Linear Quadratic Optimal Control of Continuous-time LTI Systems with Random Input Gains

Wei Chen, *Member, IEEE*, and Li Qiu, *Fellow, IEEE*

***Abstract*—This note studies the linear quadratic (LQ) optimal control of continuous-time linear time-invariant (LTI) systems with random gains imposed on the input channels. We start from the indefinite LQ problem, in which the cost weighting matrix can be indefinite. The definite LQ problem is discussed as a special case. The main novelty originates from the point of view that in networked control, designing the channels and controller jointly often leads to an easier problem and achieves better performance than designing them separately. Specifically, we formulate the LQ problem as a channel/controller co-design problem assuming that the channel capacities can be allocated among the input channels subject to an overall capacity constraint. Necessary and sufficient conditions are obtained for the well-posedness and the attainability of the indefinite LQ problem under a given channel capacity allocation satisfying the stabilization requirement. The optimal controller is given by a linear state feedback associated with the mean-square stabilizing solution of a modified algebraic Riccati equation (MARE).**

***Index Terms*—Networked control system, LQ optimal control, channel/controller co-design, channel resource allocation, modified algebraic Riccati equation.**

I. Introduction

Recently, research progresses in different fields such as networked control systems (NCSs), financial engineering, and economic systems sparkle a common interest in a class of LTI systems with random input gains. Formulating and studying control problems for such systems is both theoretically interesting and practically useful. In particular, much effort has already been devoted to the stabilization problem, leading to a fundamental limitation on the channel quality required for state feedback stabilization [12], [25], [26]. In this work, we concentrate on the LQ optimal control of such systems which is of fundamental importance and serves as a starting point to investigate the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems.

As a motivating example, the NCSs with packet-dropping channels have been extensively studied in the literature, where the packet drop in the channels is modeled as a special type of random gain, namely, a Bernoulli process. The LQ optimal control of NCSs are often treated as part of the Linear Quadratic Gaussian (LQG) control. For instance, the works in [17], [22] study the LQG control of a multi-input-multi-output (MIMO) NCS with the measurements being transmitted via a single packet-dropping channel and the control signals being transmitted via another single packet-dropping channel. It is shown therein that the optimal stabilizing controller exists, if and only if the packet arrival rates in the input and output channels are larger than certain critical values, respectively. A similar approach is adopted in [14] to study the LQG control of MIMO NCSs over multiple parallel packet-dropping channels.

Realizing that the systems with random input gains are a special type of stochastic systems, we wish to review some results from stochastic LQ optimal control. The works in [3], [20] investigate the indefinite LQ optimal control of a stochastic system with scalar multiplicative state and control dependent noise for the finite-horizon case and infinite-horizon case, respectively. It is shown in [20] that the infinite-horizon indefinite LQ problem is solvable if and only if a non-standard algebraic Riccati equation has a mean-square (MS) stabilizing solution. Similar approaches can be found in [11], where the indefinite stochastic LQ optimal control with multidimensional state and control dependent noise is investigated.

Inspired by these results, we aim to study the LQ optimal control of continuous-time LTI systems with random input gains. We start from the general indefinite case where the cost weighting matrix can be indefinite. The definite LQ problem is then discussed as a special case. Partial results of this note have been reported in the conference papers [5], [6]. A parallel study of the discrete-time counterpart can be seen in [27], [28] which focus on the definite LQ problem. The main novelty of this work is to treat the LQ optimal control under the framework of channel/controller co-design. It is assumed that the controller designer also has the freedom to design the channels. Due to this additional design freedom, the objective becomes to simultaneously design the controller and channels so as to minimize the cost function while ensuring the stability of the closed-loop system. The well-posedness and attainability of the indefinite LQ problem concerned is nicely addressed under the channel/controller co-design. The optimal controller is given by a linear state feedback associated with the MS stabilizing solution of a MARE. Note that the channel/controller co-design framework was initiated in [19] to study the multi-input networked stabilization. Several other works have been carried out following this framework, e.g., [4], [5], [6], [25], [26], [27].

The remainder of this note is organized as follows. The LQ optimal control problem to be studied is formulated in Section II. Some preliminary knowledge is presented in Section III. Section IV investigates the well-posedness of the indefinite LQ problem. The attainability of a well-posed problem is studied in Section V, where the optimal controller and the minimum value of the cost function is obtained. The note is concluded in Section VI.

The notation in this note is more or less standard and will be made clear as we proceed. The symbol $\odot$ means Hadamard product. Denote by $\mathcal{S}_n$ the space of $n\times n$ real symmetric matrices. Denote by $\mathcal{S}_n^{\mathbb{C}}$ the complexification of $\mathcal{S}_n$, i.e., the space of $n\times n$ complex symmetric matrices. The spectrum of a linear operator $\mathscr{L}$ from $\mathcal{S}_n$ to $\mathcal{S}_n$ is defined to be $\sigma(\mathscr{L})=\{\lambda\in\mathbb{C}:\exists X\in\mathcal{S}_n^{\mathbb{C}}, X\neq 0, \mathscr{L}(X)=\lambda X\}$. The open (closed, respectively) left-half complex plane is denoted by $\mathbb{C}^-$ ($\mathbb{C}^{-,0}$, respectively).

