LQG Control of LTI Systems with Random Input and Output Gains

Wei Chen, Jianying Zheng, and Li Qiu

***Abstract*— In this paper, the Linear Quadratic Gaussian (LQG) control of linear time invariant (LTI) systems with random input and output gains is studied. One main novelty of this work is that we study the problem under the framework of channel/controller co-design which allows the control designer to have the additional freedom to design the communication channels. With the channel/controller co-design, the optimal control problem studied is feasible if and only if the system is mean-square stabilizable and detectable. Moreover, we show that the separation principle partially holds under the TCP-like protocols. The optimal controller is an estimated state feedback, combining the optimal state feedback design and the optimal state estimation design. However, there exists certain asymmetry. The optimal state feedback gain does not depend on the estimator design, while the optimal estimator does depend on the optimal state feedback gain.**

Keywords: Networked control system, LQG control, stochastic systems, channel/controller co-design, channel resource allocation.

I. Introduction

The LTI systems with random gains, which are a specific type of stochastic control systems, have attracted much attention recently. They have wide range of applications in many areas, including networked control systems (NCSs) [7], [16], [13], economic stability [10], and financial engineering [19], etc. One can refer to [5], [17] and the references therein for a general study of the stochastic control systems.

What we concern in this work is the LQG control of LTI systems with random gains imposed on the control input and measurement output, respectively. Better results can be obtained for this specific type of systems compared to the study on the general stochastic systems. Below we partially review some results that are pertinent to our work in this paper, mainly in the context of NCSs. For convenience, we name the channels through which the plant output is sent to the controller as the output channels and the channels through which the controller output is sent to the plant as the input channels. The work in [7] considers the LQG control with packet dropping in the output channels. There it is shown that the separation principle holds under the TCP-like protocols. In [13], [8], the LQG control of a multi-input-multi-output (MIMO) system with a single packet dropping input channel and a single packet dropping output channel is considered. The authors point out that under the TCP-like protocols, the optimal LQG control is a linear function of the estimated state and depends on the packet dropping probabilities.

Researchers have also studied the LQG control over multiple parallel communication channels for MIMO NCSs. One such example can be seen in [4]. The objective there is finding the optimal controller for a given set of packet dropping probabilities. The separation principle is shown to hold under the TCP-like protocols and the optimal control law is obtained in the finite-horizon case. For the infinite-horizon case, a sufficient condition on the stability of the closed-loop system is given by a set of linear matrix inequalities (LMIs).

Inspired by the above results, we further study the LQG control of LTI systems with random input and output gains in this paper. Different from the setting in [4], we put the problem under the framework of channel/controller co-design which is one main novelty of this paper. We assume that the controller designer also has the freedom to participate in the channel design. Due to this additional design freedom, the objective now becomes to simultaneously design the controller and channels such that the cost function is minimized. By this channel/controller co-design, the problem can be nicely solved, as elaborated in the rest of this paper.

The framework of channel/controller co-design is first proposed in [6], which studies the stabilization of multi-input NCSs with the signal-to-error ratio (SER) channel model. The work in [6] is extended in [11] where a more complete study is carried out on stabilization of multi-input NCSs for three different channel models. Several other works [3], [15], [16], [18] have been carried out following this framework. In particular, the work in [18], which is most related to this paper, investigates the Linear Quadratic Regulator (LQR) problem of LTI systems with random input gains. By channel/controller co-design, the LQR problem is shown to be solvable if and only if the overall input channel capacity is greater than the topological entropy of the plant.

The remainder of this paper is organized as follows. The problem is formulated and some concepts, especially the channel/controller co-design framework are introduced in Section II. The mean-square stabilizability and detectability are discussed in Section III. The optimal estimator is derived in Section IV. Section V first solves the finite-horizon LQG control problem by dynamic programming and then investigates the convergence issue in the infinite-horizon case under the framework of channel/controller co-design. Conclusions follow in Section VI. Some proofs in this paper are presented in the Appendix.

Most notations in this paper are more or less standard and will be made clear as we proceed. The symbol $\odot$ means

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Hadamard product. The identity under Hadamard product, denoted by E, is a matrix with all elements equal to 1.

II. Problem Formulation

The setup of the feedback system studied in this work is shown in Fig. 1.

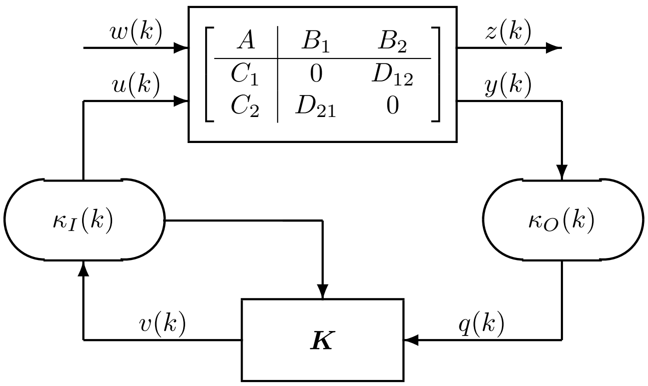

Fig. 1. LTI systems with random input and output gains.

