Linear Quadratic Optimal Control for Discrete-time LTI Systems with Random Input Gains

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Abstract: In this paper, the linear quadratic (LQ) optimal control of discrete-time linear time-invariant (LTI) systems with random input gains is studied. We define the capacity of each input channel whose sum yields the total capacity of all input channels. Different from the finite-horizon case which can be solved by dynamic programming, the infinite-horizon case may be unsolvable if the capacities of the individual channels are fixed a priori. The main novelty of this work is that we put the problem under the framework of channel/controller co-design which allows the control designer to have the additional freedom to design the channels. We assume that the overall channel capacity is constrained which can be allocated to the individual channels. By channel/controller co-design, it is shown that the infinite-horizon case is solvable if and only if the overall capacity of the input channels is greater than the topological entropy of the open-loop plant. Moreover, the optimal control signal is a linear state feedback.

Key Words: LQ optimal control, Stochastic systems, Channel resource allocation, Modified algebraic Riccati equation.

1 Introduction

The growing developments in networked control systems (NCSs), financial engineering and other related topics have stimulated great interest in stochastic systems. Parallel to the control theory for deterministic systems, stabilization as well as optimal control of stochastic systems have been investigated widely. One can refer to [5, 9, 20] for a general study of stochastic control systems.

The LQ optimal control is one of the most important control problems. In [9], the infinite-horizon LQ optimal control for a general stochastic system with both state and control-dependent noise is studied. The authors derive the optimal control law under certain assumptions of mean-square stabilizability and exact observability.

Instead of studying the general stochastic systems, we focus on LTI systems with random input gains in this paper. Such systems arise frequently in the area such as networked control, financial engineering and economic stability [12], etc. In particular, in the context of NCSs, much research has been done recently, most of which treats the LQ optimal control as part of the Linear Quadratic Gaussian (LQG) control problem. For convenience, in an NCS, the channels through which the plant output is sent to the controller is named as the output channels and the channels through which the controller output is sent to the plant is named as the input channels. The work in [7] considers the LQG control with packet dropping in the output channels. It is shown that the separation principle holds under the TCP-like protocol and the control law derived is optimal for an arbitrary packet dropping pattern. The LQG control of a multi-input-multi-output (MIMO) system with a single packet dropping input channel and a single packet dropping output channel is considered in [11, 16]. The packet dropping channel is modeled as an i.i.d Bernoulli process. The authors point out that the optimal LQG control is a linear function of the estimated state which depends on the packet dropping probabilities.

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Researchers have also studied the LQG control over multiple parallel communication channels for a MIMO NCS. One such example is given by [8]. The objective there is to find the optimal control law assuming that the packet dropping probabilities are given a priori. The separation principle is shown to hold under the TCP-like protocol and the optimal control law is obtained in the finite-horizon case. For the infinite-horizon case, a sufficient condition on the stability of the closed-loop system is given by a set of linear matrix inequalities (LMIs).

Inspired by the aforementioned results, we study the infinite-horizon LQ optimal control for LTI systems with random input gains. Different from the setting in [8], we put the problem under the framework of channel/controller co-design which is the main novelty of this paper. We assume that the controller designer has the freedom to participate in the channel design. Due to this additional design freedom, the objective now becomes to simultaneously design the control signal and channels such that the cost function is minimized. More specifically, we assume that the channel capacities can be allocated as desired subject to an overall capacity constraint. Different from the finite-horizon case where the optimal control signal can be easily obtained by dynamic programming, the infinite-horizon case requires more effort, where the channel/controller co-design plays a crucial role. By channel/controller co-design, under certain assumptions on the system parameters, the infinite-horizon case is solvable if and only if the overall capacity of the input channels is greater than the topological entropy of the open-loop plant. Moreover, the optimal control signal is a linear state feedback.

The framework of channel/controller co-design is first proposed in [6], which studies the stabilization of multi-input NCSs with the signal-to-error ratio (SER) channel model. The work in [6] is extended in [14] where a more complete study is carried out on stabilization of multi-input NCSs. Besides the SER channel model, another two channel models are considered, i.e., the received signal-to-error ratio

(R-SER) model and the AWGN channel model. With the channel/controller co-design, a uniform analytic solution is obtained for the minimum total channel capacity required for stabilization with each channel model given in terms of the topological entropy of the plant. Several other works [2, 18] have been carried out following this framework. In [2], it points out that the continuous-time NCS with multirate sampling can be stabilized by state feedback under the channel/controller co-design framework if and only if the total network capacity is greater than the topological entropy of the plant. In [18], it is shown that a multi-input system over parallel stochastic multiplicative channels can be mean-square stabilized by state feedback under the channel/controller co-design framework if and only if the overall mean-square channel capacity is greater than the topological entropy.

