MIMO Control Using MIMO Communication: A Majorization Condition for Networked Stabilizability

Wei Chen, Songbai Wang, and Li Qiu

***Abstract*— In this paper, we initiate the study of networked stabilization via a MIMO communication scheme between the controller and the plant. Specifically, the communication system is modeled as a MIMO transceiver, which consists of three parts: an encoder, a MIMO channel, and a decoder. In the spirit of MIMO communication, the number of SISO subchannels in the transceiver is often greater than the number of data streams to be transmitted. Moreover, the subchannel capacities are assumed to be fixed a priori. In this case, the encoder/decoder pair gives an additional design freedom on top of the controller, leading to a stabilization problem via coding/control co-design. The controller designer needs to design the encoder/decoder pair and the controller jointly so as to stabilize the system. We arrive at a necessary and sufficient condition on the subchannel capacities under which the coding/control co-design problem is solvable. Quite surprisingly, the condition is given in terms of a majorization type relation. As we go along, a systematic procedure is also put forward to perform the coding/control co-design. A numerical example is presented to illustrate our results.**

I. INTRODUCTION

As is well known, a MIMO control system refers to a multi-input multi-output physical system interconnected with a multi-input multi-output controller, while a MIMO communication system refers to a MIMO communication structure deployed to break the capacity limit of the conventional SISO communication scheme. What will happen if MIMO control meets MIMO communication? Inspired by this concern, we investigate in this paper a particular networked stabilization problem wherein a MIMO communication system is utilized to transmit the control signals.

Generally speaking, a networked control system (NCS) is a feedback system wherein the feedback loop is closed over a communication network. For a better understanding of the background, we briefly review the state of the art as below. It has been well recognized that in networked control, whether stabilization can be achieved or not critically depends on the information constraints in the communication network. As such, a primary concern of networked stabilization is to find a fundamental limitation on the information constraints so as to render stabilization possible. For a single-input system, the networked stabilization problem has been extensively studied under different information constraints. See [1], [16], [20] for data rate constraint, [7], [8] for quantization, [6] for fading effect, and [2] for signal-to-noise ratio constraint, etc. All these studies converge to a unified fundamental limitation on the information constraints required for stabilization given in terms of the topological entropy of the open-loop plant, i.e., the logarithm of the absolute product of unstable poles for a discrete-time plant, or the sum of the unstable poles for a continuous-time plant.

The story gets more complicated when it comes to multi-input systems. In many existing works, e.g., [8], [10], [21], a mere controller design problem is formulated assuming that the communication network is given a priori. It turns out that problems formulated in this way are usually very hard to solve. To mitigate this difficulty, the idea of channel resource allocation is proposed in [17] and followed by several other works such as [5], [23], etc. Specifically, it is assumed therein that the channel capacities can be allocated among different input channels subject to a total capacity constraint. This in turn results in a channel/controller co-design problem that is shown to be solvable, if and only if the total channel capacity is greater than the topological entropy of the open-loop plant. A similar idea, although not stated explicitly, can be seen in [19] which considers networked stabilization over parallel Gaussian channels subject to a total power constraint.

As remarked in the very beginning, one main motivation of this work is from the MIMO technology recently developed in communication theory. It has been widely used in wireless communication where spacial diversity can be exploited to increase the data transmission capacity. In this paper, we are driven to explore the potential advantanges of utilizing MIMO communication in networked control. In particular, we shall investigate the stabilization of an NCS wherein a MIMO transceiver is utilized to transmit the control signals. The MIMO transceiver has three parts: an encoder, a MIMO channel, and a decoder. One essential feature of MIMO communication is that the number of SISO subchannels in the transceiver is often greater than the number of data streams. When applied to networked control, this means that the number of subchannels is greater than the number of control inputs. We assume that the subchannel capacities are fixed a priori and, thus, cannot be freely allocated as in [17], [5], [23]. Nevertheless, the encoder/decoder pair now gives a substituted design freedom. The controller designer needs to design the encoder/decoder pair and the controller jointly so as to stabilize the system. This gives rise to a stabilization problem via coding/control co-design.

