Feasible Channel Capacity Region for MIMO Stabilization via MIMO Communication

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Abstract— **In this paper, we study the stabilization of MIMO networked control systems over MIMO communication systems. Here the parallel additive white Gaussian noise subchannels in the MIMO transceiver are used to express the spatial freedom of communication. In addition, the number of subchannels in general can be greater than that of control inputs. The aim is to find the feasible capacity region rendering stabilization possible. We also wish to examine how to design the controller and transceiver jointly. A super-region and a sub-region for the feasible channel capacity region are obtained which are characterized in terms of two majorization relations, respectively. The results are demonstrated by numerical examples.**

I. Introduction

Networked control systems (NCSs) have been greatly investigated due to their wide applications in many areas such as wireless sensor networks, multi-agent systems, robotics, unmanned aerial vehicles, distributed computing, etc. They are feedback systems in which the controller and the plant are often geographically separated, requiring communication systems in between for information exchange. The communication systems can be wired computer networks or wireless communication networks, transmitting the measurement and control signals in a shared or dedicated, collaborative or competitive, centralized or distributed manner. These communication channels have brought great advantages, including low cost, real-time and flexible remote control and estimation [1], [2], and some side effects, including quantization errors [3], [4], packet dropping [5], [6], time delay [7], [8], low signal to noise ratio (SNR) [9], etc.

The study of NCSs involves tremendous interactions between control theory, information theory, and communication theory. It has been well recognized that the classical Shannon information theory is in general not enough to characterize the real-time information constraints in a feedback loop. The research on how to modify the classical information theory in the context of networked control is still in the beginning stage. Several preliminary studies have been reported in the literature, attempting to define suitable capacity notions for communication channels in a feedback system. For instance, the work in [10] introduces the notion of anytime capacity, while the works in [11], [12] suggest the potential of defining capacity notions from a deterministic point of view. On the other side of the story, there have been continuing efforts in defining an entropy concept for a dynamic system to capture its complexity, ergodicity, information content, or expansion rate. Of particular interest to networked control is the metric-theoretical topological entropy defined in [13]. When applied to discrete-time linear time-invariant systems, the topological entropy is simply expressed as the logarithm of the absolute products of unstable poles of the open-loop system.

Through years of study, researchers have archived a good understanding of the interplay between the communication quality and system dynamics for the control of a single-input system under different information constraints. See [14], [15] for data rate constraint, [3], [6], [16] for quantization, [9] for signal-to-noise ratio constraint, and [17] for fading, etc. All these studies converge to a unified understanding: The single-input networked stabilization can be accomplished if and only if the channel capacity is greater than the topological entropy of the open-loop plant. Such fundamental limitation, on the other hand, justifies the use of topological entropy as a measure of the degree of instability of a single-input linear system, as suggested in [14], [15], [18]. Clearly, the more unstable a system is, the more communication resource it requires for stabilization.

Efforts have also been further devoted to examine whether or not such fundamental limitation on information constraints can be extended to multi-input networked stabilization. See for instance [11], [19]–[22]. In particular, the idea of channel resource allocation has been exploited in [11], [22] in order to facilitate the multi-input networked stabilization. The key philosophy is to assume that the channel capacities can be allocated subject to an overall capacity constraint, leading to a channel/controller co-design problem. It has been shown therein that a multi-input NCS can be stabilized if and only if the total channel capacity is greater than the topological entropy of the open-loop plant, consistent with the result in the single-input case.

Note that the channel resource allocation as in [11], [22] can be interpreted in a more intuitive way. One can consider the control inputs as the demand side of the communication resource. What the channel resource allocation does is simply tailoring the supplies from different channels to match the demands from different inputs. Then a question arises: What if the channel capacities are fixed a priori and not allocatable? Can we exploit some other design freedom to achieve the balance of demand and supply?

