Duration-deadline Jointly Differentiated Energy Services

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Abstract— **The demand/supply balance in a power grid faces great challenges as more and more renewables are integrated into the system. Fluctuations in the power supply increases spectacularly owing to the uncertain nature of renewables. In this case, it has been widely recognized that the conventional scheme of supply following load is neither economically efficient nor environmentally friendly. On the contrary, an alternative paradigm has been attracting more and more attention over the recent years which attempts to utilize the flexibilities in the demand side to compensate the uncertainties in the supply side. Following this direction, we propose a duration-deadline jointly differentiated energy service in this paper. Specifically, we consider a group of flexible loads with each load requiring a constant power level for a specified duration before a specified deadline. A load is indifferent of the actual time of delivery as long as the duration and deadline requirements are satisfied. We first address the adequacy problem of a given supply profile which boils down to solving a $(0,1)$-matrix feasibility problem. It turns out that the adequacy condition is given explicitly by the nonnegativity of a structure matrix. We also develop a market implementation of the proposed energy service and show the existence of an efficient competitive equilibrium.**

I. Introduction

The demand/supply balance in a power system has always been a critical issue. It becomes particularly important when large amount of renewables such as solar and wind are integrated into the system. The highly uncertain and intermittent nature of renewables imposes a big challenge to researchers and engineers. How to maintain the demand/supply balance in the presence of high uncertainties has been attracting great attention in recent years.

Conventionally, balancing the demand and supply in a power system is achieved through reserves, i.e., exploiting the reserves in order to make the supply follow the demand. Such supply following demand scheme works satisfactorily when the power is mostly generated from traditional sources such as fuels and nuclear, etc. However, as more renewables are integrated into the grid, larger amount of reserves are required to maintain the balance. This is neither economically efficient due to the high cost of reserves nor environmentally friendly because of the extra creation of green-house gases. Moreover, the fast ramping requirement is likely to increase the cost much further. Hence, the conventional scheme does not appear as a wise choice any more in the presence of high renewables.

In contrast, an alternative scheme has shown great promise in balancing the demand and supply when deep penetration of renewables is present. The underlying rationale is to exploit the flexibilities in the loads wisely so as to compensate the uncertainties in the supply, i.e., to shape the demand so as to meet the supply. Research along this direction belongs to a broad category often referred to as demand response or demand-side management. Typical examples of flexible loads include the thermostatically controlled loads (TCLs), electric vehicles (EVs), pool pumps, and smart appliances, etc. These loads can be deferred, intermitted, or modulated, depending on the nature of the particular loads. Many different schemes have been proposed to exploit such load flexibilities, see for instance [4], [9], [10], [11], [13], [14], [16], [17]. Recently, a duration-differentiated energy service has been studied in [12], wherein the loads are assumed to be indifferent of the actual delivery time provided that the duration of the power is the same. Another relevant work in [2] proposes a deadline-differentiated energy service. There the loads are assumed to be indifferent of the actual delivery time provided that a delivery deadline is guaranteed.

One critical concern in demand response is the design of the incentive mechanisms so as to motivate the consumers to elicit their flexibilities. Under a properly designed incentive mechanism, the consumers are expected to be appropriately compensated for the flexibilities they might offer. Note that in a traditional power market, electricity is more or less treated as a homogenous product with a unit price. In order for the consumers to elicit their flexibilities, the market design may need to go beyond the traditional structure. Some sort of differentiated energy service is needed so as to accommodate different flexibilities that might be offered by consumers. Several interesting attempts along this direction have been reported in the literature. See for instance [2], [5], [12], [15], [17], [18]. Of particular interest to us in the current work are the market design of duration-differentiated energy service as in [12] and the market design of deadline-differentiated energy service as in [2].

In this paper, we propose and study the co-called duration-deadline jointly differentiated energy service. Such differentiated energy service takes into account the load flexibilities exploited in [12] and [2] in a joint way. Specifically, assume that the power delivery is segmented into a series of time slots. Each flexible load demands a constant power level for a specified duration before a specified deadline. Note that a load is indifferent of the actual delivery time as long as both

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the duration and deadline requirements are satisfied. Our first concern is to characterize the adequacy condition of a given supply profile which amounts to solving a matrix feasibility problem. The adequacy condition is shown to be given by the nonnegativity of a structure matrix. Moreover, we study the market implementation of the proposed energy service and show the existence of an efficient competitive equilibrium.