Consider the following general LTI plant:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 w(k) + B_2 u(k), \\ z(k) &= C_1 x(k) + D_{12} u(k), \\ y(k) &= C_2 x(k) + D_{21} w(k), \end{aligned}$$

where $x(k) \in \mathbb{R}^n$ is the plant state, $w(k) \in \mathbb{R}^{\tilde{m}}$ is a zero-mean white noise with covariance $\boldsymbol{E}[w(k)w(l)'] = I\delta_{kl}$, $u(k) \in \mathbb{R}^m$ is the control input, $z(k) \in \mathbb{R}^{\tilde{p}}$ is the performance output and $y(k) \in \mathbb{R}^p$ is the measurement output. Let $x(0)$ be Gaussian with zero mean and variance Y_0. We denote the plant by $\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$ for simplicity. Assume that A is unstable, $[A|B_2]$ is stabilizable and $\left[\begin{array}{c} A \\ \hline C_2 \end{array}\right]$ is detectable. Also assume that $\begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}$ has full column rank and $\begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ has full row rank. Different from the setup of a traditional output feedback control system, two random gain matrices $\kappa_I(k), \kappa_O(k)$ are imposed on the control input and measurement output, respectively, i.e., $u(k) = \kappa_I(k)v(k)$, $q(k) = \kappa_O(k)y(k)$. In the sequel, we use subscript I to refer to the input channels and subscript O to refer to the output channels. In many applications, especially those in NCSs and distributed systems, very often the actuators are located far from each other. Thus, here we assume that each componet of the control signal is sent through an independent communication channel. Motivated by the wireless transmission largely used in practice, we model $\kappa_I(k) = \mathrm{diag}\{\kappa_{I_1}(k), \kappa_{I_2}(k), \dots, \kappa_{I_m}(k)\}$, where $\kappa_{I_i}(k), i = 1, 2, \dots, m$ are i.i.d white noise processes with means μ_{I_i} and variances $\sigma_{I_i}^2$, respectively. Similarly, $\kappa_O(k)$ is modeld as $\kappa_O(k) = \mathrm{diag}\{\kappa_{O_1}(k), \kappa_{O_2}(k), \dots, \kappa_{O_p}(k)\}$, where $\kappa_{O_j}(k), i = 1, 2, \dots, p$ are i.i.d white noise processes with means μ_{O_j} and variances $\sigma_{O_j}^2$, respectively. Note that there exists feedback of the outcome of $\kappa_I(k)$ to the controller $\boldsymbol{K}$. This is reasonable and can be achieved by some TCP-like communication protocols. However, we do not require that the controller knows the outcome of $\kappa_O(k)$, which is different from the setting in most current literature [14], [13], [4].

Some notations with respect to the input and output channels are defined below. First, we define the signal-to-noise ratio in the ith input channel, denoted as SNR_{I_i}, to be the ratio $\frac{\mu_{I_i}}{\sigma_{I_i}}$. Denote

$$\begin{aligned} M_I &\triangleq \mathrm{diag}\{\mu_{I_1}, \mu_{I_2}, \dots, \mu_{I_m}\}, \\ \Sigma_I^2 &\triangleq \mathrm{diag}\{\sigma_{I_1}^2, \sigma_{I_2}^2, \dots, \sigma_{I_m}^2\}, \\ W_I &\triangleq E + M_I^{-2}\Sigma_I^2 \\ &= \begin{bmatrix} 1+\mathrm{SNR}_{I_1}^{-2} & 1 & \cdots & 1 \\ 1 & 1+\mathrm{SNR}_{I_2}^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+\mathrm{SNR}_{I_m}^{-2} \end{bmatrix}. \end{aligned}$$

The mean-square capacity of the ith input channel is [15]:

$$\mathfrak{C}_{I_i} \triangleq \frac{1}{2}\log(1+\frac{\mu_{I_i}^2}{\sigma_{I_i}^2}) = \frac{1}{2}\log(1+\mathrm{SNR}_{I_i}^2).$$

The overall mean-square capacity of the input channels is given by $\mathfrak{C}_I = \sum_{i=1}^m \mathfrak{C}_{I_i}$. For the output channels, we can correspondingly define the signal-to-noise ratio $\mathrm{SNR}_{O_j}, j = 1, 2, \dots, p$, the matrices M_O, Σ_O^2, W_O, the mean-square capacity of the jth output channel $\mathfrak{C}_{O_j}$ and the overall mean-square capacity of the output channels $\mathfrak{C}_O$.

We study the LQG control problem with random gains. The purpose is to design a dynamic controller $\boldsymbol{K}$ to minimize

$$\begin{aligned} &\lim_{k\to\infty} \boldsymbol{E}[z(k)'z(k)] \\ &= \lim_{k\to\infty} \boldsymbol{E}\left[\begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix}\right], \end{aligned}$$

where $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \triangleq \begin{bmatrix} C_1' \\ D_{12}' \end{bmatrix} \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$. Notice that here $\boldsymbol{K}$ is not an LTI system with input $q(k)$ and output $v(k)$. Instead, it is a linear parameter-varying (LPV) system depending on $\kappa_I(k)$. One traditional way to deal with this problem is to fix the channel capacities $\mathfrak{C}_{I_i}, \mathfrak{C}_{O_j}$ a priori and then find the optimal $\boldsymbol{K}$ to minimize $\lim_{k\to\infty} \boldsymbol{E}[z(k)'z(k)]$. However, under this formulation, the problem is not always feasible for any given set of channel capacities.

To tackle this difficulty, the channel/controller co-design framework provides a significant insight, which is the main novelty of this work. The channel capacities are often closely related to certain physical resource available, e.g., transmission power or communication bandwidth, which can be allocated among different channels. Considering this, we assume that the individual input channel capacities $\mathfrak{C}_{I_i}$ and output channel capacities $\mathfrak{C}_{O_j}$ are not given a priori. Instead, they can be designed, or allocated under the overall constraints on $\mathfrak{C}_I$ and $\mathfrak{C}_O$. The allocation of the overall capacity to the individual channels, called channel resource allocation, can be formally given by two probability vectors

$$\pi_I = \begin{bmatrix}\pi_{I_1} & \pi_{I_2} & \dots & \pi_{I_m}\end{bmatrix}',$$
$$\pi_O = \begin{bmatrix}\pi_{O_1} & \pi_{O_2} & \dots & \pi_{O_p}\end{bmatrix}',$$

where $0 \leq \pi_{I_i} \leq 1$, $\sum_{i=1}^m \pi_{I_i} = 1$, $0 \leq \pi_{O_j} \leq 1$, $\sum_{j=1}^p \pi_{O_j} = 1$, such that $\mathfrak{C}_{I_i} = \pi_{I_i}\mathfrak{C}_I$, $\mathfrak{C}_{O_j} = \pi_{O_j}\mathfrak{C}_O$. With the channel/controller co-design, our problem becomes to simultaneously design the probability vectors π_I, π_O and the optimal controller $\boldsymbol{K}$ to minimize $\lim_{k\to\infty} \boldsymbol{E}[z(k)'z(k)]$. Thanks to the additional design freedom given by the channel resource allocation, the problem can be nicely solved. It is shown that the separation principle partially holds under the TCP-like protocols. On one hand, the optimal controller is still an estimated state feedback, combining the optimal state feedback design and the optimal state estimation design. On the other hand, although the optimal state feedback gain does not depend on the estimator design, the optimal estimator does depend on the optimal state feedback gain.