Motivated by this concern, we have recently proposed an alternative scheme in [23] which does the exact opposite: shaping the demands to meet the fixed supplies. The demand shaping is made possible thanks to the coding mechanism in

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a MIMO transceiver that is utilized for information exchange between the controller and the plant. While the continuous-time NCSs are considered in [23], we in this paper proceed to investigate the case of discrete-time NCSs. Despite some parallel results as one would expect, we also observe some interesting phenomenon that is different from the continuous-time case. A super-region and a sub-region for the feasible channel capacity region rendering stabilization possible are obtained which are characterized by two majorization relations.

Notation: The super-scripts $(\cdot)'$ and $(\cdot)^*$ denote transpose and complex conjugate transpose, respectively. The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$ represent the set of non-negative and positive numbers, respectively. Here $d(X)$ and $\lambda(X)$ denote the vectors constituted by the diagonal elements and eigenvalues of a Hermitian matrix X in decreasing order, respectively. The ith diagonal element of a matrix is denoted by $\{\cdot\}_{ii}$. The relation $\leq$ between two vectors denotes component-wise less than. The logarithm is assumed to have base 2; hence, the unit of channel capacity is bits/transmission. Other notation is more or less standard, and will be made clear as we proceed.

II. Problem Formulation

Consider a discrete-time linear time-invariant system described by the state model

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0,$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$. The plant is denoted by $[A|B]$ for simplicity. Assume that $[A|B]$ is unstable but stabilizable. Also assume that the state vector $x(k)$ is available for feedback control. A static state feedback controller is employed with the state feedback gain $F \in \mathbb{R}^{m\times n}$. The control signal $v(k) = Fx(k)$ is sent through a communication network to the plant actuators. The closed-loop feedback system is now depicted in Fig. 1 (a).

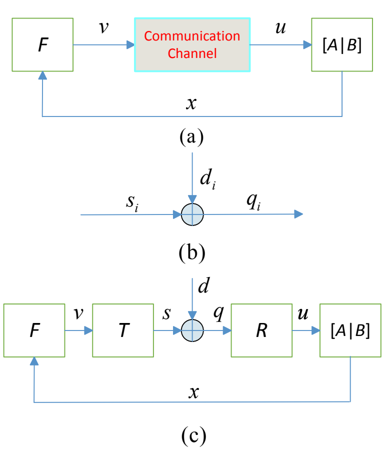

Fig. 1. (a) State feedback via a communication channel. (b) The AWGN channel model. (c) NCS – control over a communication system.

How to model the communication channels is a big issue in networked control. As a starting point, we consider the parallel AWGN channels, as shown in Fig. 1 (b). Let l denote the total number of subchannels, and assume that $l \geq m$, i.e., the number of subchannels can be greater than the number of control inputs. The input signal of the channel is $s \in \mathbb{R}^l$, and the output signal $q \in \mathbb{R}^l$ is given by $q = s + d$, where d is the Gaussian noise vector, with zero mean and covariance $\Sigma^2 = \text{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_l^2\}$. To mitigate the distortion caused by the channel noise, a transmitter $T \in \mathbb{R}^{l\times m}$ and a receiver $R \in \mathbb{R}^{m\times l}$ are to be designed subject to a mild constraint:

$$RT = I.$$

Thus, the control signal received at the plant input can be expressed as

$$u(k) = v(k) + Rd(k).$$

The integrated networked control system, consisting of two MIMO sub-systems, MIMO control and MIMO communication, and three physical objects, controller, transceiver, and plant, is depicted in Fig. 1 (c).

