gaussian (LQG) control problem. In [12], [19], the LQG control of a discrete-time multi-input-multi-output (MIMO) NCS with a single packet-dropping input channel and a single packet-dropping output channel is studied under the TCP-like protocols. It is shown therein that the optimal stabilizing controller exists if and only if the packet arrival rates in the input and output channels are larger than certain critical values respectively. The latest work in [22], [4] studies the definite LQ optimal control of LTI systems with random input gains for the discrete-time and continues-time case, respectively. By virtue of the channel/controller co-design, the infinite-horizon LQ problem is solved with the optimal control law given by a linear state feedback.

Researchers have also made effort to study the indefinite stochastic LQ optimal control, which plays a significant role in stochastic $\mathcal{H}_\infty$ control and robust control as well as various financial engineering problems such as investment optimization. The work in [17] investigates the indefinite LQ optimal control of a continues-time stochastic system with scalar multiplicative state and control dependent noise. It is shown therein that the indefinite LQ problem is solvable if and only if a non-standard algebraic Riccati equation has a mean-square (MS) stabilizing solution. The results for the discrete-time counterpart are reported in [15]. Similar approaches can be found in [5], [6], where the indefinite stochastic LQ optimal control with multidimensional state and control dependent noise is investigated for the continuous-time and discrete-time case, respectively.

Inspired by the above results, we study the indefinite LQ optimal control of continuous-time LTI systems with random input gains. The problem we consider is much more involved than a special case of the LQ stochastic optimal control studied in [5], [6]. In our setup, motivated from numerous applications in NCSs, distributed systems and economic systems, each component of the control signal is subject to independent stochastic perturbation. More insights can be gained for such systems with this special structure compared to the general stochastic systems. Indeed, we put the LQ problem under the framework of channel/controller co-design which is one main novelty of this work. It is assumed that the controller designer also has the freedom to participate in the channel design. Due to this additional design freedom, the objective becomes to simultaneously design the controller and channels so as to minimize the cost function. The well-posedness and attainability of the indefinite LQ problem concerned is studied with the channel/controller co-design, as elaborated in the rest of this paper.

Note that the framework of channel/controller co-design is first proposed in the conference paper [10] to study the stabilization of multi-input NCSs and then extended in [16]. Following this framework, several other works have been carried out, e.g., [20], [21], [22], [4].

The remainder of this paper is organized as follows. The indefinite stochastic LQ optimal control problem is formulated and some concepts, especially the channel/controller co-design framework, are introduced in Section II. Some

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preliminary knowledge on the MS stabilizability is presented in Section III. Section IV investigates the well-posedness of the indefinite LQ problem. The attainability of a well-posed problem is studied in Section V, where the optimal controller and the minimum value of the cost function is obtained. Finally, conclusions follow in Section VI.

Most notations in this paper are more or less standard and will be made clear as we proceed. The symbol $\odot$ means Hadamard product. The identity under Hadamard product, denoted by E, is a matrix with all elements equal to 1. The set of $n \times n$ symmetric matrices are denoted by $\mathcal{S}_n$.

II. Problem Formulation

Consider the system described by the following stochastic differential equation:

$$\dot{x}(t) = Ax(t) + B\kappa(t)u(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $\kappa(t) = \text{diag}\{\kappa_1(t), \kappa_2(t), \dots, \kappa_m(t)\}$ is a random matrix process consisting of diagonal mutually independent white noise process elements $\kappa_i(t)$ with mean $\mu_i = \boldsymbol{E}[\kappa_i(t)]$ and variance $\sigma_i^2 = \boldsymbol{E}[(\kappa_i(t) - \mu_i)^2]$. The real matrices A and B are assumed to have compatible dimensions. Let the state $x(t)$ be available for feedback. The control signal $u(t)$ is generated by a finite-dimensional LTI controller $\boldsymbol{K}$ with a state space realization:

$$\begin{aligned} \dot{x}_K(t) &= A_K x_K(t) + B_K x(t), \\ u(t) &= C_K x_K(t) + D_K x(t). \end{aligned}$$

The dimension of the controller state $x_K(t)$ is not specified a priori. Note that the system (1) reduces to a deterministic LTI system when $\kappa(t)$ is a constant diagonal matrix. Therefore, it can be regarded as an LTI system with independent random input gains. Such systems are motivated from many practical applications. For example, in NCSs, very often the actuators are located far away from each other and the controller. Thus, it is reasonable to assume that each component of the control signal is sent through an independent communication channel to the corresponding actuator.