Before proceeding, recall that the topological entropy [2] of a matrix $A \in \mathbb{R}^{n\times n}$ is given by $h(A) = \sum_{|\lambda_i|>1} \log|\lambda_i|$, where λ_i are the eigenvalues of A.

III. Mean-square Stabilizability and Detectability

In this section, the concepts of mean-square stabilizability and detectability are introduced. First, we consider the state feedback stabilization of $[A|B_2]$ with the random gain $\kappa_I(k)$ imposed on the input. Given an overall input channel capacity $\mathfrak{C}_I$, we want to design an allocation π_I and a state feedback gain F such that the closed-loop networked system

$$x(k+1) = (A + B_2\kappa_I(k)F)x(k), \tag{1}$$

is mean-square stable, i.e., for any initial state $x(0)$, $V(k) \triangleq \boldsymbol{E}[x(k)x'(k)]$ is well-defined for any $k > 0$ and $\lim_{k\to\infty} V(k) = 0$.

Definition 1: $[A|B_2]$ is said to be mean-square stabilizable with capacity $\mathfrak{C}_I$ if there is an allocation π_I and a feedback gain F such that the closed-loop networked system (1) with $\mathfrak{C}_{I_i} = \pi_{I_i}\mathfrak{C}_I$ is mean-square stable.

When $\mathfrak{C}_I = \infty$, this definition reduces to the classical stabilizabilty. The following lemma is shown in [15], [16].

Lemma 1: The following assertions are equivalent:

(a) $[A|B_2]$ is mean-square stabilizable with capacity $\mathfrak{C}_I$.

(b) There exist an allocation π_I, a feedback gain F and a matrix $V > 0$ such that

$$(A+B_2M_IF)V(A+B_2M_IF)' - V + B_2[\Sigma_I^2 \odot (FVF')]B_2' < 0.$$

In this case, F is mean-square stabilizing.

(c) $\mathfrak{C}_I > h(A)$.

One may expect that the mean-square detectability can be exactly defined in a dual way, as in the traditional control theory. However, this is not true mainly due to the fact that the outcome of $\kappa_O(k)$ is not known to the estimator. We consider the system $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0\end{array}\right]$ with $\kappa_I(k)$ imposed on the input and $\kappa_O(k)$ imposed on the output. The necessity of introducing the control input to form a closed-loop system will be appreciated more as we proceed.

As in the traditional case, we attempt to use the Leunberger observer architecture but in a slightly different way:

$$\hat{x}(k+1) = A\hat{x}(k) + K(\kappa_O(k)y(k) - M_OC_2\hat{x}(k)) + B_2\kappa_I(k)v(k).$$

Denote $e(k) \triangleq x(k) - \hat{x}(k)$, $Y(k) \triangleq \boldsymbol{E}[e(k)e(k)']$, $V(k) \triangleq \boldsymbol{E}[x(k)x'(k)]$, $\hat{V}(k) \triangleq \hat{x}(k)\hat{x}(k)'$, then

$$e(k+1) = (A - KM_OC_2)e(k) - K(\kappa_O(k) - M_O)C_2x(k),$$
$$Y(k+1) = (A - KM_OC_2)Y(k)(A - KM_OC_2)' + K[\Sigma_O^2 \odot (C_2V(k)C_2')]K'.$$

In view of the above equation, since A is unstable, if no control $v(k)$ is applied, $Y(k)$ goes to infinity when $\mathfrak{C}_O < \infty$ since $V(k)$ goes to infinity. Therefore, it makes more sense to introduce the control signal which is an estimated state feedback, i.e., $v(k) = F\hat{x}(k)$. Apparently, we assume that F is mean-square stabilizing with $\mathfrak{C}_I$. For the closed-loop system, it is easy to verify that $V(k) = Y(k) + \hat{V}(k)$ and

$$(A+B_2M_IF)\hat{V}(k)(A+B_2M_IF)' - \hat{V}(k+1) + B_2[\Sigma_I^2 \odot (F\hat{V}(k)F')]B_2' + AY(k)A' - Y(k+1) = 0.$$

Then, for given feasible allocation π_I and mean-square stabilizing F, it is possible to design an allocation π_O and an innovation injection matrix K such that $\lim_{k\to\infty} Y(k) = 0$.

Definition 2: Given feasible input capacity allocation π_I and mean-square stabilizing F, $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0\end{array}\right]$ is said to be mean-square detectable with capacity $\mathfrak{C}_O$ if there exist an allocation π_O and an innovation injection matrix K such that $\lim_{k\to\infty} Y(k) = 0$ for all initial error covariance $Y(0)$.

When $\mathfrak{C}_O = \infty$, this definition reduces to the classical detectability. The lemma below gives a necessary and sufficient condition on the mean-square detectability.

Lemma 2: The following assertions are equivalent:

(a) Given feasible π_I and mean-square stabilizing F, $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0\end{array}\right]$ is mean-square detectable with capacity $\mathfrak{C}_O$.