II. Problem Formulation

Consider the following continuous-time LTI system with random input gains:

$$\dot{x}(t) = Ax(t) + B\kappa(t)u(t), x(0) = x_0, \tag{1}$$

where $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $x(t)\in\mathbb{R}^n$ is the system state, and $u(t)\in\mathbb{R}^m$ is the control input. A random gain matrix $\kappa(t)$ is imposed on the input channels, where $\kappa(t)=\operatorname{diag}\{\kappa_1(t),\kappa_2(t),\dots,\kappa_m(t)\}$ consists of diagonal random process elements $\kappa_i(t), i=1,2,\dots,m$. Consider the scenario when $\kappa_i(t)=\mu_i+\xi_i(t)$, where μ_i is a real positive constant and $\xi_i(t)$ is a zero-mean white noise with autocorrelation $\boldsymbol{E}[\xi_i(t)\xi_i(t+\tau)]=\sigma_i^2\delta(\tau)$. Assume that $\xi_i(t), i=1,2,\dots,m$, are mutually independent. The system (1) can now be written into the standard Itô form:

$$dx(t) = Ax(t)dt + \sum_{i=1}^{m} B_i u_i(t)(\mu_i dt + \sigma_i d\omega_i(t)),$$

where B_i is the ith column of B and $\omega_i(t), i=1,2,\dots,m$, are independent Wiener processes. Assume that $x(t)$ is available for feedback. The control signal u is generated by a causal state feedback controller $\boldsymbol{K}$, i.e., $u=\boldsymbol{K}x$, such that u is a measurable

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W. Chen and L. Qiu are with the Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China eeweichen@ust.hk, eeqiu@ust.hk

stochastic process satisfying $\boldsymbol{E}\int_0^\infty u'(t)u(t)dt < \infty$. In view of [11, p. 26], this setup ensures that there exists a unique solution to the stochastic differential equation (1). In general, the controller $\boldsymbol{K}$ can be nonlinear, time-varying, and dynamic. It has a state space realization as follows:

$$\begin{aligned}\dot{x}_{\boldsymbol{K}}(t) &= f(t, x_{\boldsymbol{K}}(t), x(t)),\\ u(t) &= g(t, x_{\boldsymbol{K}}(t), x(t)),\end{aligned}$$

where $x_{\boldsymbol{K}}(t)$ is the controller state whose dimension is not specified a priori, f is piecewise continuous in t and locally Lipschitz in $(x_{\boldsymbol{K}}, x)$, g is piecewise continuous in t and continuous in $(x_{\boldsymbol{K}}, x)$. Suppose $x_{\boldsymbol{K}}(t) = 0$ is the unique equilibrium point of the controller. Note that in this setup, the controller is allowed to have memory instead of being restricted to a function of the current state as in [10], [11], [27], [28]. Denote the overall system state by $x_c(t) = \begin{bmatrix} x(t) \\ x_{\boldsymbol{K}}(t)\end{bmatrix}$. A controller $\boldsymbol{K}$ is said to be MS stabilizing if for every initial condition $x_c(0)$, there holds $\lim_{t\to\infty} \boldsymbol{E}[x_c(t)x_c'(t)] = 0$.

Consider the input channels connecting the controller to the plant actuators. The signal-to-noise ratio of the ith channel is defined to be $\mathrm{SNR}_i = \frac{\mu_i}{\sigma_i}$. Accordingly, the MS capacity of the channel is defined to be $\mathfrak{C}_i = \frac{1}{2}\frac{\mu_i^2}{\sigma_i^2} = \frac{1}{2}\mathrm{SNR}_i^2$. The overall channel capacity, denoted as $\mathfrak{C}$, is then given by $\mathfrak{C} = \sum_{i=1}^m \mathfrak{C}_i$. It is clear that the capacity of an ideal channel without any transmission error is infinity. In general, larger capacity implies that more reliable information can be transmitted through the channel. For future use, let us define some notations:

$$\begin{aligned} M &= \mathrm{diag}\{\mu_1, \mu_2, \dots, \mu_m\},\\ \Sigma^2 &= \mathrm{diag}\{\sigma_1^2, \sigma_2^2, \dots, \sigma_m^2\},\\ W &= M^{-2}\Sigma^2 = \mathrm{diag}\{\mathrm{SNR}_1^{-2}, \mathrm{SNR}_2^{-2}, \dots, \mathrm{SNR}_m^{-2}\}.\end{aligned}$$

The systems described above are motivated from applications in different areas such as networked control [17], [22], [14], financial engineering [29] and economic systems [18], [23].

We study the LQ optimal control with random input gains. Given an initial state x_0, consider the following cost function:

$$J(x_0, u(t)) = \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ \kappa(t)u(t)\end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R\end{bmatrix} \begin{bmatrix} x(t) \\ \kappa(t)u(t)\end{bmatrix} dt, \quad (2)$$

where $\begin{bmatrix} Q & S \\ S' & R\end{bmatrix}$ is a real symmetric weighting matrix that can be indefinite. Rewriting $\kappa(t)u(t)dt$ as $Mu(t)dt + \Sigma d\omega(t)u(t)$ with $\omega(t) = \mathrm{diag}\{\omega_1(t), \omega_2(t), \dots, \omega_m(t)\}$ and applying Itô's formula [13], [9] yields

$$J(x_0, u(t)) = \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t)\end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R\end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t)\end{bmatrix} dt,$$

where E is an $m\times m$ matrix with all elements equal to 1. We are aware that a different cost function that replaces $\kappa(t)u(t)$ in (2) by $u(t)$ is used in many studies of LQ stochastic optimal control [10], [11], [20]. However, for the current problem, the cost function (2) is more suitable since $\kappa(t)u(t)$ is the real control signal received by the plant and, thus, the function (2) contains the real energy expenditure in stabilizing the system. The discrete-time version of this cost function is used in [14], [17], [22] for studies of LQG control over packet-dropping channels, where the random gains are specified as Bernoulli processes.