For the i-th input channel of system (1), the signal-to-noise ratio is defined to be $\text{SNR}_i = \frac{\mu_i}{\sigma_i}$. Denote

$$\begin{aligned} &M \triangleq \text{diag}\{\mu_1, \mu_2, \dots, \mu_m\}, \Sigma^2 \triangleq \text{diag}\{\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2\}, \\ &W \triangleq M^{-2}\Sigma^2 = \text{diag}\{\text{SNR}_1^{-2}, \text{SNR}_2^{-2}, \dots, \text{SNR}_m^{-2}\}. \end{aligned}$$

The MS capacity of the ith input channel is defined as

$$\mathfrak{C}_i \triangleq \frac{1}{2}\frac{\mu_i^2}{\sigma_i^2} = \frac{1}{2}\text{SNR}_i^2.$$

The overall channel capacity is then given by $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

We study the state feedback indefinite LQ optimal control of system (1). Denote $x_c(t) = \begin{bmatrix} x(t)' & x_K(t)' \end{bmatrix}'$, then the closed-loop system can be written as

$$\dot{x}_c(t) = [\tilde{A} + \tilde{B}\kappa(t)\tilde{C}]x_c(t),$$

where $\tilde{A} = \begin{bmatrix} A & 0 \\ B_K & A_K \end{bmatrix}, \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tilde{C} = \begin{bmatrix} D_K & C_K \end{bmatrix}$. A controller $\boldsymbol{K}$ is said to be MS stabilizing if for every initial state x_0, the closed-loop state $x_c(t)$ of (1) satisfies $\lim_{t\to\infty} \boldsymbol{E}[x_c(t)x_c'(t)] = 0$. For a given initial state x_0, consider the following cost function:

$$\begin{aligned} J(x_0, u(t)) &= \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ \kappa(t)u(t) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ \kappa(t)u(t) \end{bmatrix} dt \\ &= \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R \end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix} dt, \end{aligned} \tag{2}$$

where $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ is a symmetric weighting matrix that can be indefinite. One traditional way to formulate the LQ optimal control problem is to fix the channel capacities a priori and then find a stabilizing state feedback controller such that $J(x_0, u(t))$ is minimized for every initial state x_0. However, fixing the channel capacities a priori may not be desirable since the problem may be unsolvable under certain given set of capacities. It is a wise expectation that the existence of an MS stabilizing controller is much relevant to the input channel capacities. If the channels are too bad with too much noise, even a stabilizing controller does not exist, let alone the optimal one. The minimum capacities required for stabilization is discussed in the next section.

To tackle this difficulty, the channel/controller co-design framework provides a significant insight, which is one main novelty of this work. In this case, the individual channel capacities $\mathfrak{C}_i$ are not assumed to be given. Instead, they can be designed, or allocated under an overall capacity constraint $\mathfrak{C}$. The allocation of the overall capacity to the individual channels, called channel resource allocation, can be formally given by a probability vector $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_m \end{bmatrix}'$, where $0 \le \pi_i \le 1$, $\sum_{i=1}^m \pi_i = 1$, such that $\mathfrak{C}_i = \pi_i \mathfrak{C}$. With the channel/controller co-design, we formulate the optimal control problem as to simultaneously design an allocation π and an optimal MS stabilizing controller $\boldsymbol{K}$ to minimize the cost function $J(x_0, u(t))$ for every initial state x_0.

We define the value function V under a feasible allocation π as

$$V(x_0) = \inf_{\boldsymbol{K} \text{ is MS stabilizing}} J(x_0, u(t)).$$

If the matrix $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ is positive semi-definite, as in [4], then $J(x_0, u(t))$ is nonnegative by nature. However, here we allow $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ to be indefinite. In this case, it may happen that $J(x_0, u(t))$ is not bounded from below. The indefinite LQ problem considered is said to be well-posed if

$$-\infty < V(x_0) < +\infty, \quad \forall x_0 \in \mathbb{R}^n.$$

A well-posed problem is said to be attainable if there exists an MS stabilizing controller, referred to as the optimal controller, that achieves the infimum.