Quite surprisingly, a nice analytic solution is obtained for the solvability of the above coding/control co-design problem given explicitly in terms of a majorization type condition.

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The authors are with the Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China eeweichen@ust.hk, swangas@ust.hk, eeqiu@ust.hk

Majorization is a rather old topic in mathematics [11] and has been frequently used in statistics in the past 100 years. However, its engineering applications only appeared recently, notably in wireless communication, information theory, operations research, and smart grid, etc. The application of majorization in control theory remains quite scattered in the literature. One relevant work can be seen in [13], in which majorization is utilized to investigate remote state estimation with communication costs.

The rest of this paper is organized as follows. Section II formulates the problem to be studied. Section III provides some preliminary knowledge. The main result of this work is presented in Section IV. A numerical example is worked out in Section V. Finally, some concluding remarks follow in Section VI. Most notations in this paper are more or less standard and will be made clear as we proceed.

II. Problem Formulation

Consider the NCS depicted in Fig. 1. Here, the plant $[A|B]$ is a continuous-time linear time invariant system described by the state space model

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume that $[A|B]$ is unstable but stabilizable. Let the state $x(t)$ be available for feedback. If the communication network between the controller and the plant is ideal, i.e., $u(t) = v(t)$, one can easily design a state feedback controller $v(t) = Fx(t)$ so that the closed-loop system is stable. However, such state feedback design faces challenges when the communication network is not ideal, i.e., $u(t)$ is only a distorted version of $v(t)$. In this case, the achievability of stabilization will depend on the transmission accuracy of the communication network. In fact, a general concern of networked stabilization is to find a fundamental limitation on the quality of the communication network so as to render stabilization possible.

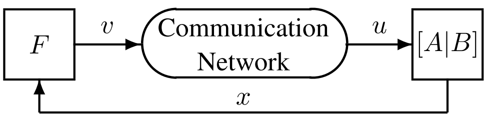

Fig. 1. State feedback via communication network.

Notice that almost all existing studies, for instance, [5], [8], [10], [17], [19], [21], [23], assume a SISO communication scheme between the controller and the plant, i.e., each control input is to be transmitted through a dedicated SISO channel. As remarked before, one main motivation of this work arises from the MIMO technology recently developed in communication theory. A typical MIMO communication system is shown in Fig. 2, which is also called a MIMO transceiver. Here the system between q and p is called a MIMO channel characterized by a channel matrix H and an additive white Gaussian noise d. The communication engineers are dedicated to designing the transmitter matrix T, also called an encoder, and the receiver matrix R, also called a decoder, so as to make the received signal u approximate the transmitted signal v as accurately as possible. Note that the MIMO transceiver is often built in such a way that the dimensions of q and p are much higher than the dimension of v and u. In connection with the NCS as in Fig. 1, we ask out of curiosity the following questions: What will happen if MIMO communication is used in networked control? Does it offer new advantages? Does it lead to new design flexibilities?

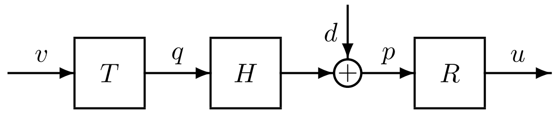

Fig. 2. MIMO transceiver, a typical MIMO communication system.

We are also motivated by the following concern. Recall that the channel resource allocation as in [17], [5], [23] is based on the crucial assumption that the channel capacities can be allocated among different input channels subject to a total capacity constraint. What if the individual channel capacities are indeed given a priori and not allocatable? In that case, is it possible to explore some other design freedoms so as to stabilize the NCSs?

