LQR PROBLEM OF CONTINUOUS-TIME LTI SYSTEMS WITH RANDOM GAINS*

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EXTENDED ABSTRACT. This paper studies the Linear Quadratic Regulator (LQR) problem of continuous-time LTI systems with random gains. The main novelty of this work is the use of the channel/controller co-design framework which bridges and integrates the design of the channels and the controller. The co-design is carried out by the twist of channel resource allocation, i.e., the channel capacities can be allocated by the control designer subject to an overall capacity constraint. By virtue of this additional design freedom, under certain conditions, a nice analytic solution is obtained for the LQR problem with random gains. The optimal control law is a linear state feedback.

Problem Formulation. The system studied in this work is shown in Fig. 0.1.

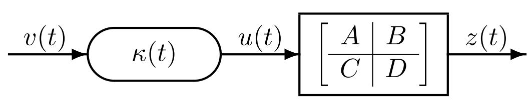

FIG. 0.1. *LTI systems with random input gains.*

Consider the plant:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ z(t) &= Cx(t) + Du(t), \end{aligned}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, z(t) \in \mathbb{R}^p$. We denote the plant by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for simplicity. Assume that $[A|B]$ is stabilizable and $x(t)$ is available for feedback control. Different from the classical setup of an LTI system, a random gain matrix $\kappa(t)$ is imposed on the control input, i.e., $u(t) = \kappa(t)v(t)$, where $\kappa(t) = \text{diag}\{\kappa_1(t), \kappa_2(t), \ldots, \kappa_m(t)\}$ is a random matrix consisting of diagonal white noise process elements with mean $\mu_i = \boldsymbol{E}[\kappa_i(t)]$ and variance $\sigma_i^2 = \boldsymbol{E}\left[(\kappa_i(t) - \mu_i)^2\right]$. The ratio $\frac{\mu_i}{\sigma_i}$, denoted as SNR_i, is the signal-to-noise ratio in the ith input channel. Such LTI systems with random gains have wide applications in different areas such as networked control, economic stability and financial engineering, etc. Denote

$$\begin{aligned} M &\triangleq \text{diag}\{\mu_1, \mu_2, \ldots, \mu_m\}, \\ \Sigma^2 &\triangleq \text{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\}, \\ W &\triangleq M^{-2}\Sigma^2 = \text{diag}\{\text{SNR}_1^{-2}, \text{SNR}_2^{-2}, \ldots, \text{SNR}_m^{-2}\}. \end{aligned}$$

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The mean-square capacity of the ith input channel is defined in [4] as $\mathfrak{C}_i \triangleq \frac{1}{2}\frac{\mu_i^2}{\sigma_i^2} = \frac{1}{2}\mathrm{SNR}_i^2$. The overall channel capacity is then given by $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

We study the state feedback LQR problem with random gains. One traditional way is to fix the channel capacities a priori and then find the optimal control law to minimize

$$\boldsymbol{E}[\|z\|_2^2] = \boldsymbol{E}\left[\int_0^\infty z(t)'z(t)dt\right] = \boldsymbol{E}\left[\int_0^\infty \begin{bmatrix} x(t) \\ \kappa(t)v(t)\end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R\end{bmatrix} \begin{bmatrix} x(t) \\ \kappa(t)v(t)\end{bmatrix} dt\right],$$

where $\begin{bmatrix} Q & S \\ S' & R\end{bmatrix} \triangleq \begin{bmatrix} C' \\ D'\end{bmatrix} \begin{bmatrix} C & D\end{bmatrix}$. However, under this formulation, the problem is not always feasible for any given set of channel capacities.

To tackle this difficulty, the channel/controller co-design framework provides a significant insight, which is the main novelty of this work. In this case, the individual channel capacities $\mathfrak{C}_i$ are not assumed to be given. Instead, they are designed, or allocated under an overall capacity constraint $\mathfrak{C}$. The allocation of the overall capacity to the individual channels, called channel resource allocation, can be formally given by a probability vector $\pi = \begin{bmatrix}\pi_1 & \pi_2 & \dots & \pi_m\end{bmatrix}'$, where $0 \leq \pi_i \leq 1$, $\sum_{i=1}^m \pi_i = 1$, such that $\mathfrak{C}_i = \pi_i\mathfrak{C}$. With the channel/controller co-design, our problem becomes to simultaneously design a probability vector π and the optimal control law to minimize $\boldsymbol{E}[\|z\|_2^2]$.