Before proceeding, recall that the topological entropy [2] of a matrix $A \in \mathbb{R}^{n\times n}$ is given by $h(A) = \sum_{|\lambda_i|>1} \ln|\lambda_i|$, where λ_i are the eigenvalues of A. Based on this, we define the topological entropy of the continuous-time system $\dot{x}(t) = Ax(t)$ as $H(A) = h(e^A) = \sum_{\Re(\lambda_i)>0} \lambda_i$, where λ_i are the eigenvalues of A.

III. PRELIMINARY

In this section, we present some preliminary knowledge on the MS stabilizability.

Consider the following stochastic system

$$\dot{x}(t) = (A+\sum_{i=1}^{m} A_i\omega_i(t))x(t), \tag{3}$$

where $\omega_1(t),\dots,\omega_m(t)$ are independent zero-mean white noise with variance 1.

Definition 1: The stochastic system (3) is said to be MS stable if for any initial state $x(0)$, $N(t) \triangleq \boldsymbol{E}[x(t)x'(t)]$ is well-defined for any $t>0$ and $\lim_{t\to\infty} N(t)=0$.

Several criterions in verifying the MS stability are given in the following lemma.

Lemma 1: The following assertions are equivalent:

(a) The stochastic system (3) is MS stable.

(b) There exists a matrix $X>0$ such that

$$A'X+XA+\sum_{i=1}^{m} A_i'XA_i<0.$$

(c) For an arbitrary $P\in\mathcal{S}_n$, there exists a unique $X\in\mathcal{S}_n$ such that

$$A'X+XA+\sum_{i=1}^{m} A_i'XA_i+P=0.$$

Moreover, if $P>0$ (respectively, $P\geq 0$), then $X>0$ (respectively, $X\geq 0$).

Proof: The equivalence of (a) and (b) can be referred to [3]. The equivalence of (a) and (c) can be shown by applying Theorem A.1 in [9]. ■

Back to the system (1), as mentioned before, state feedback stabilizing controller may not exist if the channel capacities $\mathfrak{C}_i$ are fixed a priori. To cope with this difficulty, the channel resource allocation provides a significant insight. In this case, the controller designer not only designs the controller but also chooses a feasible allocation vector π such that $\mathfrak{C}_i=\pi_i\mathfrak{C}$ so as to stabilize the system (1) in the MS sense. In the following definition of MS stabilizability, we focus on the static state feedback. In fact, it has been shown in [20] that MS stabilization can be accomplished by dynamic state feedback if and only if it can be accomplished by static state feedback.

Definition 2: The system (1) is said to be MS stabilizable with capacity $\mathfrak{C}$ if there is an allocation π and a state feedback gain F such that the closed-loop system

$$\begin{aligned}\dot{x}(t) &= (A+B\kappa(t)F)x(t) \\ &= (A+BMF+\sum_{i=1}^{m}\sigma_iB_iF_i\omega_i(t))x(t),\end{aligned} \tag{4}$$

with $\mathfrak{C}_i=\pi_i\mathfrak{C}$ is MS stable.

Remark 1: When $\mathfrak{C}=\infty$, Definition 2 reduces to that of classical stabilizability of $[A|B]$.

The next lemma gives a necessary and sufficient condition on the MS stabilizability.

Lemma 2 ([20]): The system (1) is MS stabilizable with capacity $\mathfrak{C}$ if and only if $[A|B]$ is stabilizable and $\mathfrak{C}>H(A)$.

When the conditions in Lemma 2 are satisfied, how to judiciously choose the allocation π and simultaneously design the state feedback gain F so that the system (1) is MS stabilized is also discussed in [20].

Throughout the rest of this paper, it is always assumed that the system (1) is MS stabilizable with capacity $\mathfrak{C}$.

IV. WELL-POSEDNESS OF INDEFINITE LQ PROBLEM

In this section, we investigate the condition under which the indefinite stochastic LQ optimal control problem is well-posed. The attainability is addressed in the next section.

First, we give a lemma that will be frequently used in the developments to follow.