When the whole control-communication system is operated in closed-loop mode, the transfer function from the noise d to the signal s is the complementary sensitivity function

$$T(z) = TF(zI - A - BRTF)^{-1}BR.$$

The power spectrum of s_i is given by $\{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\}_{ii}$ and the mean power of s_i is

$$\mathcal{P}_i = \frac{1}{2\pi}\int_0^{2\pi} \left\{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\right\}_{ii} d\omega.$$

Thus the SNR of the ith channel can be represented by

$$\begin{aligned}\mathcal{S}_i &= \frac{1}{2\pi}\frac{\int_0^{2\pi}\left\{T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\right\}_{ii} d\omega}{\sigma_i^2}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left\{\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1}\right\}_{ii} d\omega.\end{aligned}$$

Consequently, the corresponding ith channel capacity, from the Shannon information theory[24], can be expressed as

$$\mathfrak{C}_i = \frac{1}{2}\log\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1} d\omega\right\}_{ii},$$

yielding the total channel capacity

$$\begin{aligned}\mathfrak{C}_{\text{total}} &= \mathfrak{C}_1 + \cdots + \mathfrak{C}_l\\ &= \frac{1}{2}\log\prod_{i=1}^{l}\left\{I + \frac{1}{2\pi}\int_0^{2\pi}\Sigma^{-1}T(e^{j\omega})\Sigma^2 T(e^{j\omega})^*\Sigma^{-1} d\omega\right\}_{ii}.\end{aligned}$$

Here, we are interested in not only the total channel capacity $\mathcal{C}_{\text{total}}$ required for stabilization, but also the individual channel capacity and potential trade-offs between the individual channel capacities. Here we borrow the notion of channel capacity region from network information theory [25]. Let $\mathcal{S} = \begin{bmatrix}\mathcal{S}_1 & \ldots & \mathcal{S}_l\end{bmatrix}'$ denote the channel SNR vector and $\mathfrak{C} = \begin{bmatrix}\mathfrak{C}_1 & \cdots & \mathfrak{C}_l\end{bmatrix}' \in \mathbb{R}^l$ denote the channel capacity vector. The set of all channel capacity vectors that render networked stabilization possible is referred to as the feasible channel capacity region. The ultimate objective is to find a full characterization of such feasible channel capacity region. As a starting point, a super-region and a sub-region for the feasible channel capacity region are obtained which are characterized by two majorization type relations.

Besides the characterization of feasible channel capacity region, we are also interested in the co-design of the controller and transceiver. Given a feasible capacity vector, how to design the controller and the transceiver cooperatively so as to stabilize the system? This will be answered as we go along.

III. Preliminaries

A. Mahler measure and topological entropy

The Mahler measure of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $M(A)$, is defined as the product of the absolute value of its unstable eigenvalues, i.e., $M(A) = \prod_{i=1}^{n} \max\{1, |\lambda_i(A)|\}$. The topological entropy [26] of A, denoted by $H(A)$, is simply the logarithm of $M(A)$, i.e., $H(A) = \log M(A)$. Clearly, we have $M(A) \geq 1$ and $H(A) \geq 0$. These two concepts have been studied in mathematics and dynamical system theory for a long time. Recently it has received considerable attention in the control literature in the study of information theoretical limitations in networked control [14], [15], [18], [27].

B. Majorization theory

For any vector $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}' \in \mathbb{R}^n$, let $x^{\downarrow} = \begin{bmatrix} x_{[1]} & \dots & x_{[n]} \end{bmatrix}'$ denote the rearranged version of x such that the elements are in non-increasing order. Similarly, let $x^{\uparrow} = \begin{bmatrix} x_{(1)} & \dots & x_{(n)} \end{bmatrix}'$ denote the rearranged version of x such that the elements are in non-decreasing order.

Definition 1: For two vectors $x, y \in \mathbb{R}^n$, we say that x is majorized by y, denoted by $x \preccurlyeq y$, if

$$\begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & \text{for } k = 1, \dots, n-1, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{cases} \tag{1}$$

or equivalently,

$$\begin{cases} \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, & \text{for } k = 1, \dots, n-1, \\ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}. \end{cases} \tag{2}$$

When the last equality in (1) is replaced by inequality $\leq$, x is said to be weakly majorized by y from below, denoted by $x \preccurlyeq_w y$. Similarly, if we replace the last equality in (2) by inequality $\geq$, x is said to be weakly majorized by y from above, denoted by $x \preccurlyeq^w y$.