The rest of this paper is organized as follows. Section II formulates the problem to be studied. Section III gives the adequacy condition as well as the adequacy gap. Section IV discusses the market implementation of the duration-deadline differentiated energy service. Finally, the paper is concluded in Section V. Most notation in this paper is more or less standard and will be made clear as we proceed. A matrix S is said to be nonnegative, denoted as $S \geq 0$, if all the entries of S are nonnegative.

II. PROBLEM FORMULATION

In this paper, we shall propose a duration-deadline jointly differentiated energy service as an attempt to elicit the load flexibilities to compensate the supply uncertainties. This new scheme of energy service features the load flexibilities discussed in [12] and [2] in a joint manner. Specifically, a flexible load here is assumed to require a constant level of power for a specified duration delivered before a specified deadline. A typical motivating example is the electrical vehicle charging. A customer may wish to specify a charging duration as well as a deadline before which the charging should be completed. To highlight the key idea and simplify the presentation, we shall start with a simple yet representative case in which there are only two different deadlines to choose. The extension to the general case with arbitrary number of different deadlines would be rather straightforward.

Assume that the power is delivered over T time slots. The power available at time slot j is given by $p_j, j = 1, 2, \ldots, T$. Consider N flexible loads, indexed by $i = 1, 2, \ldots, N$. For technical simplicity, assume that load i requires 1kW of power for a total duration of h_i time slots delivered before the d_ith time slot, where $h_i \leq d_i$. Here the flexibility resides in the fact that any h_i time slots before the deadline d_i, whether continuous or not, will satisfy the requirement of load i. In this case, the energy services are differentiated by both the service duration and deadline. Such energy services are referred to as duration-deadline jointly differentiated energy services herinafter.

For the brevity of presentation, we concentrate on the case when there are only two different deadlines for the loads to choose. In particular, assume that

$$\begin{aligned} d_1 = d_2 = \cdots = d_{N_1} = T, \\ d_{N_1+1} = d_{N_1+2} = \cdots = d_N = T_1, \end{aligned}$$

where $T_1 \leq T$. In words, due to the two different deadlines, the loads are naturally divided into two groups: $[1, N_1]$ and $[N_1{+}1, N]$, where the loads of the first group require deadline T while those of the second group require deadline T_1. On the other hand, these two deadlines divide the time slots into two intervals: $[1, T_1]$ and $[T_1+1, T]$. Power supplies from the first time interval can be allocated to both group of the loads, while those from the second time interval can only be allocated to the first group of loads.

Denote the power supply profile by

$$p = \begin{bmatrix} p_1 & p_2 & \ldots & p_T \end{bmatrix}',$$

and the power demand profile by

$$h = \begin{bmatrix} h_1 & h_2 & \ldots & h_N \end{bmatrix}',$$

respectively. A given supply profile p is said to be adequate if there exits an allocation of power such that all the load requirements are satisfied. Furthermore, a supply profile p is said to be exactly adequate if it is adequate and there holds $\sum_{j=1}^{T} p_j = \sum_{i=1}^{N} h_i$, i.e., there will be no excess supply after allocation.

Our first concern is the adequacy, i.e., to find a necessary and sufficient condition under which a given supply profile is adequate. The second concern then follows naturally: the adequacy gap, i.e., if the given supply profile is inadequate, what is the minimum amount of additional purchase required in order that the total supply is adequate? It turns out that these concerns can be addressed by invoking our previous results [6] on some $(0, 1)$-matrix feasibility problem. A brief review on that matter will be given later. The main focus of this paper is to develop a market implementation of such duration-deadline jointly differentiated energy service. It is expected that the loads with more degree of flexibility should get more compensated via the market.

As far as market implementation is concerned, we would like to consider a large collection of loads with no single load big enough compared to the total available capacity. This will lead to a perfectly competitive market wherein every player is a price taker. In that case, the problem formulation may need to be modified slightly. Specifically, we shall consider a continuum of consumers indexed by $x \in [0, 1]$. Consumer x demands $r(x)$kW of power for a duration of $h(x)$ time slots delivered before the $d(x)$th time slot, where $h(x) \leq d(x)$. As before, consumer x is indifferent of the actual time of service as long as the total duration is $h(x)$ and the service is delivered before the deadline $d(x)$.