Lemma 3: For a given $X\in\mathcal{S}_n$, it holds

$$\boldsymbol{E}\int_0^t \begin{bmatrix} x(\tau) \\ Mu(\tau)\end{bmatrix}' \begin{bmatrix} A'X+XA & XB \\ B'X & W\odot(B'XB)\end{bmatrix}\begin{bmatrix} x(\tau) \\ Mu(\tau)\end{bmatrix} d\tau = \boldsymbol{E}[x(t)'Xx(t)]-x_0'Xx_0. \tag{5}$$

Proof: Let $L(t)=x(t)'Xx(t)$. Applying Itô's formula [13] to $L(t)$, we have

$$\begin{aligned}\dot{L}(t) =& [Ax(t)+B\kappa(t)u(t)]'Xx(t) \\ &+ x(t)'X[Ax(t)+B\kappa(t)u(t)] \\ &+ [B(\kappa(t)-M)u(t)]'X[B(\kappa(t)-M)u(t)].\end{aligned}$$

Then the desired result follows from integrating both sides of the above equation and taking expectations. ■

Define the operator $\mathscr{L}_F(\cdot):\mathcal{S}_n\to\mathcal{S}_n$ as

$$\begin{aligned}\mathscr{L}_F(X) \triangleq& (A+BMF)'X+X(A+BMF) \\ &+ F'(\Sigma^2\odot(B'XB))F.\end{aligned}$$

Also denote

$$\Psi_F \triangleq \begin{bmatrix} I \\ MF\end{bmatrix}'\begin{bmatrix} Q & S \\ S' & (E+W)\odot R\end{bmatrix}\begin{bmatrix} I \\ MF\end{bmatrix}.$$

The next lemma is useful to establish the upper-boundedness of the value function $V(x_0)$.

Lemma 4: Let $u(t)=Fx(t)$ be MS stabilizing under a feasible allocation π. Then the corresponding cost (2) is given by $J(x_0,u(t))=x_0'Xx_0$, where X is the unique solution to the matrix equation

$$\mathscr{L}_F(X)=-\Psi_F. \tag{6}$$

Proof: Since $u(t)=Fx(t)$ is MS stabilizing, in view of Lemma 1 (c), the matrix equation (6) has a unique solution X. By some simple calculations, we have

$$\begin{aligned} & J(x_0,u(t)) \\ &= \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t)\end{bmatrix}'\begin{bmatrix} Q & S \\ S' & (E+W)\odot R\end{bmatrix}\begin{bmatrix} x(t) \\ Mu(t)\end{bmatrix} dt \\ &= \boldsymbol{E}\int_0^\infty x(t)'\Psi_F x(t)dt \\ &= -\boldsymbol{E}\int_0^\infty x(t)'\mathscr{L}_F(X)x(t)dt \\ &= -\boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t)\end{bmatrix}'\begin{bmatrix} A'X+XA & XB \\ B'X & W\odot(B'XB)\end{bmatrix}\begin{bmatrix} x(t) \\ Mu(t)\end{bmatrix} dt.\end{aligned}$$

Applying Lemma 3 yields

$$J(x_0,u(t)) = x_0'Xx_0 - \lim_{t\to\infty} \boldsymbol{E}[x(t)'Xx(t)] = x_0'Xx_0,$$

that completes the proof. ■

Recall that the indefinite LQ problem of our concern is well-posed if $-\infty < V(x_0) < +\infty$ for every initial condition x_0. Under the assumption that the system (1) is MS stabilizable with capacity $\mathfrak{C}$, there exist a feasible allocation π together with an MS stabilizing state feedback control $u(t) = Fx(t)$. Then Lemma 4 implies that $V(x_0)$ is upper-bounded by $x_0'Xx_0$, where X is the unique solution to the matrix equation (6). Hence, $V(x_0) < +\infty$ is automatically satisfied and we only need to concern whether $V(x_0)$ is bounded from below.

To establish the main result on the well-posedness, another lemma will be useful.

Lemma 5: The indefinite stochastic LQ problem concerned is well-posed if and only if there exists a unique $X \in \mathcal{S}_n$ such that $V(x_0) = x_0'Xx_0$ for all x_0.

Lemma 5 can be proved analogously to Proposition 2 in [14]. The details are omitted here for brevity.