The remainder of this paper is organized as follows. The problem is formulated and the channel/controller co-design framework are introduced in Section 2. The finite-horizon case is studied in Section 3. Section 4 first presents some useful preliminaries, then investigates the infinite-horizon case under the framework of channel/controller co-design. Conclusions follow in Section 5.

The notation in this paper is more or less standard and will be made clear as we proceed. The symbol $\odot$ means Hadamard product and denote $\mathrm{Tr}(A)$, $\lambda_{\max}(A)$ as the trace, maximal eigenvalue of a matrix A, respectively.

2 Problem Formulation

Consider the following system as shown in Fig. 1.

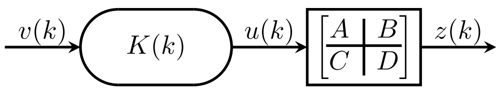

Fig. 1. Discrete-time LTI Systems with Random Input Gains

The plant is described by the following state-space equations

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k), \\ z(k) &= Cx(k)+Du(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $z(k) \in \mathbb{R}^p$. We denote the plant by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ for simplicity. Assume that $[A|B]$ is stabilizable, D has full column rank and $x(k)$ is available for feedback control. Different from the classical setup of an LTI system, a random input gain $K(k)$ is imposed on the control input:

$$u(k) = K(k)v(k),$$

where $K(k) = \mathrm{diag}\{\kappa_1(k), \kappa_2(k), \ldots, \kappa_m(k)\}$ is a random matrix consisting of diagonal white noise elements with mean $\mu_i = \boldsymbol{E}\left[\kappa_i(k)\right]$ and variance $\sigma_i^2 = \boldsymbol{E}\left[(\kappa_i(k)-\mu_i)^2\right]$. The signal-to-noise ratio of ith input channel is denoted by $\mathrm{SNR}_i \triangleq \frac{\mu_i}{\sigma_i}$. Such LTI systems with random input gains have wide applications in different areas such as networked control, financial engineering and economic stability, etc. The random gain $K(k)$ has different physical interpretations in different situations. In the contexts of NCSs, $K(k)$ can be used to model the multiplicative noises in the input channels while in the economic problems, it represents the multiplicative uncertainty existing in the economic parameters. Denote

$$\begin{aligned} M &\triangleq \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_m\}, \\ \Sigma^2 &\triangleq \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2\}, \\ W &\triangleq \begin{bmatrix} 1+\mathrm{SNR}_1^{-2} & 1 & \cdots & 1 \\ 1 & 1+\mathrm{SNR}_2^{-2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+\mathrm{SNR}_m^{-2} \end{bmatrix}. \end{aligned}$$

Recall that the mean-square capacity of the ith input channel is defined as [3]:

$$\mathfrak{C}_i \triangleq \frac{1}{2}\log(1+\frac{\mu_i^2}{\sigma_i^2}) = \frac{1}{2}\log(1+\mathrm{SNR}_i^2).$$

The overall mean-square capacity is given by $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$.

The aim of this work is to find a control signal which makes the closed-loop system stable in some sense and minimizes the LQ cost function

$$\begin{aligned} \boldsymbol{E}[\|z\|_2^2] &= \boldsymbol{E}\left[\sum_{k=0}^{\infty} z'(k)z(k)\right] \\ &= \boldsymbol{E}\left[\sum_{k=0}^{\infty} \begin{bmatrix} x(k) \\ K(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ K(k)v(k) \end{bmatrix}\right], \end{aligned}$$

where

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \triangleq \begin{bmatrix} C' \\ D' \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix}.$$

One traditional way to handle this problem, as shown in some current literature, is to fix the individual channel capacities a priori and then find the optimal control signal to minimize $\boldsymbol{E}[\|z\|_2^2]$. However, under this formulation, the problem is not always well-posed for any given channel capacities, i.e., the cost function might be always infinity no matter what control signal is used or the control signal that minimizes the cost function does not stabilize the system.