Note that when $T_1 = T$, the scenario reduces to the case of a single deadline which has been treated in [12]. In that case, the services are only differentiated by the duration and, thus, are referred to as the duration differentiated energy services.

III. ADEQUACY AND ADEQUACY GAP

The theme of this section is to characterize the adequacy condition and the adequacy gap. To feature the essential idea, we present the results for the case of finitely many consumers in this section. The extension to the case of a continuum of consumers will be discussed later.

The first step in characterizing the adequacy is to translate the problem to a $(0, 1)$-matrix feasibility problem. It is easy to see that allocating the power to the loads is equivalent to filling an $N \times T$ $(0, 1)$-matrix A wherein each row corresponds to one flexible load and each column corresponds to one time slot. Observe that the two different deadlines divide the time slots into two intervals: $[1, T_1]$ and $[T_1{+}1, T]$, where the power supplies from the first interval can be allocated to all the loads while those from the second interval can only be allocated to the loads requiring deadline T. Fitting into the picture of filling a $(0,1)$-matrix A, this imposes a structural constraint, i.e., there exists a block at the bottom right corner of A which is fixed to be zero. In other words, the matrix A can be partitioned in the form as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & O \end{bmatrix}, \tag{1}$$

where O is an $(N{-}N_1) \times (T{-}T_1)$ zero block.

It can be clearly seen that a given supply profile is exactly adequate if and only if one can find a $(0,1)$-matrix A in the form of (1) such that the row sum vector and column sum vector are given by h and p, respectively. In this way, the original adequacy problem is translated to a $(0,1)$-matrix feasibility problem. Note that the re-ordering of the rows and columns of A should not affect the solvability of the matrix feasibility problem as long as the zero block constraint is not violated. Therefore, without loss of generality, we can assume the following monotonicity on h and p:

$$\begin{aligned} &h_1 \geq h_2 \geq \cdots \geq h_{N_1}, \quad h_{N_1+1} \geq h_{N_1+2} \geq \cdots \geq h_N, \\ &p_1 \geq p_2 \geq \cdots \geq p_{T_1}, \quad p_{T_1+1} \geq p_{T_1+2} \geq \cdots \geq p_T. \end{aligned}$$

In general, such matrix feasibility problems with possibly certain fixed zeros have been attracting considerable attention from mathematics as well as many other fields. In particular, the problem at hand has been treated in [3] and revisited in our previous work [6]. A necessary and sufficient condition is obtained in [6] by exploiting a well-defined structure matrix $S(h,p)$, wherein the (k_1,k_2)th element is given by

$$\begin{aligned} S_{k_1k_2}(h,p) = &\sum_{j>k_1}^{T_1} p_j + \sum_{j>T_1+k_2}^{T} p_j \\ &- \sum_{i=1}^{N_1} [h_i - (k_1{+}k_2)]^+ - \sum_{i=N_1+1}^{N} (h_i - k_1)^+, \end{aligned} \tag{2}$$

where $0 \leq k_1 \leq T_1$ and $0 \leq k_2 \leq T{-}T_1$. The matrix S is referred to as a structure matrix since it is solely determined by the structure information: h, p, and the fixed zero block. It has nothing to do with the specific choice of A.

Lemma 1 ([6]): There exists a $(0,1)$-matrix of the form (1) with row sum vector h and column sum vector p if and only if the structure matrix $S \geq 0$.

The revealing of Lemma 1 gives a precise solution to the adequacy problem, as stated in the following theorem.

Theorem 1: A supply profile p is adequate if and only if $S \geq 0$. If further there holds $\sum_{i=1}^{N} h_i = \sum_{j=1}^{T} p_j$, then p is exactly adequate.

The mathematical expression of the structure matrix S as in (2) may look somehow complicated at first sight. However, it has a rather intuitive physical interpretation. A good one-sentence summary is simply the energy dominance in tails. Specifically, in the expression (2), the terms

$$\sum_{j>k_1}^{T_1} p_j \quad \text{and} \quad \sum_{j>T_1+k_2}^{T} p_j$$

represent the supply tails from the time interval $[1, T_1]$ and $[T_1{+}1, T]$, respectively. On the other hand, the terms

$$\sum_{i=1}^{N_1} [h_i - (k_1{+}k_2)]^+ \text{ and } \sum_{i=N_1+1}^{N} (h_i - k_1)^+$$

represent the demand tails from the group of loads $[1, N_1]$ and $[N_1{+}1, N]$, respectively. In this regard, the nonnegativity of the structure matrix is nothing but the requirement that the supply tails should be no less than the demand tails, which is quite intuitive.