One traditional way to formulate the LQ optimal control problem is to fix the channel capacities a priori and then find a stabilizing controller such that $J(x_0, u(t))$ is minimized for every initial state x_0. However, fixing the channel capacities a priori may not be desirable since a stabilizing controller may not exist when some of the channel capacities are too small, let alone the optimal one.

In this paper, we adopt the channel/controller co-design framework which provides an effective way to overcome the above difficulty. Specifically, the channel capacities $\mathfrak{C}_i$ are not assumed to be given a priori. Instead, they can be allocated subject to an overall capacity constraint $\mathfrak{C}$. The allocation of the overall capacity to the individual channels, called channel resource allocation, can be formally given by a probability vector $\pi = \begin{bmatrix}\pi_1 & \pi_2 & \dots & \pi_m\end{bmatrix}'$, where $0 \le \pi_i \le 1$, $\sum_{i=1}^m \pi_i = 1$, such that $\mathfrak{C}_i = \pi_i\mathfrak{C}$. With the channel/controller co-design, we formulate the optimal control problem as to simultaneously design an allocation vector π and an optimal MS stabilizing controller $\boldsymbol{K}$ to minimize the cost function $J(x_0, u(t))$ for every initial state x_0.

We define the value function V under a feasible allocation π as

$$V(x_0) = \inf_{\boldsymbol{K}\text{ is MS stabilizing}} J(x_0, u(t)).$$

The indefinite LQ optimal control problem is said to be well-posed if $-\infty < V(x_0) < +\infty$ for all $x_0 \in \mathbb{R}^n$, otherwise, it is called ill-posed. A well-posed problem is said to be attainable if there exists an MS stabilizing controller, referred to as the optimal controller, that achieves the infimum.

Note that the optimal controller is now searched over a broad class of controllers that can be nonlinear, time-varying and dynamic. As will be seen through the subsequent investigations, the optimal controller is given by a static linear state feedback. This means that linearity is in fact a desired property of the optimal controller among a broad class of controllers.

Before proceeding, let us define the topological entropy of a linear system $\dot{x}(t) = Ax(t)$ with $A \in \mathbb{R}^{n\times n}$ as $H(A) = \sum_{\Re(\lambda_i)>0}\lambda_i$, where λ_i are the eigenvalues of A and $\Re(\lambda_i)$ stands for the real part of λ_i.

III. Preliminary

Consider the following linear stochastic system in Itô form:

$$dx(t) = Ax(t)dt + \sum_{i=1}^m A_i x(t) d\omega_i(t), \quad x(0) = x_0, \quad (3)$$

where $A, A_i \in \mathbb{R}^{n\times n}$ and $\omega_i(t)$ are independent Wiener processes for $i = 1, 2, \dots, m$.

Definition 1: The stochastic system (3) is said to be MS stable if for any initial state x_0, the matrix $\boldsymbol{E}[x(t)x'(t)]$ is well-defined for any $t > 0$ and $\lim_{t\to\infty}\boldsymbol{E}[x(t)x'(t)] = 0$.

Several criterions in verifying the MS stability are given below. Define a linear operator $\mathscr{L}$ from $\mathcal{S}_n$ to $\mathcal{S}_n$ as

$$\mathscr{L} : X \in \mathcal{S}_n \mapsto A'X + XA + \sum_{i=1}^m A_i' X A_i.$$

Lemma 1: The following assertions are equivalent:

(a) The stochastic system (3) is MS stable.
(b) $\sigma(\mathscr{L}) \subset \mathbb{C}^-$.
(c) There exists a matrix $X > 0$ such that $\mathscr{L}(X) < 0$.
(d) For an arbitrary $P \in \mathcal{S}_n$, there exists a unique $X \in \mathcal{S}_n$ such that $\mathscr{L}(X) + P = 0$. Moreover, if $P > 0$ (respectively, $P \ge 0$), then $X > 0$ (respectively, $X \ge 0$).

Proof: The equivalence of (a), (b), and (c) can be referred to [2]. The equivalence of (a) and (d) can be shown by applying Theorem A.1 in [15]. ■

Back to the system (1), as mentioned before, the controller $\boldsymbol{K}$ and the allocation vector π are to be jointly designed so as to stabilize the closed-loop system. When defining the MS stabilizability, we tacitly consider a static linear state feedback $u(t) = Fx(t)$ that leads to the closed-loop system

$$\dot{x}(t) = (A + B\kappa(t)F)x(t). \tag{4}$$

This system can be rewritten in the form of (3), if we replace A and A_i in (3) by $A+BMF$ and $\sigma_i B_i F_i$, respectively.

Definition 2: The system (1) is said to be MS stabilizable with capacity $\mathfrak{C}$ if there exist an allocation vector π and a state feedback gain F such that the closed-loop system (4) with $\mathfrak{C}_i = \pi_i \mathfrak{C}$ is MS stable.

The minimum total channel capacity rendering MS stabilization possible is given in the following lemma.

Lemma 2 ([25]): The system (1) is MS stabilizable with capacity $\mathfrak{C}$ if and only if $[A|B]$ is stabilizable and $\mathfrak{C} > H(A)$.