To tackle this difficulty, the channel/controller co-design framework provides a significant insight, which is the main novelty of this work. In this case, the individual channel capacities $\mathfrak{C}_i$ are not assumed to be given. Instead, they are to be designed, or allocated under an overall capacity constraint $\mathfrak{C}$. The allocation of the overall capacity to the individual channels, called channel resource allocation, is formally given by a probability vector

$$\pi = \begin{bmatrix} \pi_1 & \pi_2 & \ldots & \pi_m \end{bmatrix}',$$

where $0 \le \pi_i \le 1$, $\sum_{i=1}^m \pi_i = 1$, such that $\mathfrak{C}_i = \pi_i \mathfrak{C}$. With the help of the channel/controller co-design, our problem becomes to simultaneously design a probability vector π and the optimal control law to minimize $\boldsymbol{E}[\|z\|_2^2]$ under a certain overall capacity constraint.

3 Finite-horizon LQ Optimal Control

In the deterministic LQ optimal control problem, the infinite-horizon case, if it is well-posed, can be solved by taking the result to the finite-horizon case as the horizon length goes to infinity. Similarly, we first study the finite-horizon LQ optimal control of discrete-time LTI systems with random input gains in this section. The infinite-horizon case will be investigated in the next section.

In the finite-horizon case, the cost function for the horizon length N with the initial state $x(0)$ and the control signal $v_N \triangleq \{v(0), v(1), \ldots, v(N)\}$ is given in the following quadratic form:

$$\begin{aligned} &J_N(x(0), v_N) \\ &= \boldsymbol{E}\left[\sum_{k=0}^{N} \begin{bmatrix} x(k) \\ K(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ K(k)v(k) \end{bmatrix}\right]. \end{aligned}$$

The objective is to find the optimal control signal $v_N^{opt} \triangleq \{v^{opt}(0), v^{opt}(1), \ldots, v^{opt}(N)\}$ to minimize $J_N(x(0), v_N)$. The following theorem shows that the optimal control signal is a linear function of the state and depends on the mean-square capacity of each input channel. The proof which is based on dynamic programming is presented in the Appendix.

Theorem 1. For every initial state $x(0)$, the optimal control signal for the horizon length N is given by

$$v^{opt}(k) = F(k)x(k), \tag{2}$$

where

$$\begin{aligned} F(k) =& -M^{-1}[W \odot (R + B'X_N(k+1)B)]^{-1} \\ &\times [B'X_N(k+1)A + S'], \\ X_N(k) =& A'X_N(k+1)A + Q - [A'X_N(k+1)B + S] \\ &\times [W \odot (B'X_N(k+1)B + R)]^{-1} \\ &\times [B'X_N(k+1)A + S'], \\ X_N(N+1) =& \, 0, \end{aligned} \tag{3}$$

for $k = N, N-1, \ldots, 0$, and the optimal cost is given by

$$x'(0)X_N(0)x(0).$$

4 Infinite-horizon LQ Optimal Control

In this section, we study the infinite-horizon LQ optimal control of discrete-time LTI systems with random input gains. This infinite-horizon case, as we mentioned before, may not be well-posed if the individual channel capacities are fixed a priori. However, by virtue of channel/controller co-design, it becomes well-posed under certain assumptions.

4.1 Preliminary

Before proceeding, some useful preliminaries will be presented.

Recall the following two concepts. One is the Mahler measure [13] of an $n \times n$ matrix A, denoted by $M(A)$, which is the product of the absolute value of the unstable eigenvalues of A, i.e.,

$$M(A) = \prod_{i=1}^{n} \max\{1, |\lambda_i(A)|\}.$$

The second is the topological entropy [1] of A, denoted by $h(A)$, which is the logarithm of $M(A)$, i.e.,

$$h(A) = \log M(A).$$

Consider a discrete-time stochastic system

$$x(k+1) = \left(A_0 + \sum_{i=1}^{K} A_i p_i(k)\right) x(k),$$

where $p_1(k), \ldots, p_K(k)$ are independent random variables. It is said to be mean-square stable if for any initial state $x(0)$, $\boldsymbol{E}[x(k)x'(k)]$ is well-defined for any $k > 0$ and $\lim_{k\to\infty} \boldsymbol{E}[x(k)x'(k)] = 0$.

Definition 1. $[A|B]$ is said to be stabilizable with capacity $\mathfrak{C}$ if there exists an allocation π and a feedback gain F such that the closed-loop system

$$x(k+1) = (A + BK(k)F)\, x(k) \tag{4}$$

with $\mathfrak{C}_i = \pi_i \mathfrak{C}$ is mean-square stable.