One can refer to [6] for the details on the derivation of the adequacy condition. The main idea is sketched as below. The key is to translate the matrix feasibility problem to a network flow feasibility problem [8] and then use the celebrated max-flow min-cut theorem [7]. However, a direct application of the max-flow min-cut theorem gives an exponential number of inequalities which are of limited practical use. However, thanks to the nice structure of the fixed zeros in the current problem, it turns out that among those exponential number of inequalities, most of them are redundant and, thus, can be excluded. In this way, we manage to reduce the number of inequalities substantially and obtain the condition given in terms of the nonnegativity of the structure matrix S.

Remark 1: When $T_1 = T$, the scenario simplifies to the case of one single deadline. Associated with it is a matrix feasibility problem with given row and column sums but no fixed zeros. In that case, the nonnegativity of S reduces to the well-known majorization condition [8], [12].

After obtaining the adequacy condition, we raise a natural follow-up question: If a supply profile p is not adequate, what is the minimum amount of additional purchase required to make the total supply be adequate? This amounts to solving the following optimization problem:

$$\min_{a} \sum_{j=1}^{T} a_j, \text{ subject to } S(h, p+a) \geq 0,$$

where $a = \begin{bmatrix} a_1 & a_2 & \cdots & a_T \end{bmatrix}'$ is a non-negative integer vector representing the additional purchase. Note that the network flow approach sketched in the above not only reveals the adequacy condition, but also gives an explicit solution to the adequacy gap as a by-product. See the following theorem. The details of the proof can be referred to [6] and is omitted here for brevity.

Theorem 2: When p is not adequate, the minimum total amount of additional purchase required for adequacy is

$$\left| \min_{0 \leq k_1 \leq T_1, 0 \leq k_2 \leq T-T_1} S_{k_1k_2}(h,p) \right|.$$

IV. Market Implementation

This section is dedicated to investigating a forward market implementation of the duration-deadline jointly differentiated energy service.

As reasoned before, here we shall consider a continuum of consumers such that each consumer can be viewed as a price taker. A forward market structure is considered, i.e., all the transactions are completed before the time of delivery. The market consists of three elements:

- Services: The power delivery is segmented into T time slots. The energy services are differentiated by both the duration and deadline requirements. Suppose there are two different deadlines, T_1 and T, for the consumers to choose. The service of duration h and deadline d has a price π_h^d.
- Consumers: There is a continuum of consumers indexed by $x \in [0,1]$. The utility function of consumer x who receives $r(x)$ kW of power for $h(x)$ time slots delivered before deadline $d(x)$ is given by $U(x, r(x), h(x), d(x))$. Assume that $U(x, 0, h(x), d(x)) = 0$.
- Supplier: Consider an aggregate supplier who has available for free a supply profile $p = \begin{bmatrix} p_1 & p_2 & \dots & p_T \end{bmatrix}'$.

The information flow of the market is as follows: Facing a menu of services associated with prices $\mathcal{M} = \{h, d, \pi_h^d\}$, consumer x chooses a service $\{h(x), d(x)\}$ and a level $r(x)$ kW in order to maximize his/her benefit, while the supplier decides the amount n_h^d of each service $\{h, d\}$ to produce so as to maximize its revenue.

In the sequel, we first approach the social welfare problem and proceed to show that the optimal social allocation can be sustained as a competitive equilibrium.

Before proceeding, let us revisit the adequacy concern in the scenario of a continuum of consumers. To one's delight, it turns out that the results obtained in the previous section can be extended here with a slight modification on the definition of the structure matrix:

$$\begin{aligned} S_{k_1 k_2}^c(h,p) = & \sum_{j>k_1}^{T_1} p_j + \sum_{j>T_1+k_2}^{T} p_j \\ & - \int_0^1 r(x)\left[(h(x)-(k_1+k_2))^+ \mathbb{1}(d(x)=T)\right] dx \\ & - \int_0^1 r(x)\left[(h(x)-k_1)^+ \mathbb{1}(d(x)=T_1)\right] dx, \end{aligned}$$

where $0 \le k_1 \le T_1$ and $0 \le k_2 \le T - T_1$. The superscript c in $S_{k_1 k_2}^c(h,p)$ is used to refer to the continuum case.