Both motivations lead us to the following problem. Instead of using a SISO communication scheme as in the literature, we use a MIMO transceiver as shown in Fig. 2 to transmit the control signals. For simplicity, we assume that the channel matrix H in the transceiver is identity. In fact, all the developments can be extended straightforwardly to the case of a known nonsingular H. When H is the identity, the MIMO channel in the transceiver becomes a collection of l parallel SISO subchannels. To keep the essence of MIMO technology, we assume that the number of SISO subchannels in the transceiver is greater than or equal to the number of data streams to be transmitted, i.e., $l \geq m$. Later we will see that $l < m$ may also be valid in some cases. The encoder matrix $T \in \mathbb{R}^{l\times m}$ and the decoder matrix $R \in \mathbb{R}^{m\times l}$ are free to be designed subject to a mild constraint $RT = I$. The current communication system is shown in Fig. 3.

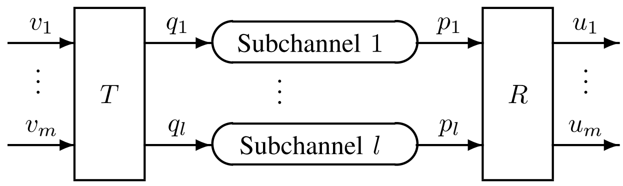

Fig. 3. A MIMO transceiver as a MIMO communication system in MIMO control.

In this paper, each SISO subchannel is modeled as an AWGN channel, as shown in Fig. 4, where d_i is a zero-mean white Gaussian noise with power spectral density N_i. The channel input signal q_i is a stationary process satisfying the power constraint:

$$\boldsymbol{E}[q_i^2] < P_i, \tag{1}$$

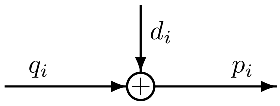

Fig. 4. An AWGN subchannel.

with some predetermined admissible power level $P_i > 0$. The capacity of such an AWGN channel with input power constraint P_i is defined as

$$\mathfrak{C}_i = \frac{1}{2}\frac{P_i}{N_i}. \tag{2}$$

Here we do not assume any kind of monotonicity among the subchannel capacities $\mathfrak{C}_i, i = 1, 2, \ldots, l$. The total channel capacity is given by the sum of all the SISO subchannel capacities, i.e., $\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots + \mathfrak{C}_l$.

Due to the predetermined admissible power levels, the subchannel capacities are now fixed a priori and thus, cannot be freely allocated among the subchannels as in [17], [5], [23] any more. Nevertheless, the encoder matrix T and the decoder matrix R are free to be designed. With this additional design freedom, the controller designer has to simultaneously design the controller and the encoder/decoder pair so as to stabilize the system subject to the input power constraints (1). This gives rise to a stabilization problem via coding/control co-design.

Now consider the closed-loop system as in Fig. 5 with a state feedback gain F such that $A + BF$ is stable. Assume that the closed-loop system has reached its steady state and all the signals are wide sense stationary. According to our setup, the total noise $d = \begin{bmatrix} d_1 & d_2 & \cdots & d_l \end{bmatrix}'$ is a vector white Gaussian noise with power spectral density

$$N = \mathrm{diag}\{N_1, N_2, \ldots, N_l\}.$$

The complementary sensitivity function, i.e., the closed-loop transfer function from d to q, is given by

$$\boldsymbol{T}(s) = TF(sI - A - BF)^{-1}BR. \tag{3}$$

Then, the power spectrum density of q_i has the expression

$$\{\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*\}_{ii},$$

where $\{\cdot\}_{ii}$ stands for the ith diagonal element of a matrix. Consequently, the power of q_i is given by

$$\boldsymbol{E}[q_i^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*\}_{ii}d\omega.$$

It follows that the input power constraint (1) can be rewritten as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\{\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*\}_{ii}d\omega < P_i.$$

In view of (2), such constraint can be further translated into

$$\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{N^{-\frac{1}{2}}\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^*N^{-\frac{1}{2}}\right\}_{ii}d\omega < \mathfrak{C}_i. \tag{4}$$

The objective of this study is to find requirements on the given subchannel capacities $\mathfrak{C}_i, i = 1, 2, \ldots, l$, such that the networked stabilization can be accomplished subject to the constraints (4) via a judicious coding/control co-design. We also wish to come up with a systematic procedure on how to jointly design the controller and the encoder/decoder pair.