(b) There exist an allocation π_O, an innovation injection matrix K and two matrices $Y > 0$, $\hat{V} \geq 0$ such that

$$(A - KM_OC_2)Y(A - KM_OC_2)' - Y + K[\Sigma_O^2 \odot (C_2(Y + \hat{V})C_2')]K' < 0$$
$$(A+B_2M_IF)\hat{V}(A+B_2M_IF)' - \hat{V} + B_2[\Sigma_I^2 \odot (F\hat{V}F')]B_2' + AYA' - Y = 0.$$

Lemma 2 can be proved analogously to Lemma 1 with the same Lyapunov function based approach in [16]. Comparing the mean-square stabilizability and detectability, we have some taste of the asymmetry between these two concepts. The mean-square stabilizability can be studied with no concern on the estimation issue, while the study of the mean-square detectability takes advantage of the feedback of the control signal which helps to make the error covariance converge. Due to this asymmetry, the minimum total output channel capacity $\mathfrak{C}_O$ required for mean-square detectability is more difficult to find. In view of Lemma 1 (c), one thing for sure is the necessity of $\mathfrak{C}_O > h(A)$. However, at this stage, no exact closed-form solution on the minimal $\mathfrak{C}_O$ has been obtained. Numerically, it can be solved by convex optimization technique in a similar way to that in [16].

IV. Optimal Estimator

This section is to derive the optimal state estimator under the TCP-like protocols. The objective is to find an estimate which is linear in the noise-corrupted measurement $q(0), q(1), \ldots, q(k)$, such that the error covariance is minimized. A similar estimation problem is studied in [12] for the case when $p = 1$. Let $\mathcal{I}(k)$ be the information set

$$\{q(0), \ldots, q(k), v(0), \ldots, v(k-1), \kappa_I(0), \ldots, \kappa_I(k-1)\}.$$

With a little abuse of notation, denote $\hat{x}(k) \triangleq \boldsymbol{E}[x(k)|\mathcal{I}(k)]$, $e(k) \triangleq x(k) - \hat{x}(k)$, $Y(k) \triangleq \boldsymbol{E}[e(k)e(k)'|\mathcal{I}(k)]$ and $V(k) \triangleq \boldsymbol{E}[x(k)x(k)'|\mathcal{I}(k)]$, $\hat{V}(k) \triangleq \hat{x}(k)\hat{x}(k)'$. Before proceeding, we state several useful facts.

Lemma 3: The following statements hold:

(a) $\boldsymbol{E}[e(k)\hat{x}(k)'] = 0$,
(b) $V(k) = Y(k) + \hat{V}(k)$,
(c) $\boldsymbol{E}[x(k)'Tx(k)] = \hat{x}(k)'T\hat{x}(k) + \mathrm{tr}(TY(k)), \forall T \geq 0$.

The proof of Lemma 3 is presented in the Appendix.

To derive the optimal estimator, the innovation process, which is a white sequence obtained by a causal, linear, and causally invertible operation on the sequence $q(k)$, is quite important. Denote $\begin{bmatrix} P & T \\ T' & H \end{bmatrix} \triangleq \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1' & D_{21}' \end{bmatrix}$. The innovation process $\eta(k)$ for the problem at hand is given in the following lemma.

Lemma 4: The innovation process is given by $\eta(k) = q(k) - M_O C_2 \hat{x}(k)$ with zero mean and covariance $\boldsymbol{E}[\eta(k)\eta(l)'] = R_\eta(k)\delta_{kl}$, where

$$\begin{aligned} R_\eta(k) =& M_O(C_2Y(k)C_2' + H)M_O \\ &+ \Sigma_O^2 \odot (C_2Y(k)C_2' + H + C_2\hat{V}(k)C_2'). \end{aligned}$$

Lemma 4 can be proved by some straightforward computations and using Lemma 3 (b). The details of the proof are presented in the Appendix. With the innovation process $\eta(k)$, following a similar argument as in [9], the optimal estimator under the TCP-like protocols admits the form

$$\begin{aligned} \hat{x}(k+1) =& A\hat{x}(k) + K(k)\eta(k) + B_2\kappa_I(k)v(k) \\ =& A\hat{x}(k) + K(k)M_OC_2e(k) \\ &+ K(k)\kappa_O(k)D_{21}w(k) + B_2\kappa_I(k)v(k) \\ &+ K(k)(\kappa_O(k) - M_O)C_2x(k). \end{aligned} \tag{2}$$

Then

$$\begin{aligned} e(k+1) =& (A - K(k)M_OC_2)e(k) + [B_1 - K(k)\kappa_O(k)D_{21}]w(k) \\ &- K(k)(\kappa_O(k) - M_O)C_2x(k), \\ Y(k+1) =& \begin{bmatrix} I \\ M_O'K(k)' \end{bmatrix}' \begin{bmatrix} AY(k)A' + P & -(AY(k)C_2' + T) \\ -(C_2Y(k)A' + T') & M_O^{-1}R_\eta(k)M_O^{-1} \end{bmatrix} \\ &\cdot \begin{bmatrix} I \\ M_O'K(k)' \end{bmatrix}. \end{aligned}$$

By completing the squares, the optimal $K(k)$ minimizing $Y(k+1)$ is given by

$$\begin{aligned} K(k) =& (AY(k)C_2' + T)[W_O \odot (C_2Y(k)C_2' + H) \\ &+ (W_O - E) \odot (C_2\hat{V}(k)C_2')]^{-1}M_O^{-1}. \end{aligned} \tag{3}$$

Substituting (3) into the expression of $Y(k+1)$ yields the following iterative function:

$$\begin{aligned} Y(k+1) =& AY(k)A' + P - (AY(k)C_2' + T) \\ &\cdot [W_O \odot (C_2Y(k)C_2' + H) \\ &+ (W_O - E) \odot (C_2\hat{V}(k)C_2')]^{-1}(C_2Y(k)A' + T'). \end{aligned} \tag{4}$$

The initial condition for the estimator iteration is $\hat{x}(0) = 0$, $Y(0) = Y_0$.

V. LQG Controller

In the standard LQG control problem, the solution to the infinite-horizon LQG control is obtained by taking the limit of the finite-horizon result as the horizon length goes to infinity. Similarly, here we first study the finite-horizon LQG control with random input and output gains, then investigate the convergence issue when taking the limit to infinity. We want to stress that the study on the infinite-horizon case is the main concern of this paper, where the channel/controller co-design contributes to obtain the convergence condition.