Theorem 3: A supply profile p is adequate to satisfy the requirements of a continuum of consumers if and only if $S^c(h,p) \ge 0$. Moreover, in the event of an inadequate supply, the adequacy gap is given by

$$\left| \min_{0 \le k_1 \le T_1, 0 \le k_2 \le T - T_1} S_{k_1 k_2}^c(h,p) \right|.$$

The proof of the above theorem follows straightforwardly from the results in the previous section together with some limit process. The details are omitted here for brevity.

A. Social welfare optimization

The social welfare problem aims at finding an allocation of the services that maximizes the overall benefit of the society. Given a supply profile p and a continuum of consumers indexed by $x \in [0,1]$, a social allocation of the services $x \mapsto (r(x), h(x), d(x))$ is said to be feasible if p is adequate to meet the associated demand profile. Since we assume that the supply is available for free, together with the fact that the total costs of the consumers and the revenue of the supplier will cancel each other, maximizing the overall benefit of the society boils down to maximizing the integral of the utility of all the individual consumers. To be more precise, the social welfare problem is formulated as the following optimization problem:

$$\begin{aligned} \max_{r(x),h(x),d(x)} & \int_0^1 U(x, r(x), h(x), d(x)) dx \\ \text{s.t.} \quad & r(x) \ge 0, h(x) \le d(x), d(x) \in \{T_1, T\} \\ & S^c(h,p) \ge 0 \end{aligned}$$

The main result of this subsection is stated in the following theorem.

Theorem 4: The social welfare optimization problem has a solution for any type of utility function $U(x, r, h, d)$.

Proof: Define $Z(x) = [z_{k_1 k_2}(x)]$ with $Z(0) = 0$ and

$$\dot{Z}(x) = F(x) = [f_{k_1 k_2}(x)], \tag{3}$$

where

$$f_{k_1 k_2}(x) = \begin{cases} r(x)[(h(x)-(k_1+k_2))^+ \mathbb{1}(d(x)=T) \\ \qquad + (h(x)-k_1)^+ \mathbb{1}(d(x)=T_1)], \\ \qquad \text{when } 0 \le k_1 \le T_1, 0 \le k_2 \le T - T_1, \\ U(x, r(x), h(x), d(x)), \\ \qquad \text{when } k_1 = T_1 + 1, 0 \le k_2 \le T - T_1. \end{cases}$$

For $x \in [0,1]$, let

$$\mathcal{F}(x) = \{F(x) | r(x) \ge 0, h(x) \le d(x), d(x) \in \{T_1, T\}\}.$$

Then $x \mapsto \mathcal{F}(x)$ in fact gives a set-valued correspondence and, thus, the differential equation (3) can be written as a differential inclusion:

$$\dot{Z}(x) \in \mathcal{F}(x). \tag{4}$$

Let G be the integral of the set-valued correspondence $\mathcal{F}(x)$:

$$G = \int_0^1 \mathcal{F}(x) dx.$$

In view of (4), it follows that

$$G = \{Z(1) | Z(1) \text{ is reached by a service allocation } x \mapsto (r(x), h(x), d(x))\}.$$

With the aid of the set G, the welfare optimization problem can be restated as

$$\max_{Z(1)\in G} \quad z_{T_1+1\ 0}(1) \tag{5}$$

$$\text{s.t.} \quad z_{k_1k_2}(1) \le \sum_{j>k_1}^{T_1} p_j + \sum_{j>T_1+k_2}^{T} p_j \tag{6}$$
$$\text{for } 0 \le k_1 \le T_1, 0 \le k_2 \le T - T_1.$$

By a theorem of Lyapunov on the convexity of the range of a set-valued integral [1], we know that G is convex and closed. In addition, as clearly seen, the adequacy constraint is nothing but a set of linear inequalities. Therefore, the optimal solution $Z^*(1)$ exists regardless of the form of the utility function which concludes the proof. ■

Remark 2: The solvability of the social welfare problem regardless of the form of the utility function as indicated in Theorem 4 relies critically on the assumption of a continuum of consumers. If finitely many consumers are considered, it is expected that the solvability would require certain convexity properties of the utility functions.