Remark 1: The implementation of the channel/controller co-design is also discussed in [25]. Briefly, such co-design is performed based on the Wonham decomposition that was first put forward in [24] to solve the multi-input pole-placement problem. For a stabilizable system $[A|B]$, the Wonham decomposition is of the form:

$$\left[\begin{bmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_m \end{bmatrix} \middle| \begin{bmatrix} b_1 & * & \cdots & * \\ 0 & b_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & b_m \end{bmatrix}\right],$$

where each pair $[A_i|b_i]$ is stabilizable. The topological entropy $H(A_i)$ of the ith subsystem can be regarded as a measure of its degree of instability and, consequently, determines the amount of resource that has to be allocated to the ith input channel for stabilization. More specifically, given a total capacity $\mathfrak{C} > H(A)$, a feasible allocation of $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_m$ such that $\sum_{i=1}^m \mathfrak{C}_i = \mathfrak{C}$ is to make $\mathfrak{C}_i > H(A_i)$. The channel/controller co-design is then carried out in a sequential way: choose $\mathfrak{C}_1 > H(A_1)$ and design f_1 so that the first input is used to stabilize all unstable modes controllable from the first input; choose $\mathfrak{C}_2 > H(A_2)$ and design f_2 so that the second input is used to stabilize the additional unstable modes controllable from the second input excluding the ones that are already stabilized by the first input; ...; finally, choose $\mathfrak{C}_m > H(A_m)$ and design f_m so that the last input can be used to stabilize the remaining unstable modes that are not stabilized by the other inputs.

Throughout the rest of this note, we always assume that the system (1) is MS stabilizable with capacity $\mathfrak{C}$.

IV. Well-posedness of Indefinite LQ Problem

In this section, we investigate the condition under which the indefinite LQ optimal control with random input gains is well-posed. The attainability is addressed in the next section.

First, we give a lemma that will be frequently used in later developments. It can be shown in analogy to Lemma 4 in [20]. The details of the proof are omitted for brevity.

Lemma 3: Let $x(t)$ be the solution of (1) corresponding to the control input $u(t)$. For a given $X \in \mathcal{S}_n$, it holds

$$\boldsymbol{E}\int_0^t \begin{bmatrix} x(\tau) \\ Mu(\tau) \end{bmatrix}' \begin{bmatrix} A'X+XA & XB \\ B'X & W\odot(B'XB) \end{bmatrix} \begin{bmatrix} x(\tau) \\ Mu(\tau) \end{bmatrix} d\tau = \boldsymbol{E}[x(t)'Xx(t)] - x_0'Xx_0,$$

for all $t \geq 0$.

The next lemma is useful to establish the upper-boundedness of the value function $V(x_0)$. Define a linear operator $\mathscr{L}_F$ from $\mathcal{S}_n$ to $\mathcal{S}_n$ as

$$\mathscr{L}_F : X \in \mathcal{S}_n \mapsto (A+BMF)'X + X(A+BMF) + F'(\Sigma^2 \odot (B'XB))F. \tag{5}$$

Also, denote

$$\Psi_F = \begin{bmatrix} I \\ MF \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R \end{bmatrix} \begin{bmatrix} I \\ MF \end{bmatrix}.$$

Lemma 4: Let $u(t) = Fx(t)$ be MS stabilizing under a feasible allocation π (see Remark 1). Then the corresponding cost is given by $J(x_0, u(t)) = x_0'Xx_0$, where $X \in \mathcal{S}_n$ is the unique solution to

$$\mathscr{L}_F(X) = -\Psi_F. \tag{6}$$

Proof: Since $u(t) = Fx(t)$ is MS stabilizing, in view of Lemma 1 (d), the matrix equation (6) has a unique solution $X \in \mathcal{S}_n$. By some simple calculations, we have

$$\begin{aligned} & J(x_0, u(t)) \\ &= \boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R \end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix} dt \\ &= \boldsymbol{E}\int_0^\infty x(t)'\Psi_F x(t) dt \\ &= -\boldsymbol{E}\int_0^\infty x(t)'\mathscr{L}_F^*(X)x(t) dt \\ &= -\boldsymbol{E}\int_0^\infty \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix}' \begin{bmatrix} A'X+XA & XB \\ B'X & W\odot(B'XB) \end{bmatrix} \begin{bmatrix} x(t) \\ Mu(t) \end{bmatrix} dt. \end{aligned}$$

Applying Lemma 3 yields

$$J(x_0, u(t)) = x_0'Xx_0 - \lim_{t\to\infty} \boldsymbol{E}[x(t)'Xx(t)] = x_0'Xx_0,$$

that completes the proof. ∎

Recall that the indefinite LQ problem of our concern is well-posed if $-\infty < V(x_0) < +\infty$ for every initial condition x_0. In view of Lemma 4, $V(x_0) < +\infty$ is automatically satisfied when $[A|B]$ is MS stabilizable with capacity $\mathfrak{C}$. However, since the cost weighting matrix can be indefinite, the boundedness from below, i.e., $V(x_0) > -\infty$, may not hold in general.

We attempt to find a theoretical condition under which the cost function can be bounded from below and, thus, the indefinite LQ problem is well-posed. For this purpose, the following lemma is needed which can be proved analogously to Theorem 2 in [7]. The details of the proof are omitted here for brevity.

Lemma 5: The indefinite LQ problem concerned is well-posed if and only if there exists a unique $X \in \mathcal{S}_n$ such that $V(x_0) = x_0'Xx_0$ for all x_0.