A. Finite-horizon case

In the finite-horizon case, we consider the following cost function for N steps:

$$J(N) = \boldsymbol{E}\left[\sum_{k=0}^{N} \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix} \middle| \mathcal{I}(N)\right],$$

where $v(N) = 0$. The objective is to find the optimal control sequence $v(0), v(1), \ldots, v(N-1)$ to minimize $J(N)$. To this end, the dynamic programming is used. Define the optimal value function $L(k)$ as

$$\begin{aligned} L(N) =& \boldsymbol{E}[x(N)'Qx(N)|\mathcal{I}(N)], \\ L(k) =& \min_{v(k)} \boldsymbol{E}\left[\begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix} \right. \\ &\left. + L(k+1) \middle| \mathcal{I}(k)\right]. \end{aligned}$$

By the dynamic programming theory [1], we have $J(N)^* \triangleq \min J(N) = L(0)$.

We claim that $L(k)$ has the form

$$L(k) = \boldsymbol{E}[x(k)'X(k)x(k)|\mathcal{I}(k)] + c(k), \tag{5}$$

where $X(k)$ and $c(k)$ are to be determined. The proof is carried out by induction. Apparently, the claim is true for $k = N$ with $X(N) = Q$, $c(N) = 0$. Now assume that the claim is true for $k+1$, i.e., $L(k+1) = \boldsymbol{E}[x(k+1)'X(k+1)x(k+1)|\mathcal{I}(k+1)]+c(k+1)$, then

$$
\begin{aligned}
L(k) =& \min_{v(k)} \boldsymbol{E}\Bigg[\begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix} \\
&+ \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix}' \begin{bmatrix} A' \\ B_2' \end{bmatrix} X(k+1) \begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} x(k) \\ \kappa_I(k)v(k) \end{bmatrix} \\
&+ \mathrm{tr}(X(k+1)P)+c(k+1)\Big|\mathcal{I}(k)\Bigg] \\
=& \min_{v(k)} \boldsymbol{E}\Bigg[\begin{bmatrix} x(k) \\ M_I v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ M_I v(k) \end{bmatrix} \\
&+ \begin{bmatrix} x(k) \\ M_I v(k) \end{bmatrix}' \begin{bmatrix} A' \\ B_2' \end{bmatrix} X(k+1) \begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} x(k) \\ M_I v(k) \end{bmatrix} \\
&+ v(k)'[\Sigma_I^2 \odot (B_2'X(k+1)B_2+R)]v(k) \\
&+ \mathrm{tr}(X(k+1)P)+c(k+1)\Big|\mathcal{I}(k)\Bigg] \\
=& \min_{v(k)} \boldsymbol{E}\Bigg[\begin{bmatrix} x(k) \\ M_I v(k) \end{bmatrix}' \begin{bmatrix} Z_1 & Z_2 \\ Z_2' & Z_3 \end{bmatrix} \begin{bmatrix} x(k) \\ M_I v(k) \end{bmatrix} \\
&+ \mathrm{tr}(X(k+1)P)+c(k+1)\Big|\mathcal{I}(k)\Bigg],
\end{aligned} \tag{6}
$$

where

$$
\begin{bmatrix} Z_1 & Z_2 \\ Z_2' & Z_3 \end{bmatrix} = \begin{bmatrix} A'X(k+1)A+Q & A'X(k+1)B_2+S \\ B_2'X(k+1)A+S' & W_I \odot (B_2'X(k+1)B_2+R) \end{bmatrix}.
$$

By completing the squares with respect to $v(k)$, we obtain the optimal control law $v_{opt}(k) = F(k)\hat{x}(k)$, where

$$
\begin{aligned}
F(k) =& -M_I^{-1}Z_3^{-1}Z_2' \\
=& -M_I^{-1}[W_I \odot (B_2'X(k+1)B_2+R)]^{-1} \\
& \cdot (B_2'X(k+1)A+S').
\end{aligned} \tag{7}
$$

Substituting $v_{opt}(k)$ into (6) and using Lemma 3 (c) indicates that $L(k)$ indeed admits the form in (5), where

$$
\begin{aligned}
X(k) =& A'X(k+1)A+Q-(A'X(k+1)B_2+S)[W_I \odot \\
& (B_2'X(k+1)B_2+R)]^{-1}(B_2'X(k+1)A+S'), \\
c(k) =& \mathrm{tr}[(A'X(k+1)A+Q-X(k))Y(k)] \\
& + \mathrm{tr}(X(k+1)P)+\boldsymbol{E}[c(k+1)|\mathcal{I}(k)].
\end{aligned} \tag{8}
$$

Therefore, the optimal cost $J(N)^*$ is given by

$$
\begin{aligned}
J(N)^* =& L(0) = \boldsymbol{E}[x(0)'X(0)x(0)|\mathcal{I}(0)]+c(0) \\
=& \mathrm{tr}(X(0)Y_0)+\sum_{k=0}^{N-1}\mathrm{tr}(X(k+1)P) \\
& + \sum_{k=0}^{N-1}\mathrm{tr}[(A'X(k+1)A+Q-X(k))Y(k)].
\end{aligned} \tag{9}
$$

It can be seen that the optimal $F(k)$ does not depend on the estimator design. In fact, it is the same as the optimal state feedback gain derived for the LQR problem studied in [18]. The computation of the error covariance $Y(k)$ needs more effort since the iterative equation (4) involves $\hat{V}(k)$. With the designed $F(k)$ in (7), we have

$$
x(k+1) = Ax(k)+B_1w(k)+B_2\kappa_I(k)F(k)\hat{x}(k).
$$

After some calculations and using Lemma 3 (b), we get

$$
\begin{aligned}
\hat{V}(k+1) =& (A+B_2M_IF(k))\hat{V}(k)(A+B_2M_IF(k))' \\
& + B_2[\Sigma_I^2 \odot (F(k)\hat{V}(k)F'(k))]B_2' \\
& + AY(k)A' - Y(k+1)+P.
\end{aligned} \tag{10}
$$

The equations (4) and (10) constitute a coupling iteration, from which $Y(k)$ can be computed with initial condition $\hat{V}(0) = 0$, $Y(0) = Y_0$.