Define the linear operator $\mathscr{D}(\cdot) : \mathcal{S}_n \to \mathcal{S}_n$ as

$$\mathscr{D}(X) \triangleq W \odot (B'XB) + (W+E) \odot R.$$

We are now in a position to present the following theorem which gives a necessary and sufficient condition for the well-posedness.

Theorem 1: Under a feasible allocation π, the indefinite stochastic LQ problem concerned is well-posed if and only if there exists $X \in \mathcal{S}_n$ satisfying the linear matrix inequality (LMI):

$$\begin{bmatrix} A'X+XA+Q & XB+S \\ B'X+S' & \mathscr{D}(X) \end{bmatrix} \geq 0. \tag{7}$$

Proof: To prove the sufficiency, assume that there exists a matrix $X \in \mathcal{S}_n$ such that (7) holds. Then for any $u(t)$ generated by an MS stabilizing controller, we have

$$\begin{aligned} &J(x_0,u(t)) \\ &= \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R \end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix} dt \\ &\geq -\boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix}' \begin{bmatrix} A'X+XA & XB \\ B'X & W\odot(B'XB) \end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix} dt \\ &= x_0'Xx_0 - \lim_{t\to\infty} \boldsymbol{E}[x(t)'Xx(t)] = x_0'Xx_0, \end{aligned}$$

where the second equality follows from Lemma 3. This implies that $V(x_0) \geq x_0'Xx_0$ and thus, the indefinite LQ problem is well-posed.

To prove the necessity, by Lemma 5, if the indefinite LQ problem is well-posed, then there exists a unique $X \in \mathcal{S}_n$ such that $V(x_0) = x_0'Xx_0$ for all x_0. By the knowledge of dynamic programming [1], it holds

$$x_0'Xx_0 \leq \boldsymbol{E}\left[\int_0^t \begin{bmatrix} x(\tau) \\ Mu(\tau) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R \end{bmatrix} \begin{bmatrix} x(\tau) \\ Mu(\tau) \end{bmatrix} d\tau + x(t)'Xx(t) \right],$$

for all $t \geq 0$ and any $u(t)$ generated by an MS stabilizing controller. Applying the above inequality together with Lemma 3 leads to

$$\boldsymbol{E}\int_0^t \begin{bmatrix} x(\tau) \\ Mu(\tau) \end{bmatrix}' \begin{bmatrix} A'X+XA+Q & XB+S \\ B'X+S' & \mathscr{D}(X) \end{bmatrix} \begin{bmatrix} x(\tau) \\ Mu(\tau) \end{bmatrix} d\tau \geq 0.$$

Dividing both sides by t and letting $t \to 0$ implies

$$\begin{bmatrix} x(0) \\ Mu(0) \end{bmatrix}' \begin{bmatrix} A'X+XA+Q & XB+S \\ B'X+S' & \mathscr{D}(X) \end{bmatrix} \begin{bmatrix} x(0) \\ Mu(0) \end{bmatrix} \geq 0.$$

Since the choice of $x(0)$ is arbitrary, it follows that the matrix X indeed satisfies the LMI (7). This completes the proof. ■

V. Attainability of Indefinite LQ Problem

This section studies the attainability of the indefinite stochastic LQ optimal control problem. Theorem 1 shows that the wellposedness of the LQ problem is determined by the solvability of the LMI (7). In this section, for simplicity, we confine our attention to a broad class of well-posed problems for which LMI (7) has a solution X such that $\mathscr{D}(X)$ is nonsingular, or equivalently, the following LMIs

$$\begin{cases} \begin{bmatrix} A'X+XA+Q & XB+S \\ B'X+S' & \mathscr{D}(X) \end{bmatrix} \geq 0, \\ \mathscr{D}(X) > 0. \end{cases} \tag{8}$$

has a solution. A similar approach can be used to address the general case which will be a bit more complex due to the possible singularity of $\mathscr{D}(X)$.

In the sequel, a type of modified algebraic Riccati equation (MARE) is first investigated and then employed to study the attainability of the indefinite stochastic LQ optimal control.