Define a linear operator $\mathscr{D}$ from $\mathcal{S}_n$ to $\mathcal{S}_n$ as

$$\mathscr{D} : X \in \mathcal{S}_n \mapsto W\odot(B'XB) + (W+E)\odot R.$$

We are now in a position to present the main theorem of this section which gives a necessary and sufficient condition for the well-posedness.

Theorem 1: Under a feasible allocation π, the indefinite LQ problem concerned is well-posed if and only if there exists $X \in \mathcal{S}_n$ satisfying the following LMI:

$$\begin{bmatrix} A'X + XA + Q & XB + S \\ B'X + S' & \mathscr{D}(X) \end{bmatrix} \geq 0. \tag{7}$$

Proof: To show the sufficiency, assume that there is a matrix $X \in \mathcal{S}_n$ satisfying the LMI (7). Then for any $u(t)$ generated by an

MS stabilizing controller, we have

$$\begin{aligned}&J(x_0,u(t))\\&=\boldsymbol{E}\int_0^\infty\begin{bmatrix}x(t)\\Mu(t)\end{bmatrix}'\begin{bmatrix}Q & S\\S' & (E+W)\odot R\end{bmatrix}\begin{bmatrix}x(t)\\Mu(t)\end{bmatrix}dt\\&\geq-\boldsymbol{E}\int_0^\infty\begin{bmatrix}x(t)\\Mu(t)\end{bmatrix}'\begin{bmatrix}A'X+XA & XB\\B'X & W\odot(B'XB)\end{bmatrix}\begin{bmatrix}x(t)\\Mu(t)\end{bmatrix}dt\\&=x_0'Xx_0-\lim_{t\to\infty}\boldsymbol{E}[x(t)'Xx(t)]=x_0'Xx_0,\end{aligned}$$

where the second equality follows from Lemma 3. This implies that $V(x_0)\geq x_0'Xx_0$ and, thus, the indefinite LQ problem is well-posed.

To show the necessity, by Lemma 5, if the indefinite LQ problem is well-posed, there exists a unique $X\in\mathcal{S}_n$ such that $V(x_0)=x_0'Xx_0$ for all x_0. By the knowledge of dynamic programming [1], it holds

$$\begin{aligned}x_0'Xx_0\leq\boldsymbol{E}\Bigg[&\int_0^t\begin{bmatrix}x(\tau)\\Mu(\tau)\end{bmatrix}'\begin{bmatrix}Q & S\\S' & (E+W)\odot R\end{bmatrix}\begin{bmatrix}x(\tau)\\Mu(\tau)\end{bmatrix}d\tau\\&+x(t)'Xx(t)\Bigg],\end{aligned}$$

for all $t\geq 0$ and any $u(t)$ generated by an MS stabilizing controller. This inequality together with Lemma 3 implies

$$\boldsymbol{E}\int_0^t\begin{bmatrix}x(\tau)\\Mu(\tau)\end{bmatrix}'\begin{bmatrix}A'X+XA+Q & XB+S\\B'X+S' & \mathscr{D}(X)\end{bmatrix}\begin{bmatrix}x(\tau)\\Mu(\tau)\end{bmatrix}d\tau\geq 0.$$

Dividing both sides by t and letting $t\to 0$ implies

$$\begin{bmatrix}x(0)\\Mu(0)\end{bmatrix}'\begin{bmatrix}A'X+XA+Q & XB+S\\B'X+S' & \mathscr{D}(X)\end{bmatrix}\begin{bmatrix}x(0)\\Mu(0)\end{bmatrix}\geq 0$$

for all $x(0)$ and $u(0)$. It follows that the matrix X satisfies the LMI (7). ■

V. Attainability of Indefinite LQ Problem

This section studies the attainability of the indefinite LQ optimal control problem. For simplicity, we confine our attention to a class of well-posed problems for which the LMI (7) has a solution X such that $\mathscr{D}(X)$ is nonsingular, or equivalently, the LMIs

$$\begin{cases}\begin{bmatrix}A'X+XA+Q & XB+S\\B'X+S' & \mathscr{D}(X)\end{bmatrix}\geq 0,\\ \mathscr{D}(X)>0.\end{cases}\tag{8}$$

has a solution. Similar to the treatment in [21], the approach presented here can be extended to address the general case which will be a bit more complex due to the possible singularity of $\mathscr{D}(X)$.

A. MARE

The following MARE plays an essential role in later developments:

$$A'X+XA+Q-(XB+S)\mathscr{D}(X)^{-1}(B'X+S')=0.\tag{9}$$

For a given solution X to this MARE, the associated state feedback gain is given by

$$F=-M^{-1}\mathscr{D}(X)^{-1}(B'X+S').\tag{10}$$

A solution X is said to be MS stabilizing (MS semi-stabilizing, respectively) if $\sigma(\mathscr{L}_F)\subset\mathbb{C}^-$ ($\sigma(\mathscr{L}_F)\subset\mathbb{C}^{-,0}$, respectively), where $\mathscr{L}_F$ is the linear operator (5) with F given by (10).

Remark 2: When $\mathfrak{C}=\infty$, the MARE (9) reduces to a standard continuous-time algebraic Riccati equation.