B. Competitive equilibrium analysis

One fundamental issue in market implementation concerns the question whether or not the optimal social allocation can be sustained as a competitive equilibrium. If the answer is affirmative, it means that the market is able to operate in a decentralized manner under the guidance of the price signal.

For the proposed duration-deadline jointly differentiated energy service, a competitive equilibrium has to satisfy three conditions:

1) Consumers maximize their welfare. Consumer x selects $r(x)$ kW of a service $(h(x), d(x))$ in order to maximize his/her welfare, i.e., to solve the following optimization problem:
$$\max_{r,h,d} U(x, r, h, d) - r\pi_h^d.$$
2) Supplier maximizes revenue. The supplier sees a market of different energy services with associated prices $\mathcal{M} = \{h, d, \pi_h^d\}$. It then uses its available supply profile p to produce such services. The question facing the supplier is to decide how much it will produce for each service to maximize its revenue. Denote the amount of service $\{h, d\}$ to be produced by n_h^d, where $h \le d, d = \{T_1, T\}$. To ensure the feasibility of this production bundle, the implicit constraint on the adequacy of the supply profile must be taken into account. Adapting the adequacy condition here, the supplier revenue optimization problem can be mathematically stated as:
$$\max_{n_h^T, n_h^{T_1}} \sum_{h=1}^{T} n_h^T \pi_h^T + \sum_{h=1}^{T_1} n_h^{T_1} \pi_h^{T_1}$$
$$\text{s.t.} \sum_{j>k_1}^{T_1} p_j + \sum_{j>T_1+k_2}^{T} p_j - \sum_{j>k_1+k_2}^{T} \delta_j^T - \sum_{j>k_1}^{T_1} \delta_j^{T_1} \ge 0$$
$$\text{for } 0 \le k_1 \le T_1, 0 \le k_2 \le T - T_1,$$
where $\delta_j^T = \sum_{j\le h\le T} n_h^T$, and $\delta_j^{T_1} = \sum_{j \le h \le T_1} n_h^{T_1}$.
3) The market clears, i.e.,
$$n_h^T = \int_0^1 r(x)\mathbb{1}(h(x)=h, d(x)=T)dx, \text{ for } 1\le h\le T,$$
$$n_h^{T_1} = \int_0^1 r(x)\mathbb{1}(h(x)=h, d(x)=T_1)dx, \text{ for } 1\le h\le T_1.$$
In words, the amount of each service produced is equal to the amount demanded.

As described above, decisions are made in a decentralized manner. The individual consumers choose to maximize their own welfare while the supplier aims to maximize its revenue. When market clears, a competitive equilibrium is reached. If the resulting allocation of the services maximizes the social welfare as well, the competitive equilibrium is said to be efficient.

The next theorem establishes the existence of an efficient competitive equilibrium.

Theorem 5: There exists an efficient competitive equilibrium in a forward market for duration-deadline jointly differentiated energy services.

Proof: Let $Z^*(1)$ be the optimal solution to the social welfare problem (5) and $x \mapsto (r^*(x), h^*(x), d^*(x))$ be the corresponding optimal social allocation. Dualizing the social welfare problem with respect to (6) implies that there exists $\lambda_{k_1k_2} \ge 0, 0 \le k_1 \le T_1, 0 \le k_2 \le T - T_1$, such that $Z^*(1)$ is also the solution to the following optimization problem:

$$\max_{Z(1)\in G} z_{T_1+1\ 0}(1) - \sum_{k_1,k_2} \lambda_{k_1k_2} z_{k_1k_2}(1). \tag{7}$$

In addition, complementary slackness indicates that

$$\lambda_{k_1k_2}\left[z_{k_1k_2}^*(1) - \left(\sum_{j>k_1}^{T_1} p_j + \sum_{j>T_1+k_2}^{T} p_j\right)\right] = 0, \tag{8}$$

for $0 \le k_1 \le T_1, 0 \le k_2 \le T - T_1$. In view of (3), the term being maximized in (7) can be rewritten as

$$\int_0^1 \{U(x, r, h, d) - r\pi_h^d\}dx,$$

where

$$\pi_h^d = \sum_{k_1,k_2} \lambda_{k_1k_2}[h - (k_1+k_2))^+ \mathbb{1}(d=T) + (h-k_1)^+ \mathbb{1}(d=T_1)]. \tag{9}$$

Since $Z^*(1)$ maximizes this term, it follows that

$$(r^*(x), h^*(x), d^*(x)) = \arg\max_{r,h,d} U(x, r, h, d) - r\pi_h^d.$$

Interpreting the quantity π_h^d as in (9) as the price of service $\{h, d\}$, the above identity shows that under such prices, the consumption pattern chosen by the consumers to maximize their welfare is consistent with the optimal social allocation.