Remark 1. When $\mathfrak{C} = \infty$, this definition reduces to that of classic stabilizability.

Several sufficient and necessary conditions on the stabilizability of $[A|B]$ with capacity $\mathfrak{C}$ is given in the following lemma. In particular, it indicates that the minimum total channel capacity required for mean-square stabilization with channel resource allocation is equal to the topological entropy of the open-loop plant.

Lemma 1 ([18, 19]). The following statements are equivalent:

i) $[A|B]$ is stabilizable with capacity $\mathfrak{C}$.
ii) $\mathfrak{C} > h(A)$.
iii) There exist an allocation π, matrices F and $X > 0$ such that

$$\begin{aligned} X >& (A + BMF)'X(A + BMF) \\ &+ F'[\Sigma^2 \odot (B'XB)]F. \end{aligned} \tag{5}$$

Moreover, the closed-loop system (4) is mean-square stable with $v(k) = Fx(k)$.

Remark 2. It is worthwhile emphasizing the basic idea in the proof of Lemma 1. Without loss of generality, $[A|B]$ can be assumed to have the following Wonham decomposition [17]:

$$A = \begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix}, \; B = \begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix},$$

where each pair $[A_i|b_i]$ is stabilizable. It is clear from the Wonham decomposition that A_i contains all the unstable eigenvalues of A which are controllable by the ith input but not controllable by any previous inputs. For a given overall capacity $\mathfrak{C}$, a feasible allocation π can always be found such that $\mathfrak{C}_i = \pi_i \mathfrak{C} > h(A_i)$. With this allocation, we sequentially design f_i such that $[A_i|b_i]$ is stabilized with capacity $\mathfrak{C}_i$. The existence of such f_i is guaranteed by the result in [3] for the state feedback mean-square stabilization of a single-input system over a fading channel. By such a sequential design, $[A|B]$ can be stabilized with capacity $\mathfrak{C}$.

4.2 Main Result

In the infinite-horizon case, the cost function is

$$J_\infty(x(0),v_\infty)=\boldsymbol{E}[\|z\|_2^2],$$

where $v_\infty \triangleq \{v(0),v(1),\ldots,v(\infty)\}$. With the finite-horizon result as shown in Theorem 1, the infinite-horizon case can be solved by taking the horizon length $N\to\infty$. However, this requires that as $N\to\infty$, the matrix $X_N(0)$ solved by the backward iteration of $X_N(k)$ in (3) converges to a matrix X which satisfies the following discrete-time Modified algebraic Riccati equation (MARE):

$$\begin{aligned} X=&A'XA+Q-(A'XB+S)\\ &\times[W\odot(R+B'XB)]^{-1}(B'XA+S'). \end{aligned} \tag{6}$$

Moreover, the matrix X is required to be mean-square stabilizing in the sense that with the associated state feedback gain

$$F_X=-M^{-1}[W\odot(R+B'XB)]^{-1}(B'XA+S'), \tag{7}$$

the closed-loop system (4) is mean-square stable. Unfortunately, the above requirements are not necessarily satisfied for any channel capacities $\mathfrak{C}_1,\mathfrak{C}_2,\ldots,\mathfrak{C}_m$. If the individual capacities are given a priori, in view of Lemma 1, even the mean-square stabilizability cannot be guaranteed, let alone the existence of the optimal control signal.

To tackle this difficulty, we study the infinite-horizon LQ optimal control with random input gains under the framework of channel/controller co-design. We make the following assumption:

Assumption 1. $\begin{bmatrix} Q & S\\ S' & R\end{bmatrix}>0.$

With the additional design freedom given by channel/controller co-design, the infinite-horizon case can be nicely solved. In the following theorem, which is the main result of this work, we find a necessary and sufficient condition under which the MARE (6) has a mean-square stabilizing solution with a feasible allocation π. Thus the infinite-horizon case is solvable.

Theorem 2. For every initial state $x(0)$, the optimal control signal together with a feasible allocation π exist if and only if $[A|B]$ is stabilizable with capacity $\mathfrak{C}$. Then for the designed π, the optimal control signal is given by

$$v^{opt}(k)=F_X x(k),$$

where X is the mean-square stabilizing solution to (6) and F_X is the associated state feedback gain as in (7). The optimal cost is given by $x'(0)Xx(0)$.