A. MARE

The following MARE will play an essential role in later developments:

$$A'X+XA+Q-(XB+S)\mathscr{D}(X)^{-1}(B'X+S') = 0. \tag{9}$$

A solution X to (9) is said to be MS stabilizing if the associated state feedback gain

$$F = -M^{-1}\mathscr{D}(X)^{-1}(B'X+S') \tag{10}$$

makes the closed-loop system (4) MS stable.

Remark 2: When $\mathfrak{C} = \infty$, the MARE (9) reduces to the classical continuous-time algebraic Riccati equation.

The MARE (9) is closely related to the LMIs (8). Define the operator $\mathscr{R}(\cdot) : \mathcal{S}_n \to \mathcal{S}_n$ as

$$\mathscr{R}(X) \triangleq A'X+XA+Q-(XB+S)\mathscr{D}(X)^{-1}(B'X+S')$$

and the sets

$$\Omega \triangleq \{X | X \in \mathcal{S}_n, \mathscr{R}(X) \geq 0, \mathscr{D}(X) > 0\},$$
$$\Gamma \triangleq \{X | X \in \mathcal{S}_n, \mathscr{R}(X) > 0, \mathscr{D}(X) > 0\}.$$

By the knowledge of Schur complement, Ω is in fact the solution set to (8). Apparently, $\Gamma \subset \Omega$. The LMIs (8) is said to be feasible if $\Omega \neq \emptyset$ and is said to be strictly feasible if $\Gamma \neq \emptyset$. Convex optimization technique can be used to check numerically whether the LMIs (8) is feasible (respectively, strictly feasible) or not. The maximal solution to feasible LMIs (8), denoted as X_+, is the maximal element in Ω in the sense that $X_+ \geq X$ for all $X \in \Omega$. The maximal solution, if exists, is unique.

The following lemma investigates the existence of the maximal solution X_+ and establishes a link between X_+ and the MS stabilizing solution to the MARE (9).

Lemma 6: If $\Omega \neq \emptyset$ under a feasible allocation π, then the LMIs (8) has a maximal solution X_+. Moreover, X_+ is a solution to the MARE (9). In this case, the MARE (9) has at most one MS stabilizing solution, which coincides with X_+.

Lemma 6 has been proved in [4] for the case when $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ is positive semi-definite. A closer look reveals that the proof there does not involve any property of the definiteness and therefore can be applied to the indefinite case here. The details of the proof are omitted for brevity.

Numerically, X_+ can be computed by solving the following convex optimization problem:

$$\begin{aligned} \max \quad & \mathrm{tr}(X), \\ \text{subject to} \quad & \text{constraints } (8). \end{aligned}$$

The next issue of interest is to investigate the condition under which the MARE (9) indeed has an MS stabilizing solution. See the following lemma.

Lemma 7: The following assertions are equivalent:

(a) $\Gamma \neq \emptyset$.

(b) The MARE (9) has an MS stabilizing solution X such that $\mathscr{D}(X) > 0$.

Proof: The implication (a)⇒(b) has been shown in [4] for the definite case. The proof there does not involve any property of the definiteness and therefore can be applied to the indefinite case here. Hence, it suffices to show (b)⇒(a), as elaborated below.

By Lemma 6, if the MS stabilizing solution exists and satisfies the required inequality, it must be X_+ which is the maximal solution to the LMIs (8). Consider an open subset $\mathcal{U} = \{X \in \mathcal{S}_n | \mathscr{D}(X) > 0\}$. One can see that $X_+ \in \mathcal{U}$. Let $\Psi : \mathcal{U} \times R \to \mathcal{S}_n$ be defined by $\Psi(X, \delta) = \mathcal{R}(X) + \delta I$. It is clear that $(X_+, 0)$ is a solution to

$$\Psi(X, \delta) = 0. \tag{11}$$

We shall apply the implicit function theorem [18] to the equation (11) to show that there exists $\tilde{\delta} > 0$ and a smooth function $\delta \to X_\delta : (-\tilde{\delta}, \tilde{\delta}) \to \mathcal{U}$ such that $\Psi(X_\delta, \delta) = 0$. To this end, one needs to show that $\frac{\partial \Psi}{\partial X}(X_+, 0)$ is an isomorphism. Indeed, we have