From the above analysis, we can see that the estimator design, i.e., the optimal $K(k)$ as in (3), indeed depends on the design of $F(k)$. This indicates that the separation principle only partially holds. There is certain asymmetry here. The design of $F(k)$ does not depend on $K(k)$, however, the converse is not true.

The above results on finite-horizon LQG control with random gains can be summarized in the next theorem.

Theorem 1: The optimal control law for N step is $v_{opt}(k) = F(k)\hat{x}(k)$. The feedback gain $F(k)$ is given by (7), where the matrix $X(k)$ can be computed iteratively using (8). The dynamics of $\hat{x}(k)$ is given by (2), where $K(k)$ is given by (3). The error covariance $Y(k)$ can be computed iteratively using (4) and (10). The optimal cost is given by (9).

B. *Infinite-horizon case*

In the infinite-horizon case, the cost function becomes

$$
\lim_{k\to\infty} \boldsymbol{E}[z(k)'z(k)] = \lim_{N\to\infty} \frac{1}{N} J(N).
$$

We can solve the infinite-horizon case by taking the limit of the horizon length to infinity. However, this requires that the iteration of $X(k)$ in (8) as well as the iteration of $Y(k)$ in (4) and (10) converge as $N \to \infty$, which is not necessarily true for any given channel capacities. In other words, if the capacities $\mathfrak{C}_{I_i}, i = 1, 2, \ldots, m$ and $\mathfrak{C}_{O_j}, j = 1, 2, \ldots, p$ are fixed a priori, the LQG control problem we consider may be infeasible.

To mitigate this difficulty, as mentioned before, the problem is studied under the framework of channel/controller co-design. We only assume that the total input channel capacity $\mathfrak{C}_I$ and output channel capacity $\mathfrak{C}_O$ are constrained. Now the controller designer has the additional freedom to design the allocation vectors π_I and π_O. In this case, surprisingly, the problem can be nicely solved.

Inherited from the finite-horizon result, the separation principle only partially holds for the LQG control problem of our concern. The design of the optimal feedback gain F is independent of the design of the estimator, while the converse is not true. To simplify the problem, we assume that $S = 0$, $Q > 0$ and $T = 0$, $P > 0$.

For the control part, convergence of $X(k)$ in (8) yields the following control modified algebraic Riccati equation (CMARE):

$$X = A'XA + Q - A'XB_2[W_I \odot (B_2'XB_2 + R)]^{-1}B_2'XA. \quad (11)$$

Note that (11) is exactly the same as the CMARE associated with the LQR problem studied in [18] which is not surprising since the optimal feedback gain is independent of the estimator design.

Lemma 5: The CMARE (11) has a unique positive-definite solution X if and only if $[A|B_2]$ is mean-square stabilizable with capacity $\mathfrak{C}_I$. Moreover, in this case, the optimal feedback gain is given by

$$F = -M_I^{-1}[W_I \odot (B_2'XB_2 + R)]^{-1}B_2'XA. \quad (12)$$

The proof to Lemma 5 can be found in [18].

For the estimator part, with a designed π_I and the associated optimal F in (12), the convergence of $Y(k)$ in (4) and (10) yields the following coupled matrix equations:

$$\begin{aligned} &AYA' - Y + P - AYC_2'[W_O \odot (C_2YC_2' + H) \quad (13)\\ &\quad + (W_O - E) \odot (C_2\hat{V}C_2')]^{-1}C_2YA' = 0,\\ &(A + B_2M_IF)\hat{V}(A + B_2M_IF)' + B_2[\Sigma_I^2 \odot (F\hat{V}F')]B_2' \quad (14)\\ &\quad + AYA' - Y + P = 0. \end{aligned}$$

By Lemma 1 (b), we can solve $\hat{V}$ from (14) in terms of a linear function of $AYA' - Y + P$, i.e., $\hat{V} = f(AYA' - Y + P)$. Substituting this into (13) yields

$$\begin{aligned} &AYA' - Y + P - AYC_2'[W_O \odot (C_2YC_2' + H) + (W_O - E) \quad (15)\\ &\odot (C_2 f(AYA' - Y + P)C_2')]^{-1}C_2YA' = 0. \end{aligned}$$

Apparently the coupled equations (13), (14) have a solution if and only if the filtering modified algebraic Riccati equation (FMARE) (15) has a solution. Define the operator

$$\begin{aligned} \varphi(K, Y, P, H) = &(A - KM_OC_2)Y(A - KM_OC_2)' + P\\ &+ K[\Sigma_O^2 \odot (C_2(Y + f(AYA' - Y + P))C_2')]K'\\ &+ KM_O(W_O \odot H)M_OK'. \end{aligned}$$

Also define the operator

$$\begin{aligned} g(Y, P, H) = &AYA' + P - AYC_2'[W_O \odot (C_2YC_2' + H) + (W_O - E)\\ &\odot (C_2 f(AYA' - Y + P)C_2')]^{-1}C_2YA'. \end{aligned}$$

Then the FMARE (15) can be rewritten as $Y = g(Y, P, H)$. The next lemma establishes the condition for the existence of a positive-definite solution to the FMARE (15) in terms of the mean-square detectability of $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$.

Lemma 6: For a designed π_I and the associated optimal gain F, the following assertions are equivalent:

(a) $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$ is mean-square detectable with output channel capacity $\mathfrak{C}_O$.

(b) There exist an allocation π_O and a matrix $Y > 0$, such that $g(Y, P, H) - Y < 0$.