Majorization, defined as above, is based on partial sums. We also need the so-called log-majorization for later use which is defined in a multiplicative manner.

Definition 2: For $x, y \in \mathbb{R}^n_+$, we say that x is log-majorized by y, denoted by $x \preccurlyeq_{\log} y$, if

$$\begin{cases} \prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, & \text{for } k = 1, \dots, n-1, \\ \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}, \end{cases} \tag{3}$$

or equivalently,

$$\begin{cases} \prod_{i=1}^{k} x_{(i)} \geq \prod_{i=1}^{k} y_{(i)}, & \text{for } k = 1, \dots, n-1, \\ \prod_{i=1}^{n} x_{(i)} = \prod_{i=1}^{n} y_{(i)}. \end{cases} \tag{4}$$

Again, when the last equality in (3) is replaced by inequality $\leq$, x is said to be weakly log-majorized by y from below, denoted by $x \preccurlyeq_{w\log} y$. Similarly, when the last inequality in (4) is replaced by inequality $\geq$, x is said to be weakly log-majorized by y from above, denoted by $x \preccurlyeq^{w\log} y$. In particular, if $x, y \in \mathbb{R}^n_{++}$, then $x \preccurlyeq_{w\log} y$ and $x \preccurlyeq^{w\log} y$ are equivalent to $\log x \preccurlyeq_w \log y$, and $\log x \preccurlyeq^w \log y$, respectively.

The following lemmas would be very useful in later developments.

Lemma 1 ([28]): There exists a real symmetric matrix X with eigenvalues $\lambda = [\lambda_1 \ \cdots \ \lambda_n]'$, and diagonal elements $d = [d_1 \ \cdots \ d_n]'$, if and only if $d \preccurlyeq \lambda$.

Lemma 2 ([28]): Let X be an $n \times n$ positive semi-definite Hermitian matrix with diagonal elements $d = [d_1 \ \cdots \ d_n]'$ and eigenvalues $\lambda = [\lambda_1 \ \cdots \ \lambda_n]'$, then we have $d \preccurlyeq^{w\log} \lambda$.

C. Cyclic decomposition

Let A be an $n \times n$ real matrix. The minimal polynomial of A is the monic polynomial $\alpha(\lambda)$ of least degree such that $\alpha(A) = 0$. It is unique, and is a factor of the characteristic polynomial of A. Generally speaking, the degree of the minimal polynomial is less than or equal to n. If the degree of the minimal polynomial is indeed n, we say the matrix is cyclic. In this case, the minimal polynomial coincides with the characteristic polynomial.

For a given real square matrix, one can always carry out a cyclic decomposition. When applied to linear dynamical system, this leads to a cyclic decomposition of the system, as shown in the following lemma.

Lemma 3 ([29]): Given a stabilizable linear system $[A|B]$ with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let the monic polynomial of A be α. Then there exists a nonsingular matrix $V \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{m\times m}$ such that $[A|B]$ can be transformed into

$$\begin{aligned} &[V^{-1}AV | V^{-1}BQ] \\ &= \left[\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix} \middle| \begin{bmatrix} b_1 & * & \cdots & * & * \\ 0 & b_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & * & * \\ 0 & \cdots & 0 & b_k & * \end{bmatrix} \right], \end{aligned}$$

where $A_i, i = 1, 2, \dots, k$, are cyclic with minimal polynomials $\alpha_i(\lambda)$, such that $\alpha_1(\lambda) = \alpha(\lambda)$ and $\alpha_{i+1}(\lambda)|\alpha_i(\lambda)$

for $i = 1, 2, \ldots, k-1$. Moreover, the cyclic subsystems $[A_i|b_i], i = 1, 2, \ldots, k$, are stabilizable.