Proof: The necessity is quite straightforward by the above analysis. It suffices to show the sufficiency.

First, we will show the existence of a solution $X>0$ to the MARE (6). Denote

$$V_N(x(0))\triangleq\min_{v_N}(J_N(x(0),v_N)).$$

Then by Theorem 1, $V_N(x(0))=x'(0)X_N(0)x(0)$. For $N2>N1\geq 0$,

$$V_{N2}(x(0))\geq V_{N1}(x(0)),$$

which implies $X_{N2}(0)\geq X_{N1}(0)$, i.e., $X_N(0)$ is monotonically increasing with respect to N. Furthermore, since $\begin{bmatrix} Q & S\\ S' & R\end{bmatrix}>0$, by Schur complements, $Q-SR^{-1}S'>0$, which implies

$$X_1(x(0))=Q-S(W\odot R)^{-1}S'>0.$$

Then we have

$$X_{N2}(x(0))\geq X_{N1}(x(0))>0$$

for $N2>N1\geq 1$.

Since $[A|B]$ is stabilizable with capacity $\mathfrak{C}$, by Lemma 1, there exist an allocation π, a feedback gain F and $Y>0$ such that

$$\begin{aligned} Y=&(A+BMF)'Y(A+BMF)\\ &+F'[\Sigma^2\odot(B'YB)]F+I. \end{aligned} \tag{8}$$

Moreover, the closed-loop system (4) is mean-square stable with this feedback gain F.

Denote $G(k)\triangleq\boldsymbol{E}[x(k)x'(k)]$. It is easy to obtain

$$\begin{aligned} &\mathrm{Tr}\,[G(k+1)Y]\\ &=\mathrm{Tr}\,\{\boldsymbol{E}[(A+BK(k)F)x(k)x'(k)(A+BK(k)F)']Y\}\\ &=\mathrm{Tr}\{(A+BMF)G(k)(A+BMF)'Y\\ &\qquad+B[\Sigma^2\odot(FG(k)F')]B'Y\}\\ &=\mathrm{Tr}\{G(k)(A+BMF)'Y(A+BMF)\\ &\qquad+G(k)F'[\Sigma^2\odot(B'YB)]F\}\\ &=\mathrm{Tr}[G(k)Y]-\mathrm{Tr}[G(k)], \end{aligned}$$

where equation (8) is used to derive the last equality. By summing over $k=0,1,\ldots,N$, we obtain

$$\sum_{k=0}^{N}\mathrm{Tr}[G(k)]=\mathrm{Tr}\,[G(0)Y]-\mathrm{Tr}\,[G(N+1)Y]\,.$$

Furthermore, due to the mean-square stability of the closed-loop system (4), i.e., $\lim_{k\to\infty}G(k)=0$, we obtain

$$\sum_{k=0}^{\infty}\boldsymbol{E}[\|x(k)\|^2]=\sum_{k=0}^{\infty}\mathrm{Tr}\,[G(k)]=\mathrm{Tr}[G(0)Y]\leq r_1\|x(0)\|^2,$$

where $r_1=\lambda_{\max}(Y)$,

With $v(k)=Fx(k)$, the cost function becomes

$$\begin{aligned} J_\infty(x(0),v_\infty)=&\sum_{k=0}^{\infty}\mathrm{Tr}[\boldsymbol{E}[z(k)z(k)']]=\sum_{k=0}^{\infty}\mathrm{Tr}[G(k)U]\\ \leq& r_2\|x(0)\|^2 \end{aligned}$$

with

$$U\triangleq(C+DMF)'(C+DMF)+F'[\Sigma^2\odot R]F$$

and $r_2=\lambda_{\max}(U)\lambda_{\max}(Y)$. Then

$$\begin{aligned} &x'(0)X_N(0)x(0)=V_N(x(0))\\ &\leq V_\infty(x(0))\leq J_\infty(x(0),v_\infty)\leq r_2\|x(0)\|^2, \end{aligned}$$

which implies $0 \leq X_N(0) \leq r_2 I$. By monotone convergence theorem, $X = \lim_{N\to\infty} X_N(0) > 0$ exists.