(c) There exist an allocation π_O such that the FMARE (15) has a solution $Y > 0$. In this case, the optimal innovation injection matrix K is given by

$$\begin{aligned} K = &AYC_2'[W_O \odot (C_2YC_2' + H) + (W_O - E) \quad (16)\\ &\odot (C_2 f(AYA' - Y + P)C_2')]^{-1}M_O^{-1}. \end{aligned}$$

The proof to Lemma 6 is presented in the Appendix. With these preparations, we are now in a position to state the main theorem on the infinite-horizon LQG control with random input and output gains.

Theorem 2: The infinite-horizon problem is solvable if and only if $[A|B_2]$ is mean-square stabilizable with $\mathfrak{C}_I$ and $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$ is mean-square detectable with $\mathfrak{C}_O$ for a designed π_I and the associated optimal F. In this case, the optimal LPV controller $\boldsymbol{K}$ is given by

$$\begin{aligned} \hat{x}(k+1) &= (A - KM_OC_2 + B_2\kappa_I(k)F)\hat{x}(k) + Kq(k),\\ v(k) &= F\hat{x}(k), \end{aligned}$$

with F in (12) and K in (16). Moreover, the optimal cost is given by

$$\min \lim_{k\to\infty} \boldsymbol{E}[z(k)'z(k)] = \mathrm{tr}(XP) + \mathrm{tr}[(A'XA + Q - X)Y],$$

where X is the solution to (11) and Y is the solution to (15).

Proof: The condition on the solvability of the infinite-horizon problem follows from Lemma 5 and Lemma 6. The optimal $\boldsymbol{K}$ as well as the optimal cost can be obtained by taking the limit of the finite-horizon result as $N \to \infty$. ■

Remark 1: When $\mathfrak{C}_I = \infty$, the CMARE (11) reduces to a standard control ARE. Also, when $\mathfrak{C}_O = \infty$, the FMARE (15) reduces to a standard filtering ARE.

VI. Conclusion

In this paper, the LQG control of LTI systems with random input and output gains is studied. One main novelty of this work is that we study the problem under the framework of channel/controller co-design which allows the control designer to have the additional freedom to design the channels. With the channel/controller co-design, the optimal control problem studied is feasible if and only if the system is mean-square stabilizable and detectable. The minimum input channel capacity $\mathfrak{C}_I$ required is given by $h(A)$. The minimum output channel capacity $\mathfrak{C}_O$ depends on the design of π_I and the associated optimal F. A closed-form solution to the minimum $\mathfrak{C}_O$ is currently under our investigation.

We show that the separation principle partially holds under the TCP-like protocols. On one hand, the optimal controller is still an estimated state feedback, combining the optimal state feedback design and the optimal state estimation design. On the other hand, there exists certain asymmetry. Although the optimal state feedback gain does not depend on the estimator design, the optimal estimator does depend on the optimal state feedback gain.

Appendix

We give proofs to Lemma 3, Lemma 4 and Lemma 6 in the Appendix.

A. Proof of Lemma 3

(a) By definition, $\boldsymbol{E}[e(k)\hat{x}(k)'] = \boldsymbol{E}[(x(k) - \hat{x}(k))\hat{x}(k)'] = \boldsymbol{E}[x(k)]\hat{x}(k)' - \hat{x}(k)\hat{x}(k)' = 0$.

(b) By part (a), we have $V(k) = \boldsymbol{E}[(\hat{x}(k)+e(k))(\hat{x}(k)+e(k))'] = Y(k)+\hat{V}(k)$.

(c) Again, by part (a), we have $\boldsymbol{E}[x(k)'Tx(k)] = \boldsymbol{E}[(\hat{x}(k)+e(k))'T(\hat{x}(k)+e(k))] = \hat{x}(k)'T\hat{x}(k)+\mathrm{tr}(TY(k))+2\mathrm{tr}(T\boldsymbol{E}[e(k)\hat{x}(k)']) = \hat{x}(k)'T\hat{x}(k)+\mathrm{tr}(TY(k))$. ■

B. Proof of Lemma 4

First, it can be easily verified that

$$\boldsymbol{E}[\eta(k)] = \boldsymbol{E}[\kappa_O(k)(C_2x(k)+D_{21}w(k)) - M_OC_2\hat{x}(k)] = 0.$$

Then we compute the covariance of $\eta(k)$. When $k \neq l$, $\boldsymbol{E}[\eta(k)\eta(l)'] = \boldsymbol{E}[\eta(k)]\boldsymbol{E}[\eta(l)]' = 0$. When $k = l$,

$$\begin{aligned}\boldsymbol{E}[\eta(k)\eta(k)'] &= \boldsymbol{E}[(q(k)-M_OC_2\hat{x}(k))(q(k)-M_OC_2\hat{x}(k))'] \\ &= M_O(C_2Y(k)C_2'+H)M_O \\ &\quad + \Sigma_O^2 \odot (C_2V(k)C_2'+H) \\ &= M_O(C_2Y(k)C_2'+H)M_O \\ &\quad + \Sigma_O^2 \odot (C_2Y(k)C_2'+H+C_2\hat{V}(k)C_2'),\end{aligned}$$

which completes the proof. ■

C. Proof of Lemma 6

It can be inferred from Lemma 2 (b) that $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$ is mean-square detectable with capacity $\mathfrak{C}_O$, if and only if for sufficiently small $\alpha > 0$, there exist π_O, K and $Y > 0$ such that

$$\varphi(K, Y, \alpha P, \alpha H) - Y < 0. \tag{17}$$

For this particular Y, minimizing $\varphi(K, Y, \alpha P, \alpha H)$ with respect to K yields the optimal K as in (16). Substituting (16) into (17) indicates that the inequality (17) holds if and only if

$$g(Y, \alpha P, \alpha H) - Y < 0. \tag{18}$$

Note that (18) is equivalent to $g(\tilde{Y}, P, H) - \tilde{Y} < 0$, where $\tilde{Y} = \frac{1}{\alpha}Y$. This shows the equivalence between (a) and (b).