An interesting implication from the above lemma is that $\text{Spec}(A_1) \supset \cdots \supset \text{Spec}(A_k)$ and, thus, $H(A_1) \geq \cdots \geq H(A_k)$, where $\text{Spec}(A_i)$ denotes the spectrum of A_i.

IV. A Super-region and A Sub-region of Feasible Channel Capacity Region

In this section, we will derive a super-region and a sub-region of the feasible channel capacity region, respectively.

Firstly, we will present the super-region. Denote

$$\mathcal{H}(A) = \begin{bmatrix} H(A_1) & \cdots & H(A_k) & 0 & \cdots & 0 \end{bmatrix}' \in \mathbb{R}^l,$$
$$\mathcal{M}(A) = \begin{bmatrix} M(A_1) & \cdots & M(A_k) & 1 & \cdots & 1 \end{bmatrix}' \in \mathbb{R}^l.$$

Hereinafter, by applying a scalar function f to a vector $x \in \mathbb{R}^n$, we mean

$$f(x) = [f(x_1) \ \cdots \ f(x_n)]'.$$

The following theorem gives a super-region of the feasible channel capacity region. The proof is omitted here due to the page limit.

Theorem 1: If $\mathfrak{C}$ lies in the feasible capacity region, there holds $\mathfrak{C} \preccurlyeq^w \mathcal{H}(A)$, or equivalently, $1 + \mathcal{S} \preccurlyeq^{w \log} \mathcal{M}(A)^2$.

A corollary follows directly from Theorem 1.

Corollary 1: If $\mathfrak{C}$ lies in the feasible capacity region, then $\mathfrak{C}_{\text{total}} \geq H(A)$.

In what follows, we will focus on the sub-region. The result is stated in the following theorem. The proof is constructive and gives a guideline for the joint design of the controller and the transceiver.

Theorem 2: A sub-region of the feasible capacity region for design is $2^{2\bar{\mathfrak{C}}} \preccurlyeq^\omega 2^{2\mathcal{H}(A)}$, or equivalently, $\bar{\mathcal{S}} \preccurlyeq^\omega \mathcal{M}(A)^2$–1.

Proof: For brevity, assume that all the eigenvalues of A lie in outside of the unit circle. This assumption can be removed following the same arguments as in [9], [11].

We need to design a controller F, a transmitter T, and a receiver R so that the NCS is stabilized, simultaneously requiring that the actual SNR $\mathcal{S}$ satifies $\mathcal{S} \leq \bar{\mathcal{S}}$.

From [28], we know that if $\bar{\mathcal{S}} \preccurlyeq^\omega \mathcal{M}(A)^2 - 1$, then there exists some vector $w \leq \bar{\mathcal{S}}, w \preccurlyeq \mathcal{M}(A)^2 - 1$. Then let $\mathcal{S} = w$; hence we use $\mathcal{S}$ afterwards.

Furthermore, without loss of generality, assume that the system $[A|B]$ has been decomposed into the cyclic canonical form, in which each subsystem $[A_i|b_i], i = 1, \ldots, k$ is controllable with state dimension $n_i, \sum_{i=1}^n n_i = n$.

For each subsystem $[A_i|b_i]$, which is a single input system and we can design a stabilizing state feedback controller f_i such that $\|T_i(z)\|_2^2 = M(A_i)^2 - 1$ [9], with the transfer function,

$$T_i(z) = f_i(zI - A_i - b_i f_i)^{-1} b_i.$$

The controller f_i is an $\mathcal{H}_2$ controller. Then we let the controller F be

$$F = \begin{bmatrix} f \\ 0_{(m-k)\times n} \end{bmatrix},$$

where $f = \text{diag}\{f_1, f_2, \ldots, f_k\}$.