Before proceeding, recall that the topological entropy [1] of a matrix $A \in \mathbb{R}^{n\times n}$ is given by $h(A) = \sum_{|\lambda_i|>1} \ln|\lambda_i|$, where λ_i are the eigenvalues of A. Based on this, we define the topological entropy of the continuous-time plant as $H_c(A) = h(e^A) = \sum_{\Re(\lambda_i)>0} \lambda_i$, where λ_i are the eigenvalues of A.

Main Results. To solve the LQR problem formulated above, we first consider the finite-horizon optimal control and then take the limit as the horizon-length goes to infinity.

The finite-horizon case can be easily solved by either dynamic programming or completing squares. The cost function for horizon length T is

$$J(T) = \boldsymbol{E}\left[\int_0^T \begin{bmatrix} x(t) \\ \kappa(t)v(t)\end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R\end{bmatrix} \begin{bmatrix} x(t) \\ \kappa(t)v(t)\end{bmatrix} dt\right],$$

where $v(T) = 0$. Let E be an $m \times m$ matrix with all elements equal to 1. The optimal control law minimizing $J(T)$ is given in the following theorem.

THEOREM 0.1. *For every initial state $x(0)$, the optimal control law for horizon $[0, T]$ is $v_{opt}(t) = F(t)x(t)$, where*

$$F(t) = -M^{-1}[W \odot (B'X(t)B) + (E+W) \odot R]^{-1}(B'X(t)+S')$$

and $X(t)$ is the solution to the matrix differential equation

$$\begin{aligned} -\dot{X} &= A'X + XA + Q - (XB+S)[W\odot(B'XB)+(E+W)\odot R]^{-1}(B'X+S'), \\ X(T) &= Q. \end{aligned} \tag{0.1}$$

Moreover, the optimal cost is given by $\min J(T) = x(0)'X(0)x(0)$.

To study the infinite-horizon case, we first present some preliminary knowledge on the mean-square stabilization.

DEFINITION 0.2. *$[A|B]$ is said to be stabilizable with capacity $\mathfrak{C}$ if there is an allocation π and a state feedback gain F such that the closed-loop system*

$$\dot{x}(t) = (A+B\kappa(t)F)x(t) \tag{0.2}$$

with $\mathfrak{C}_i = \pi_i\mathfrak{C}$ is mean-square stable, i.e., for any initial state $x(0)$, $N(t) \triangleq \boldsymbol{E}[x(t)x'(t)]$ is well-defined for any $t > 0$ and $\lim_{t\to\infty} N(t) = 0$.

REMARK 1. *When $\mathfrak{C} = \infty$, the above definition reduces to that of classical stabilizability.*

The next theorem gives a necessary and sufficient condition for the mean-square stabilizability in terms of the topological entropy of the open-loop plant.

THEOREM 0.3 ([4]). *$[A|B]$ is stabilizable with capacity $\mathfrak{C}$ if and only if $[A|B]$ is stabilizable and $\mathfrak{C} > H_c(A)$.*

As mentioned before, the infinite-horizon LQR problem with random gains can be solved by taking the limit of the finite-horizon result as $T \to \infty$. However, this requires that the following continuous-time modified algebraic Riccati equation (MARE)

$$A'X+XA+Q-(XB+S)[W\odot(B'XB)+(E+W)\odot R]^{-1}(B'X+S') = 0 \tag{0.3}$$

has a mean-square stabilizing solution X in the sense that the associated state feedback gain

$$F = -M^{-1}[W\odot(B'XB)+(E+W)\odot R]^{-1}(B'X+S') \tag{0.4}$$

makes the closed-loop system (0.2) mean-square stable.