In order to show π_h^d as in (9) indeed gives the equilibrium price, it suffices to show that under such prices, the supplier revenue optimization leads to a production bundle n_h^d that clears the market. To this end, note that the supplier's revenue can be written as

$$\sum_{h=1}^{T} n_h^T \pi_h^T + \sum_{h=1}^{T_1} n_h^{T_1} \pi_h^{T_1} = \sum_{j=1}^{T} \delta_j^T (\pi_j^T - \pi_{j-1}^T) + \sum_{j=1}^{T_1} \delta_j^{T_1} (\pi_j^{T_1} - \pi_{j-1}^{T_1}),$$

where $\delta_j^T = \sum_{j \le h \le T} n_h^T, \delta_j^{T_1} = \sum_{j \le h \le T_1} n_h^{T_1}$. Substituting the expression of π_h^d into the above equation yields

$$\begin{aligned}\sum_{h=1}^{T} n_h^T \pi_h^T + \sum_{h=1}^{T_1} n_h^{T_1} \pi_h^{T_1} &= \sum_{j=1}^{T} \delta_j^T \sum_{k_1+k_2<j} \lambda_{k_1k_2} + \sum_{j=1}^{T_1} \delta_j^{T_1} \sum_{k_1<j} \lambda_{k_1k_2} \\ &= \sum_{k_1+k_2 \le T} \lambda_{k_1k_2} \left(\sum_{j>k_1+k_2}^{T} \delta_j^T + \sum_{j>k_1}^{T_1} \delta_j^{T_1} \right) \\ &\le \sum_{k_1+k_2 \le T} \lambda_{k_1k_2} \left(\sum_{j>k_1}^{T_1} p_j + \sum_{j>T_1+k_2}^{T} p_j \right) \\ &= \sum_{k_1+k_2 \le T} \lambda_{k_1k_2} z_{k_1k_2}^*(1),\end{aligned}$$

where the inequality follows from the adequacy constraint and the last equality follows from the complementary slackness as in (8). It can be easily verified that this upper bound on the revenue can be achieved when

$$n_h^T = \int_0^1 r(x) \mathbb{1}(h^*(x) = h, d^*(x) = T) dx, \text{ for } 1 \le h \le T,$$

$$n_h^{T_1} = \int_0^1 r(x) \mathbb{1}(h^*(x) = h, d^*(x) = T_1) dx, \text{ for } 1 \le h \le T_1,$$

and, thus, the market clears. This completes the proof. ∎

Remark 3: In view of the price π_h^d as in (9), two intriguing observations can be made. Firstly, for a fixed deadline d, π_h^d is non-decreasing in h with non-decreasing increments, i.e., the marginal price is increasing as h increases. Secondly, for a fixed duration h, we have $\pi_h^{T_1} \ge \pi_h^T$. In other words, one needs to pay more for the same duration of power if he/she requires a shorter deadline. Both observations coincide with the intuition that consumers with more flexility should get more compensated in such a power market.

V. Conclusion

In this paper, we investigate a duration-deadline jointly differentiated energy service. It gives a potential way of using load flexibilities to compensate the supply variations in a sustainable grid with high renewable penetrations. Specifically, we consider a group of flexible loads with each load requiring a constant power level for a specified duration before certain deadline. The flexibility resides in the assumption that a load is indifferent of the actual service time as long as the duration and deadline requirements are satisfied. We first approach the adequacy concern which amounts to solving a $(0, 1)$-matrix feasibility problem with certain fixed zeros. The adequacy condition is given explicitly in terms of the nonnegativity of a structure matrix. We then develop a forward market implementation of the proposed energy service and show the existence of an efficient competitive equilibrium.

For brevity of presentation, we concentrate on the scenario of two different deadlines in the current paper. This is a quite representative scenario and all the results presented here can be easily extended to general scenarios of multiple deadlines.

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