Then we can design the transmitter as

$$T = \Sigma U D^{-1},$$

and the receiver as

$$R = D U' \Sigma^{-1},$$

where Σ is the given noise variance matrix, $U \in \mathbb{R}^{l \times m}$ is an isometry matrix, i.e., $U'U = I_m$, and $D = \text{diag}\{1, \epsilon, \ldots, \epsilon^{m-1}\}$ is a scaling matrix. The matrix U needs to be designed and the small scaling factor ϵ needs to be chosen so that $\mathcal{S} \leq \bar{\mathcal{S}}$ is satisfied.

Let matrix $V = \text{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{k-1} I_{n_k}\}$. Then the transfer function from noise d to channel input signal s can be written as

$$\begin{aligned} T(z) &= TF(zI - A - BF)^{-1}BR \\ &= \Sigma U D^{-1} F (zI - A - BF)^{-1} B D U' \Sigma^{-1} \\ &= \Sigma U D^{-1} F V (zI - V^{-1} A V - \\ &\qquad V^{-1} B D D^{-1} F V)^{-1} V^{-1} B D U' \Sigma^{-1} \\ &= \Sigma U \hat{F} (zI - \hat{A} - \hat{B}\hat{F})^{-1} \hat{B} U' \Sigma^{-1}, \end{aligned}$$

where

$$\hat{F} = D^{-1} F V = F,$$
$$\hat{A} = V^{-1} A V = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix},$$
$$\hat{B} = V^{-1} B D = \begin{bmatrix} b_1 & o(\epsilon) & \cdots & o(\epsilon) & o(\epsilon) \\ 0 & b_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) & o(\epsilon) \\ 0 & \cdots & 0 & b_k & o(\epsilon) \end{bmatrix},$$

and $o(\epsilon)/\epsilon$ approaches to a finite constant as $\epsilon \to 0$. Then we have

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \Sigma^{-1} T(e^{j\omega}) \Sigma^2 T(e^{j\omega})^* \Sigma^{-1} d\omega \\ &= U \hat{F} (zI - \hat{A} - \hat{B}\hat{F})^{-1} \hat{B} \hat{B}' (zI - \hat{A} - \hat{B}\hat{F})^{-1^*} \hat{F}' U' \\ &= U (\text{diag}\{\|T_1(z)\|_2^2, \ldots, \|T_k(z)\|_2^2, 0, \ldots, 0\}) U' + o(\epsilon) U U' \\ &= U \big(\text{diag}\{M(A_1)^2 - 1, \ldots, M(A_k)^2 - 1, 0, \ldots, 0\}\big) U' \\ &\quad + o(\epsilon) U U'. \end{aligned} \tag{5}$$

Since

$$\mathcal{S} \preccurlyeq \mathcal{M}(A)^2 - 1,$$

by lemma 1, there exists an isometry matrix U such that

$$\mathcal{S}_i = \{U(\text{diag}\{M(A_1)^2 - 1, \ldots, M(A_k)^2 - 1, 0, \ldots, 0\}) U'\}_{ii}$$

for $i = 1, \ldots, l$. Furthermore, let the ϵ in (5) be sufficiently small, then the result can be obtained. We can use the same approach for the power vector. And the structure of the transceiver can be shown as in Fig. 2.

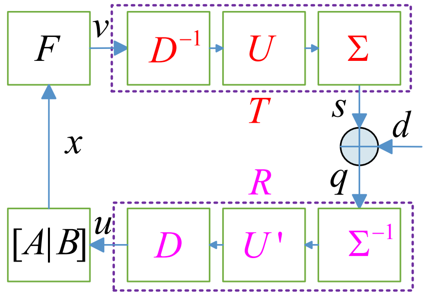

Fig. 2. Controller and transceiver structure.