$$\frac{\partial \Psi}{\partial X}(X_+, 0)Y = \lim_{\epsilon \to 0} \frac{\Psi(X_+ + \epsilon Y, 0) - \Psi(X_+, 0)}{\epsilon} = \mathcal{L}_{F_+}(Y),$$

for all $Y \in \mathcal{S}_n$, where

$$F_+ = -M^{-1}\mathscr{D}(X_+)^{-1}(B'X_+ + S'). \tag{12}$$

It is then clear that $\frac{\partial \Psi}{\partial X}(X_+, 0)$ is a linear function. In addition, since F_+ is MS stabilizing, it follows from Lemma 1 (c) that the kernel and range of $\frac{\partial \Psi}{\partial X}(X_+, 0)$ are $\{0\}$ and $\mathcal{S}_n$, respectively. This implies that $\frac{\partial \Psi}{\partial X}(X_+, 0)$ is an isomorphism. Also the continuity of $(X, \delta) \to \frac{\partial \Psi}{\partial X}(X, \delta)$ at $(X, \delta) = (X_+, 0)$ is obvious. We can now apply the implicit function theorem to deduce that there exists $\tilde{\delta} > 0$ and a smooth function $\delta \to X_\delta : (-\tilde{\delta}, \tilde{\delta}) \to \mathcal{U}$ such that $\Psi(X_\delta, \delta) = \mathcal{R}(X_\delta) + \delta I = 0$ for all $\delta \in (-\tilde{\delta}, \tilde{\delta})$ and $\lim_{\delta \to 0} X_\delta = X_+$. As a direct consequence, for an arbitrary $\delta \in (-\tilde{\delta}, 0)$, there holds $\mathcal{R}(X_\delta) = -\delta I > 0$. Since $X_\delta \in \mathcal{U}$, i.e., $\mathscr{D}(X_\delta) > 0$, it follows that $X_\delta \in \Gamma$ which completes the proof. ■

B. Attainability of Indefinite LQ Problem

We first show that the value function $V(x_0)$ of the indefinite stochastic LQ problem is given in terms of the maximal solution to the LMIs (8).

Theorem 2: If $\Omega \neq \emptyset$ under a feasible allocation π, then the value function is given by $V(x_0) = x_0' X_+ x_0$ for all x_0, where X_+ is the maximal solution to the LMIs (8).

Proof: The existence of the maximal solution X_+ is guaranteed by Lemma 6. Moreover, by the same procedure as in the sufficiency proof of Theorem 1, we can show that $V(x_0) \geq x_0' X_+ x_0$.

Now it suffices to show that $V(x_0) \leq x_0' X_+ x_0$. Assume that X is an arbitrary element in Ω. It can be easily verified that X satisfies the LMI

$$\begin{bmatrix} A'X + XA + Q + \epsilon I & XB + S \\ B'X + S' & \mathscr{D}(X) \end{bmatrix} > 0,$$

for an arbitrary $\epsilon > 0$. In view of Lemma 7, the MARE

$$A'X + XA + Q + \epsilon I - (XB + S)\mathscr{D}(X)^{-1}(B'X + S') = 0$$

has a unique MS stabilizing solution X_ϵ. The associated MS stabilizing state feedback gain is given by

$$F_\epsilon = -M^{-1}\mathscr{D}(X_\epsilon)^{-1}(B'X_\epsilon + S').$$

In addition, there holds $\mathscr{L}_{F_\epsilon}(X_\epsilon) = -\Psi_{F_\epsilon}$. Then Lemma 4 yields that $V(x_0) \leq x_0' X_\epsilon x_0$. Taking the limit as $\epsilon \to 0$, we have $V(x_0) \leq x_0' X_+ x_0$ which completes the proof. ■

The next theorem gives equivalent conditions for the attainability of the indefinite stochastic LQ problem under the assumption that $\Omega \neq \emptyset$. The optimal controller is also obtained.

Theorem 3: If $\Omega \neq \emptyset$ under a feasible allocation π, the following assertions are equivalent:

(a) The indefinite LQ problem concerned is attainable.

(b) The MARE (9) has an MS stabilizing solution X.

(c) $\Gamma \neq \emptyset$.

Moreover, for an attainable problem, the unique optimal controller is given by $u(t) = Fx(t)$, where F is the state feedback gain (10) associated with the MS stabilizing solution X to the MARE (9).