Now we show (b) $\Rightarrow$ (c). The following fact is used: for every $0 \leq Y_1 \leq Y_2$, $g(Y_1, P, H) \leq g(Y_2, P, H)$, i.e. $g(Y, P, H)$ is monotonically increasing with respect to Y. Suppose that the inequality $g(Y, P, H) - Y < 0$ holds for certain π_O and $Y > 0$. Define the iteration $\hat{Y}_k = g(\hat{Y}_{k-1}, P, H)$. For any initial condition $\hat{Y}_0 \geq 0$, we can get a sequence $\{\hat{Y}_k\}_{k \geq 0}$. Apparently, there exist $0 \leq a_1 < 1, a_2 > 0$ such that $g(Y, P, H) \leq a_1 Y$, $\hat{Y}_0 \leq a_2 Y$, and $a_1 a_2 \geq 1$. Then

$$\begin{aligned}\hat{Y}_1 &= g(\hat{Y}_0, P, H) \leq a_1 a_2 Y, \\ \hat{Y}_2 &= g(\hat{Y}_1, P, H) \leq \max\{a_1^2 a_2 Y, Y\}, \\ &\vdots \\ \hat{Y}_k &= g(\hat{Y}_{k-1}, P, H) \leq \max\{a_1^k a_2 Y, Y\},\end{aligned}$$

which implies that the sequence $\{\hat{Y}_k\}_{k \geq 0}$ is bounded from above. Now we let $\hat{Y}_0 = \epsilon I$ with sufficiently small $\epsilon > 0$ such that $\hat{Y}_1 > \hat{Y}_0$. By the monotonicity of the operator $g(Y, P, H)$, we have $\hat{Y}_k > \hat{Y}_{k-1} > 0$ for $k > 0$. Therefore by monotone convergence theorem, $\hat{Y} = \lim_{k \to \infty} \hat{Y}_k$ exists and satisfies $\hat{Y} = g(\hat{Y}, P, H) > 0$.

Finally, we show (c) $\Rightarrow$ (a). Suppose that $Y = g(Y, P, H)$ holds for certain π_O and $Y > 0$. Let K be given by (16), then $\varphi(K, Y, 0, 0) - Y < 0$. According to Lemma 2, $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array}\right]$ is mean-square detectable with $\mathfrak{C}_O$ which completes the proof. ■

REFERENCES

[1] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, Athena Scientific, Belmont, Massachusetts, 2005.

[2] R. Bowen, "Entropy for group endomorphisms and homogeneous spaces", *Transations of the American Mathematical Society*, vol. 153, pp. 401-414, 1971.

[3] W. Chen and L. Qiu, "Stabilization of networked control systems with multirate sampling", provisionally accepted by *Automatica*, 2012.

[4] E. Garone, B. Sinopoli, A. Goldsmith, and A. Casavola, "LQG control for MIMO systems over multiple erasure channels with perfect acknowledgment", *IEEE Trans. Automat. Contr.*, vol. 57, pp. 450-456, 2012.

[5] E. Gershon, U. Shaked, and I. Yaesh, *H_∞-Control and Estimation of State-multiplicative Linear Systems*, Lecture Notes in Control and Information Sciences, Springer: Berlin, 2005.

[6] G. Gu and L. Qiu, "Networked stabilization of multi-input systems with channel resource allocation", *Reprints of the 17th IFAC World Congress*, 2008.

[7] V. Gupta, B. Hassibi, and R. M. Murray, "Optimal LQG control across packet-dropping links", *Systems and Control Letters*, vol. 56, no. 6, pp. 439-446, 2007.

[8] O. C. Imer, S. Yuksel, and T. Basar, "Optimal control of LTI systems over unreliable communication links", *Automatica*, vol. 42, no. 9, pp. 1429-1440, 2006.

[9] T. Kailath, A. H. Sayed, B. Hassibi, *Linear Estimation*, Prentice Hall, Upper Saddle River, New Jersey, 2000.

[10] T. Katayama, "On matrix Riccati equation for linear systems with a random gain", *IEEE Trans. Automat. Contr.*, vol. 21, pp. 770-771, 1976.

[11] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation", to appear in *IEEE Trans. Automat. Contr.*, 2012.

[12] P. K. Rajasekaran, N. Satyanarayana, M. D. Srinath, "Optimum linear estimation of stochastic signals in the presence of multiplicative noise", *IEEE Trans. Aerospace and Electronic Systems*, vol. AES-7, pp. 462-468, 1971.

[13] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks", *Proc. of IEEE*, vol. 95, pp. 163-187, 2007.

[14] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. I. Jordan, and S. S. Sastry, "Kalman filtering with intermittent observations", *IEEE Trans. Automat. Contr.*, vol. 49, pp. 1453-1464, 2004.

[15] N. Xiao, L. Xie, and L. Qiu, "Mean square stabilization of multi-input systems over stochastic multiplicative channels", *Proc. 48th IEEE Conf. Decision Contr.*, pp. 6893-6898, 2009.

[16] N. Xiao, L. Xie, and L. Qiu, "Feedback stabilization of discrete-time networked systems over fading channels", *IEEE Trans. Automat. Contr.*, to appear, 2012.

[17] W. Zhang, Y. Huang, and H. Zhang, "Stochastic H_2/H_∞ control for discrete-time systems with state and disturbance dependent noise", *Automatica*, vol. 43, pp. 513-521, 2007.

[18] J. Zheng, W. Chen, L. Shi, and L. Qiu, "Linear quadratic optimal control for discrete-time LTI systems with random input gains", in *Proc. 31st Chinese Contr. Conf.*, pp. 5803-5808, 2012.

[19] X. Y. Zhou and D. Li, "Explicit efficient frontier of a continuous-time mean-variance portfolio selection problem", in *Control of Distributed Parameter and Stochastic Systems*, S. Chen *et al.*, Eds. Dordrecht, the Netherlands: Kluwer, pp. 323-330, 1999.