Using the relations, $1+\bar{\mathcal{S}} = 2^{2\bar{\mathfrak{C}}}$ and $\mathcal{M}(A)^2 = 2^{2\mathcal{H}(A)}$, we have $1+\bar{\mathcal{S}} \preccurlyeq^{\omega} \mathcal{M}(A)^2$, that is, $\bar{\mathcal{S}} \preccurlyeq^{\omega} \mathcal{M}(A)^2 - 1$. Then from above, we can design the controller and receiver so that the actual SNR vector $\mathcal{S} \le \bar{\mathcal{S}}$, implying $\mathfrak{C} \le \bar{\mathfrak{C}}$ with the same controller and transceiver. This finishes the proof. ∎

V. Numerical Example

An example is worked out in this section for illustration. Consider an unstable system $[A|B]$ with

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

There are two cyclic subsystems:

$$[A_1|b_1] = \left[\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right], \quad [A_2|b_2] = [\ 2 \mid 1\ ].$$

The Mahler measure is given by $\mathcal{M}(A) = \begin{bmatrix} 8 & 2 \end{bmatrix}'$, and the topological entropy is given by $\mathcal{H}(A) = \begin{bmatrix} 3 & 1 \end{bmatrix}'$.

Consider the case when there are 6 subchannels with channel capacity vector being

$$\bar{\mathfrak{C}} = \begin{bmatrix} 2.01 & 2.01 & 2.01 & 2.01 & 1.01 & 1.01 \end{bmatrix}'.$$

Note that there are no subchannels with capacity greater than $H(A_1)$ and, thus, no single channel has enough capacity to stable $[A_1|b_1]$. Nevertheless, one can verify that $2^{2\bar{\mathfrak{C}}} \preccurlyeq^{w} \mathcal{M}(A)^2$, where $\mathcal{M}(A) = \begin{bmatrix} 8 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}'$ is the augmented Mahler measure. Then we can jointly design the controller and transceiver to stabilize the NCS.

To obtain the controller, we need to solve the $\mathcal{H}_2$ optimal control problem for each cyclic subsystem. The optimal $\mathcal{H}_2$ controller for $[A_1|b_1]$ is $f_1 = \begin{bmatrix} -6.5625 & 1.3125 \end{bmatrix}'$ and that for $[A_2|b_2]$ is $f_2 = -0.5$. Then design

$$F = \mathrm{diag}\{f_1, f_2\} = \begin{bmatrix} -6.5625 & 1.3125 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}.$$

One can verify that the closed-loop poles, i.e., the eigenvalues of $A+BF$, are given by $\{0.25, 0.5, 0.5\}$, which are exactly the mirror images of the open-loop poles with respect to the unit circle.

Regarding the transceiver design, we choose the isometry matrix U and the scaling matrix D:

$$U = \begin{bmatrix} 0.4851 & 0.2425 \\ 0.4472 & -0.8944 \\ 0.4851 & 0.2425 \\ 0.4851 & 0.2425 \\ 0.2169 & 0.1085 \\ 0.2169 & 0.1085 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The actual channel capacity is now given by

$$\mathfrak{C} = \begin{bmatrix} 2.0025 & 2.0009 & 2.0025 & 2.0025 & 1.0018 & 1.0018 \end{bmatrix}',$$

which satisfies the constraint.

VI. Conclusion

In this paper, we investigate the networked stabilization of a multi-input system over a MIMO communication system. The MIMO communication system consists of a cascade of a transmitter, a collection of subchannels, and a receiver. The number of subchannels in the transceiver can be greater than the number of the control inputs. The transmitter and receiver can be freely designed subject to a mild constraint, leading to a joint design problem of the controller and the transceiver.

We wish to reveal a fundamental limitation on information constraints required for multi-input networked stabilization. At this stage, a super-region and a sub-region of the feasible channel capacity region are obtained which are characterized in terms of two majorization type relations, respectively. The majorization relations connect the subchannel capacity vector with the system topological entropy vector. To some extent, this also suggests the use of the topological entropy vector as a measure of the structured instability in a multi-input linear system. A full characterization of the feasible channel capacity region rendering stabilization possible is still under our current investigation.

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