THEOREM 0.4. *If MARE (0.3) has a mean-square stabilizing solution X, then for every initial state $x(0)$, the infinite-horizon optimal control law is $v_{opt}(t) = Fx(t)$ with F as in (0.4). The optimal cost is given by $\min \boldsymbol{E}[\|z\|_2^2] = x(0)'Xx(0)$.*

REMARK 2. *When $\mathfrak{C} = \infty$, the MARE (0.3) reduces to the classical continuous-time algebraic Riccati equation.*

Unfortunately, if the channel capacities $\mathfrak{C}_i$ are given a priori, by Theorem 0.3, even the mean-square stabilizability cannot be guaranteed, let alone the existence of the optimal control law. However, by virtue of the channel/controller co-design which provides additional design freedom to allocate the capacities among the input channels, the infinite-horizon case can be nicely solved under certain conditions.

In the sequel, we investigate the existence of the mean-square stabilizing solution to MARE (0.3) by building a connection with the following linear matrix inequalities (LMIs):

$$\begin{cases} \begin{bmatrix} A'X + XA + Q & XB + S \\ B'X + S' & W\odot(B'XB) + (E+W)\odot R \end{bmatrix} \geq 0, \\ W\odot(B'XB) + (E+W)\odot R > 0. \end{cases} \tag{0.5}$$

A similar approach has been used in [2, 3].

The LMIs (0.5) is said to be feasible if it has a solution. It is said to be strictly feasible if it has a solution such that the first inequality is strictly satisfied. The maximal solution to LMIs (0.5), denoted as X_+, is a solution which is greater than or equal to any other solution. The maximal solution, if exists, is unique.

The next theorem bridges the maximal solution to LMIs (0.5) and the mean-square stabilizing solution to MARE (0.3).

THEOREM 0.5. *Assume that $[A|B]$ is stabilizable with capacity $\mathfrak{C}$. If LMIs (0.5) is feasible, then it has a maximal solution X_+. Moreover, X_+ is a solution to the MARE (0.3). In this case, the MARE (0.3) has at most one mean-square stabilizing solution, which coincides with X_+.*

One can compute X_+ by solving the convex optimization problem below:

$$\begin{aligned} \max \quad & \operatorname{tr}(X), \\ \text{subject to} \quad & \text{constraints (0.5)}. \end{aligned}$$

A sufficient condition for the maximal solution to be indeed mean-square stabilizing is presented in the following theorem.

THEOREM 0.6. *Assume that $[A|B]$ is stabilizable with capacity $\mathfrak{C}$. If LMIs (0.5) is strictly feasible, then the MARE (0.3) has a mean-square stabilizing solution X.*

We apply the above results to some special cases. First, we consider the case when $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} > 0$. Clearly, $X = 0$ is a strictly feasible solution to LMIs (0.5). By Theorem 0.6, the MARE (0.3) has a unique mean-square stabilizing solution and the infinite-horizon LQR problem is solvable.

We proceed to look at the general case where the only assumption is $R > 0$. In this case, $X = 0$ is only a feasible solution to (0.5). By Theorem 0.5, the LMIs (0.5) has a maximal solution X_+ which is also a solution to MARE (0.3). However, whether X_+ is mean-square stabilizing is not clear. Nevertheless, we can compute X_+ by solving the convex optimization problem mentioned before and then check whether it is mean-square stabilizing. If not, the MARE (0.3) does not have a mean-square stabilizing solution in this case. On the other hand, if we let $\tilde{Q} = Q + \epsilon I$ for $\epsilon > 0$, then $\begin{bmatrix} \tilde{Q} & S \\ S' & R \end{bmatrix} > 0$ and thus the associated infinite-horizon LQR problem is solvable. Taking the limit as $\epsilon \to 0$ indicates that in the general case, $\inf \boldsymbol{E}[\|z\|_2^2] = x(0)' X_+ x(0)$. But this cost is not achieved by a mean-square stabilizing controller.

It can be seen from the above discussions that the sufficient condition given in Theorem 0.6 is quite strong. How to relax it to obtain a necessary and sufficient condition is not clear at this stage and is under our current investigation.

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