Define an operator $\mathscr{R}$ from $\mathcal{S}_n$ to $\mathcal{S}_n$ as:

$$\mathscr{R}\colon X\in\mathcal{S}_n\mapsto A'X+XA+Q-(XB+S)\mathscr{D}(X)^{-1}(B'X+S').$$

Then the MARE (9) can be rewritten in a compact form $\mathscr{R}(X)=0$. Also, define

$$\begin{aligned}\Omega&=\{X|X\in\mathcal{S}_n,\mathscr{R}(X)\geq 0,\mathscr{D}(X)>0\},\\ \Gamma&=\{X|X\in\mathcal{S}_n,\mathscr{R}(X)>0,\mathscr{D}(X)>0\}.\end{aligned}$$

By the knowledge of Schur complement [16], Ω is in fact the solution set to the LMIs (8). Apparently, $\Gamma\subset\Omega$. The LMIs (8) is said to be feasible if $\Omega\neq\emptyset$ and is said to be strictly feasible if $\Gamma\neq\emptyset$. The maximal solution to a feasible LMIs (8), denoted as X_+, is the maximal element in Ω in the sense that $X_+\geq X$ for all $X\in\Omega$. The maximal solution, if exists, is unique.

It is of particular interest to study the existence of the MS stabilizing solution to the MARE (9). Several useful lemmas are presented below. The proofs can be adapted from some other studies in the literature [8], [10], [11] wherein a general class of Riccati-type matrix equation has been studied.

Lemma 6: If $\Omega\neq\emptyset$ under a feasible allocation π, then the LMIs (8) has a maximal solution X_+. Moreover, X_+ is an MS semi-stabilizing solution to the MARE (9).

Lemma 7: If $\Omega\neq\emptyset$ under a feasible allocation π, then the MARE (9) has at most one MS stabilizing solution, which, if exists, coincides with X_+.

Lemma 8: If $\Omega\neq\emptyset$ under a feasible allocation π, then the MARE (9) has an MS stabilizing solution if and only if the LMIs (8) is strictly feasible, i.e., $\Gamma\neq\emptyset$.

Remark 3: With Lemma 6 and Lemma 7 in order, one can compute X_+ by solving the following convex optimization problem:

$$\begin{aligned}\max\quad &\operatorname{tr}(X),\\ \text{subject to}\quad &\text{constraints (8)}.\end{aligned}\tag{11}$$

and then check whether X_+ is MS stabilizing. If not, one is led to the conclusion that the MS stabilizing solution does not exist.

B. Attainability of Indefinite LQ Problem

We first give a result concerning the infimum cost $V(x_0)$, as shown below.

Theorem 2: If $\Omega\neq\emptyset$ under a feasible allocation π, then the value function is given by $V(x_0)=x_0'X_+x_0$ for all x_0, where X_+ is the maximal solution to the LMIs (8).

Proof: The existence of the maximal solution X_+ to the LMIs (8) is guaranteed by Lemma 6. Moreover, by the same procedure as in the sufficiency proof of Theorem 1, one can show that $V(x_0)\geq x_0'X_+x_0$.

Now it suffices to show $V(x_0)\leq x_0'X_+x_0$. Consider temporarily an indefinite LQ problem with a slightly modified cost function

$$\begin{aligned}J_\epsilon(x_0,u(t))&=\boldsymbol{E}\int_0^\infty\begin{bmatrix}x(t)\\\kappa(t)u(t)\end{bmatrix}'\begin{bmatrix}Q+\epsilon I & S\\S' & R\end{bmatrix}\begin{bmatrix}x(t)\\\kappa(t)u(t)\end{bmatrix}dt\\&=\boldsymbol{E}\int_0^\infty\begin{bmatrix}x(t)\\Mu(t)\end{bmatrix}'\begin{bmatrix}Q+\epsilon I & S\\S' & (E+W)\odot R\end{bmatrix}\begin{bmatrix}x(t)\\Mu(t)\end{bmatrix}dt,\end{aligned}$$

where ϵ is a small positive number. Choose an arbitrary $X\in\Omega$, then X satisfies the LMI

$$\begin{bmatrix}A'X+XA+Q+\epsilon I & XB+S\\B'X+S' & \mathscr{D}(X)\end{bmatrix}>0.$$

In view of Lemma 8, the MARE

$$A'X+XA+Q+\epsilon I-(XB+S)\mathscr{D}(X)^{-1}(B'X+S')=0$$

has a unique MS stabilizing solution X_ϵ with the associated state feedback gain given by $F_\epsilon=-M^{-1}\mathscr{D}(X_\epsilon)^{-1}(B'X_\epsilon+S')$. In addition, there holds $\mathscr{L}_{F_\epsilon}(X_\epsilon)=-\Psi_{F_\epsilon}$. It then follows from Lemma 4 that $\inf J_\epsilon(x_0,u(t))\leq x_0'X_\epsilon x_0$. It is easy to verify that X_ϵ is continuous with respect to ϵ. Therefore, taking the limit $\epsilon \to 0$ yields $X_\epsilon \to X_+$ and in addition, the infimum cost of the original LQ problem satisfies the inequality $V(x_0) \le x_0' X_+ x_0$ which completes the proof. ■

The next theorem establishes a necessary and sufficient condition for the attainability of the indefinite LQ optimal control with random input gains. Meanwhile, the optimal controller is obtained to be a static linear state feedback.

Theorem 3: If $\Omega \neq \emptyset$ under a feasible allocation π, then the indefinite LQ problem is attainable if and only if the MARE (9) has an MS stabilizing solution X. In this case, the optimal controller is uniquely given by a linear state feedback $u(t) = Fx(t)$, where F is the feedback gain (10) associated with X.