Next we show that X is a mean-square stabilizing solution. Notice that with F_X defined in (7), the MARE (6) becomes

$$X = (A+BMF_X)'X(A+BMF_X) + F_X'[\Sigma^2 \odot (B'XB)]F_X + U. \quad (9)$$

Also, U can be rewritten as follows:

$$U = Q - SR^{-1}S' + (SR^{-1} + F'M')R(SR^{-1} + F'M')' + F'[\Sigma^2 \odot R]F,$$

Since $Q - SR^{-1}S' > 0$, $U > 0$. Therefore, from (9),

$$X > (A+BMF_X)'X(A+BMF_X) + F_X'[\Sigma^2 \odot (B'XB)]F_X.$$

By Lemma 1, the closed-loop system (7) is mean-square stabilized with feedback gain F_X. Hence the optimal control signal is given by $v^{opt}(k) = F_X x(k)$ and the optimal cost is $x'(0)Xx(0)$. □

Note that to ensure the existence of the mean-square stabilizing optimal control signal, we need to impose some assumptions on the system parameters. Assumption 1 provides a sufficient assumption, but it seems to be unnecessarily strong. How to find the minimal necessary assumptions ensuring the existence of the mean-square stabilizing optimal control signal remains to be studied and is under our investigation. Nevertheless, we can get some insight from the deterministic LQ optimal control, where the minimal necessary assumptions on the system parameters are that $[A|B]$ is stabilizable, $\left[\frac{A}{C}\right]$ is detectable and $\begin{bmatrix} A - e^{j\omega}I & B \\ C & D \end{bmatrix}$ has full column rank for all $\omega \in [0, 2\pi)$. In fact, for our current problem, the detectability of $\left[\frac{A}{C}\right]$ is still necessary. Otherwise, let λ be an unstable and unobservable eigenvalue of A , then to make the cost function minimized, the best control strategy is to ignore the eigenvalue λ and thus the closed-loop system is not mean-square stabilized. We are continuing to investigate the necessary assumption that is parallel to the third one for the deterministic case. In the following, we provide two motivating examples to illustrate the gap between the sufficient assumption given by Assumption 1 and the necessary assumption on the detectability of $\left[\frac{A}{C}\right]$.

We first consider an LTI system with $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Design the mean and covariance matrix of the random input gain to be $M = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ and $\Sigma^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1.6 \end{bmatrix}$. It can be verified that $\mathfrak{C} > h(A)$ and the allocation of the channel capacities is feasible. The iteration of $X_N(k)$ defined in (3) converges to $X = \begin{bmatrix} 7.3983 & 0.7203 \\ 0.7203 & 3.1463 \end{bmatrix}$. The associated state feedback gain is given by $F = \begin{bmatrix} -0.7545 & -0.0072 \\ -0.1171 & -0.3489 \end{bmatrix}$. We can check that X and F satisfy iii) of Lemma 1 which implies the closed-loop system (4) is mean-square stable with F. Note that in this example, $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$ is not positive-definite while the mean-square stabilizing optimal control signal exists. This illustrates that Assumption 1 is indeed not necessary.

Now consider another LTI system with $A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Clearly, $\left[\frac{A}{C}\right]$ is detectable. The design of the random input gain is the same as that in the previous example. Again, $\mathfrak{C} > h(A)$ and the allocation of the channel capacities is feasible. In this case, the iteration of $X_N(k)$ defined in (3) diverges, i.e., the optimal control signal does not exist. This example illustrates that the assumption on the detectability of $\left[\frac{A}{C}\right]$ is not sufficient for the existence of the mean-square stabilizing optimal control signal.

5 Conclusion

In this paper, the LQ optimal control of discrete-time LTI systems with random input gains is studied. Different from the finite-horizon case which can be solved by dynamic programming, the infinite-horizon case may be unsolvable if the capacities of the individual channels are fixed a priori. To tackle this difficulty, we put the problem under the framework of channel/controller co-design which allows the control designer to have the additional freedom to design the channels. We assume that the overall channel capacity is constrained and can be allocated to the individual channels. By channel/controller co-design, under certain assumptions on the system parameters, it is shown that the infinite-horizon LQ optimal control is solvable if and only if the overall capacity of the input channels is greater than the topological entropy of the open-loop plant. The mean-square stabilizing optimal control signal is a linear state feedback with the feedback gain associated to the MARE (6). In the future, we wish to find the minimal assumptions under which there exists the optimal control signal.