Proof: The equivalence between (b) and (c) has been shown in Lemma 7. It suffices to show the equivalence between (a) and (b).

We first prove (a)⇒(b). Since $\Omega \neq \emptyset$, the maximal solution X_+ to (8) exists which is also a solution to the MARE (9).

Let $u^*(t)$ be generated by an optimal controller and $x^*(t)$ be the corresponding plant state. Applying Lemma 3 yields

$$\begin{aligned}
&V(x_0)\\
=&J(x_0,u^*(t))\\
=&\boldsymbol{E}\int_0^\infty \begin{bmatrix} x^*(t)\\ Mu^*(t)\end{bmatrix}' \begin{bmatrix} Q & S\\ S' & (E+W)\odot R\end{bmatrix}\begin{bmatrix} x^*(t)\\ Mu^*(t)\end{bmatrix}dt\\
=&\boldsymbol{E}\int_0^\infty \begin{bmatrix} x^*(t)\\ Mu^*(t)\end{bmatrix}' \begin{bmatrix} A'X_+ + X_+A+Q & X_+B+S\\ B'X_+ + S' & \mathscr{D}(X_+)\end{bmatrix}\begin{bmatrix} x^*(t)\\ Mu^*(t)\end{bmatrix}dt\\
&+x_0'X_+x_0.
\end{aligned}$$

By completing the squares, we have

$$\begin{aligned}
&V(x_0)\\
=&\boldsymbol{E}\int_0^\infty (u^*(t)-F_+x^*(t))'M\mathscr{D}(X_+)M(u^*(t)-F_+x^*(t))dt\\
&+x_0'X_+x_0,
\end{aligned}$$

where F_+ is given by (12). Since $V(x_0)=x_0'X_+x_0$ and $\mathscr{D}(X_+)>0$, it follows that $u^*(t)$ is uniquely given by the feedback form $u^*(t)=F_+x^*(t)$. Therefore, X_+ is the MS stabilizing solution to the MARE (9) as $u^*(t)$ is MS stabilizing.

Now we prove (b)⇒(a). Assume that the MS stabilizing solution to (9) exists. In view of Lemma 6, it coincides with the maximal solution X_+ of the LMIs (8). Hence, $u(t)=F_+x(t)$ is MS stabilizing, where F_+ is given by (12). Moreover, there holds $\mathscr{L}_{F_+}(X_+)=-\Psi_{F_+}$. Then Lemma 4 yields that $J(x_0,u(t))=x_0'X_+x_0$. Since $V(x_0)=x_0'X_+x_0$ as shown in Theorem 2, it follows that the indefinite LQ problem is indeed attainable with the optimal controller given by $u(t)=F_+x(t)$. This completes the proof. ■

Remark 3: By virtue of the equivalence between the assertions (a) and (c) in Theorem 3, one can easily check the attainability of the indefinite LQ optimal control problem with random input gains by efficient LMI solvers.

VI. Conclusion

In this paper, we study the indefinite LQ optimal control of continuous-time LTI systems with random input gains. In our setup, each element of the control signal is subject to independent stochastic multiplicative noise. One main novelty of this work is that we do not assume that the input channel capacities are fixed priori. Instead, we put the indefinite stochastic LQ problem under the framework of channel/controller co-design which bridges and integrates the design of the channels and controller. The co-design is carried out by the twist of channel resource allocation, i.e., the channel capacities can be allocated among the input channels by the control designer subject to an overall capacity constraint. With the channel/controller co-design, the well-posedness and attainability of the indefinite LQ problem concerned is nicely addressed.

The well-posedness of the indefinite stochastic LQ optimal control is more involved than the definite case, as the cost function may not be bounded from below in general. It is shown that the well-posedness of the indefinite problem is determined by the feasibility of the LMI (7). In addition, under certain mild assumptions, a well-posed problem is shown to be attainable if and only if the MARE (9) has an MS stabilizing solution. The attainability is also equivalent to the strict feasibility of LMIs (8) that can be easily verified by efficient LMI solvers. For an attainable problem, the optimal controller is given by a linear state feedback associated with the MS stabilizing solution to the MARE (9).

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