Proof: We first show the necessity. By Lemma 6, the LMIs (8) has a maximal solution X_+ that is also a solution to the MARE (9). Let $u^*(t)$ be generated by an optimal MS stabilizing controller and $x^*(t)$ be the corresponding plant state. Applying Lemma 3 yields

$$\begin{aligned} &V(x_0)\\ &=J(x_0, u^*(t))\\ &=\boldsymbol{E}\int_0^\infty \begin{bmatrix} x^*(t) \\ Mu^*(t) \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & (E+W)\odot R \end{bmatrix} \begin{bmatrix} x^*(t) \\ Mu^*(t) \end{bmatrix} dt\\ &=\boldsymbol{E}\int_0^\infty \begin{bmatrix} x^*(t) \\ Mu^*(t) \end{bmatrix}' \begin{bmatrix} A'X_+ + X_+A + Q & X_+B + S \\ B'X_+ + S' & \mathscr{D}(X_+) \end{bmatrix} \begin{bmatrix} x^*(t) \\ Mu^*(t) \end{bmatrix} dt\\ &\quad + x_0' X_+ x_0. \end{aligned}$$

By completing the squares, we have

$$\begin{aligned} V(x_0) =& \boldsymbol{E}\int_0^\infty (u^*(t) - F_+ x^*(t))' M \mathscr{D}(X_+) M (u^*(t) - F_+ x^*(t)) dt\\ &+ x_0' X_+ x_0, \end{aligned}$$

where F_+ is given by

$$F_+ = -M^{-1}\mathscr{D}(X_+)^{-1}(B'X_+ + S'). \quad (12)$$

Since $V(x_0) = x_0' X_+ x_0$ and $\mathscr{D}(X_+) > 0$, it follows that $u^*(t)$ is uniquely given by the feedback form $u^*(t) = F_+ x^*(t)$. Therefore, F_+ is MS stabilizing and, thus, X_+ is indeed the MS stabilizing solution to the MARE (9).

Now we show the sufficiency. Assume that the MS stabilizing solution to the MARE (9) exists. By Lemma 7, it coincides with the maximal solution X_+ to the LMIs (8). Hence, the linear state feedback controller $u(t) = F_+ x(t)$ with F_+ as in (12) is MS stabilizing. Also, it holds $\mathscr{L}_{F_+}(X_+) = -\Psi_{F_+}$. From Lemma 4, the corresponding cost is given by $J(x_0, u(t)) = x_0' X_+ x_0$. On the other hand, we know from Theorem 2 that the minimum cost of an attainable problem is given by $V(x_0) = x_0' X_+ x_0$. Therefore, the indefinite LQ problem is indeed attainable with the optimal controller being $u(t) = F_+ x(t)$. This completes the proof. ■

Remark 4: Theorem 2 indicates that the infimum cost for an unattainable problem takes the same form $V(x_0) = x_0' X_+ x_0$ as that for an attainable problem, except that this infimum is not achieved by any MS stabilizing controller. Furthermore, the proof there implies that although the optimal controller does not exist for an unattainable problem, one can always find MS stabilizing controllers to arbitrarily closely approach the infimum cost.

Remark 5: As in Theorem 3, the optimal controller for an attainable LQ problem is given by a linear state feedback associated with the MS stabilizing solution to the MARE (9). Such optimal design, apparently, takes the network effect into account. However, one might wish to assume that the network effect is not significant and then design an optimal controller assuming the channels are perfect. Unfortunately, when the controller designed this way is implemented with the network in place, the performance can be very bad. Therefore, the network effect has to be taken into account in networked optimal control.

Remark 6: When $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \ge 0$, the problem becomes a definite LQ optimal control problem. Assume that $R > 0$. Since $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \ge 0$, we have $X = 0 \in \Omega$. Then, Theorem 1 implies that the definite LQ problem is always well-posed. Moreover, by Lemma 6, the LMIs (8) has a maximal solution X_+ that is also an MS semi-stabilizing solution to the MARE (9). The attainability, however, is not so straightforward. A sufficient condition is found to be $\begin{bmatrix} Q & S \\ S' & (W+E)\odot R \end{bmatrix} > 0$. This can be easily shown by noting $X = 0 \in \Gamma$ and invoking Lemma 8. An immediate consequence of the above sufficient condition states that a definite LQ problem for a scalar system is always attainable provided $\mathfrak{C} < \infty$. However, for general multi-dimensional systems, such a sufficient condition seems quite strong. How to weaken this condition is under our current investigation. Numerically, one can compute X_+ by solving the convex optimization problem as in (11) and then check whether X_+ is MS stabilizing. If so, the LQ problem is attainable, otherwise, the MS stabilizing solution to the MARE (9) does not exist and the LQ problem is unattainable.

VI. Conclusion

In this note, the LQ optimal control of continuous-time LTI systems with random input gains is studied. We develop our main results for the indefinite case and then discuss the definite case as a special case. The main novelty of this work originates from the point of view that in networked control, designing the channels and controller jointly often leads to an easier problem and meanwhile achieves better performance than designing them separately. To be specific, we re-formulate the LQ optimal control as a channel/controller co-design problem assuming that the channel capacities can be allocated subject to an overall capacity constraint. With this channel/controller co-design, the well-posedness and attainability of the indefinite LQ problem concerned is nicely addressed. It is shown that the well-posedness of the indefinite problem is determined by the feasibility of the LMI (7). In addition, under certain mild assumptions, a well-posed problem is shown to be attainable if and only if the MARE (9) has an MS stabilizing solution. In that case, the optimal controller is given by a linear state feedback associated with the MS stabilizing solution.

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References

[1] D. P. Bertsekas, *Dynamic Programming and Optimal Control*, Athena Scientific, Belmont, Massachusetts, 2005.