Appendix

Proof of Theorem 1

The proof is based on dynamic programming. Define the initial condition and cost-to-go function as

$$X_N(N+1) = 0, \; L(N+1) = 0,$$

$$L(k) = \min_{v(k)} \boldsymbol{E}\left[\begin{bmatrix} x(k) \\ K(k)v(k) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(k) \\ K(k)v(k) \end{bmatrix} + L(k+1)\right],$$

for $k = N, \ldots, 1, 0$.

First, it is easy to see that

$$L(N+1) = x'(N+1)X_N(N+1)x(N+1) = 0.$$

Next we will show that if $L(k+1)$ is a quadratic function of the state $x(k+1)$, i.e., $L(k+1) = x'(k+1)X_N(k+1)x(k+1)$, then $L(k)$ is also a quadratic function of the state $x(k)$. From the definition of $L(k)$, we have

$$L(k)=\min_{v(k)} \boldsymbol{E}\left[\begin{bmatrix} x(k) \\ K(k)v(k)\end{bmatrix}'\begin{bmatrix} Q & S \\ S' & R\end{bmatrix}\begin{bmatrix} x(k) \\ K(k)v(k)\end{bmatrix} + \begin{bmatrix} x(k) \\ K(k)v(k)\end{bmatrix}'\begin{bmatrix} A' \\ B'\end{bmatrix} X_N(k+1)\begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} x(k) \\ K(k)v(k)\end{bmatrix}\right].$$

After some calculations, we have

$$\begin{aligned}&\boldsymbol{E}[v'(k)K(k)RK(k)v(k)]\\ &\quad = v'(k)\boldsymbol{E}[K(k)RK(k)]v(k)\\ &\quad = v'(k)\boldsymbol{E}[MRM + (K(k)-M)R(K(k)-M)]v(k)\\ &\quad = v'(k)[MRM+\Sigma^2\odot R]v(k).\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\boldsymbol{E}[v'(k)K(k)B'X_N(k+1)BK(k)v(k)]\\ &= v'(k)[MB'X_N(k+1)BM+\Sigma^2\odot B'X_N(k+1)B]v(k).\end{aligned}$$

Therefore $L(k)$ becomes

$$\begin{aligned}L(k) =&\min_{v(k)}\left\{\begin{bmatrix} x(k) \\ Mv(k)\end{bmatrix}'\begin{bmatrix} Q & S \\ S' & R\end{bmatrix}\begin{bmatrix} x(k) \\ Mv(k)\end{bmatrix}\right.\\ &+\begin{bmatrix} x(k) \\ Mv(k)\end{bmatrix}'\begin{bmatrix} A' \\ B'\end{bmatrix} X_N(k+1)\begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} x(k) \\ Mv(k)\end{bmatrix}\\ &\left.+v'(k)[\Sigma^2\odot(R+B'X_N(k+1)B)]v(k)\right\}\\ =&\min_{v(k)}\left\{\begin{bmatrix} x(k) \\ Mv(k)\end{bmatrix}'\begin{bmatrix} T_1 & T_2' \\ T_2 & T_3\end{bmatrix}\begin{bmatrix} x(k) \\ Mv(k)\end{bmatrix}\right\}\\ =&\min_{v(k)}\{[v(k)-F(k)x(k)]'M'T_3M[v(k)-F(k)x(k)]\}\\ &+x'(k)(T_1-T_2'T_3^{-1}T_2)x(k),\end{aligned}$$

where

$$\begin{bmatrix} T_1 & T_2' \\ T_2 & T_3\end{bmatrix}=\begin{bmatrix} A'X_N(k+1)A+Q & A'X_N(k+1)B+S \\ B'X_N(k+1)A+S' & W\odot(B'X_N(k+1)B+R)\end{bmatrix}$$

and $F(k)=-M^{-1}T_3^{-1}T_2$. From this, we obtain the optimal control law:

$$\begin{aligned}v^{opt}(k)=&F(k)x(k)\\ =&-M^{-1}[W\odot(B'X_N(k+1)B+R)]^{-1}\\ &\times(B'X_N(k+1)A+S')x(k).\end{aligned}$$

Moreover, the cost-to-go function $L(k)$ is given by

$$\begin{aligned}L(k)&=x'(k)(T_1-T_2'T_3^{-1}T_2)x(k)\\ &=x'(k)X_N(k)x(k),\end{aligned}$$

where $X_N(k)$ is given by (3), as a result of which, the optimal control cost is given by

$$L(0)=x'(0)X_N(0)x(0).$$

The proof is completed. □

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