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[3] S. Chen, X. Li, and X. Y. Zhou, "Stochastic linear quadratic regulators with indefinite control weight costs", *SIAM J. Contr. Optim.*, vol. 36, no. 5, pp. 1685-1702, 1998.

[4] W. Chen and L. Qiu, "Stabilization of networked control systems with multirate sampling", *Automatica*, vol. 49, no. 6, pp. 1528-1537, 2013.

[5] W. Chen and L. Qiu, "Continuous-time indefinite linear quadratic optimal control with random input gains", in *Proc. 12th European Control Conf.*, pp. 2392-2397, Zurich, Switzerland, 2013.

[6] W. Chen, J. Zheng, and L. Qiu, "Linear quadratic optimal control of continuous-time LTI systems with random input gains", in *Proc. 2012 IEEE Int. Conf. Information and Automation*, pp. 241-246, Shenyang, China, 2012.

[7] D. J. Clements, B. D. O. Anderson, and P. J. Moylan, "Matrix inequality solution to linear-quadratic singular control problems", in *IEEE Trans. Automat. Contr.*, vol. 22. pp. 55-57, Feb. 1977.

[8] O. L. V. Costa and R. P. Marques, "Maximal and stabilizing Hermitian solutions for discrete-time coupled algebraic Riccati equations", *Math. Control Signals Systems*, vol. 12, pp. 167-195, 1999.

[9] T. Damm, *Rational Matrix Equations in Stochastic Control*, Lecture Notes in Control and Information Sciences, vol. 297, Springer-Verlag, Berlin, 2004.

[10] V. Dragan, G. Freiling, A. Hochhaus, and T. Morozan, "A class of nonlinear differential equations on the space of symmetric matrices", *Electronic J. Differential Equations*, vol. 96, pp. 1-48, 2004.

[11] V. Dragan, T. Morozan, and A. M. Stoica, *Mathematical Methods in Robust Control of Linear Stochastic Systems*, Springer, New York, 2006.

[12] N. Elia, "Remote stabilization over fading channels", *Systems & Control Letters*, vol. 54, no. 3, pp. 237-249, 2005.

[13] A. Friedman, *Stochastic Differential Equations and Applications*, vol. 1, Academic Press, New York, 1975.

[14] E. Garone, B. Sinopoli, A. Goldsmith, and A. Casavola, "LQG control for MIMO systems over multiple erasure channels with perfect acknowledgment", *IEEE Trans. Automat. Contr.*, vol. 57, pp. 450-456, Feb. 2012.

[15] L. E. Ghaoui and M. A. Rami, "Robust state-feedback stabilization of jump linear systems via LMIs", *Int. J. Robust Nonlinear Contr.*, vol. 6, pp. 1015-1022, 1996.

[16] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[17] O. C. Imer, S. Yüksel, and T. Basar, "Optimal control of LTI systems over unreliable communication links", *Automatica*, vol. 42, no. 9, pp. 1429-1439, 2006.

[18] T. Katayama, "On the matrix Riccati equation for linear systems with a random gain", *IEEE Trans. Automat. Contr.*, vol. 21, pp. 770-771, Oct. 1976.

[19] L. Qiu, G. Gu, and W. Chen, "Stabilization of networked multi-input systems with channel resource allocation", *IEEE Trans. Automat. Contr.*, vol. 58, pp. 554-568, Mar. 2013.

[20] M. A. Rami and X. Zhou, "Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls", *IEEE Trans. Automat. Contr.*, vol. 45, pp. 1131-1143, June. 2000.

[21] M. A. Rami, X. Zhou, and J. B. Moore, "Well-posedness and attainability of indefinite stochastic linear quadratic control in infinite time horizon", *Systems & Control Letters*, vol. 41, no. 2, pp. 123-133, 2000.

[22] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks", *Proc. of IEEE*, vol. 95, pp. 163-187, Jan. 2007.

[23] F. R. Shupp, "Uncertainty and optimal policy intensity in fiscal and incomes policies", *Ann. Econ. Soc. Meas.*, vol. 5, pp. 225-237, 1976.

[24] W. M. Wonham, "On pole assignment in multi-input controllable linear systems", *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 660-665, Dec. 1967.

[25] N. Xiao and L. Xie, "Feedback stabilization over stochastic multiplicative input channels: continuous-time case", in *Proc. 11th Int. Conf. Contr., Automation, Robotics and Vision*, pp. 543-548, Singapore, 2010.

[26] N. Xiao, L. Xie, and L. Qiu, "Feedback stabilization of discrete-time networked systems over fading channels", *IEEE Trans. Automat. Contr.*, vol. 57, pp. 2176-2189, Sept. 2012.

[27] J. Zheng, W. Chen, L. Shi, and L. Qiu, "Linear quadratic optimal control for discrete-time LTI systems with random input gains", in *Proc. 31st Chinese Contr. Conf.*, pp. 5803-5808, Hefei, China, 2012.

[28] J. Zheng and L. Qiu, "Infinite-horizon linear quadratic optimal control for discrete-time LTI systems with random input gains", in *Proc. 2013 Amer. Control Conf.*, pp. 1197-1202, Washington, DC, USA, 2013.

[29] X. Y. Zhou and D. Li, "Explicit efficient frontier of a continuous-time mean-variance portfolio selection problem", in *Control of Distributed Parameter and Stochastic Systems*, S. Chen *et al.*, Eds. Dordrecht, the Netherlands: Kluwer, pp. 323-330, 1999.