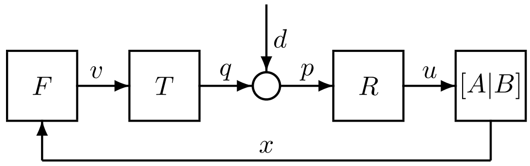

Fig. 5. NCS with MIMO communication.

Note that the subchannel capacity defined as above appears the same as the Shannon capacity of an infinite-bandwidth AWGN channel. However, it has been recognized that the Shannon capacity is in general not enough to characterize the information requirement for channels in a feedback system due to the causality constraint in the information processing in a feedback loop. Recently, several attempts have been made to define a suitable capacity for channels in a feedback system from an information-theoretic point of view. See for instance [18], [15].

Before proceeding, let us recall the notion of topological entropy [1], [2], [5], [19] of a continuous-time linear system $\dot{x}(t) = Ax(t)$, which is defined as the quantity $H(A) = \sum_{\Re(\lambda_i)>0}\lambda_i$, where λ_i are the eigenvalues of A.

III. Preliminary

For preparation, some preliminary knowledge is presented in this section.

A. Cyclic decomposition

Let A be an $n \times n$ real matrix. The minimal polynomial of A is the monic polynomial $\alpha(\lambda)$ of least degree such that $\alpha(A) = 0$. The minimal polynomial of a matrix is unique. We say that A is cyclic if its minimal polynomial has degree n, or equivalently, its minimal polynomial coincides with its characteristic polynomial. The following lemma gives the cyclic decomposition of a linear system, which plays an essential role in later developments.

Lemma 1: Given a stabilizable linear system $[A|B]$ with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, there exist nonsingular matrices P and Q such that

$$\begin{aligned}&[P^{-1}AP|P^{-1}BQ]\\ &= \left[\left.\begin{bmatrix} A_1 & 0 & \cdots & 0\\ 0 & A_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & A_k\end{bmatrix}\right| \begin{bmatrix} b_1 & * & \cdots & * & *\\ 0 & b_2 & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & * & *\\ 0 & \cdots & 0 & b_k & *\end{bmatrix}\right],\end{aligned} \tag{5}$$

where $A_i, i = 1, 2, \ldots, k$, are cyclic with minimal polynomials $\alpha_i(\lambda)$, such that $\alpha_1(\lambda) = \alpha(\lambda)$ and $\alpha_{i+1}(\lambda)|\alpha_i(\lambda)$ for $i = 1, 2, \ldots, k-1$. Moreover, the cyclic subsystems $[A_i|b_i], i = 1, 2, \ldots, k$, are stabilizable.

For the details, one can refer to [9] for the cyclic decomposition of a matrix and [22] for the cyclic decomposition of a linear system. Note that the number k as in Lemma 1 is referred to as the cyclic index of A and is unique. The minimal polynomials $\alpha_i(\lambda), i = 1, 2, \ldots, k$, are also unique. In addition, from the relation $\alpha_{i+1}(\lambda)|\alpha_i(\lambda)$, it follows that the spectrum of A_{i+1} is contained in the spectrum of A_i. Consequently, there naturally holds $H(A_1) \geq H(A_2) \geq \cdots \geq H(A_k)$.

Remark 1: The role of nonsingular matrices P and Q as in Lemma 1 can be considered as linear transformations in the state space and input space, respectively. The following implication can be inferred from Lemma 1. In the cyclic decomposition (5), A_1 contains the greatest number of unstable eigenvalues of A that can be stabilized by a single input up to linear transformations in the input space; likewise, A_1 together with A_2 contains the greatest number of unstable eigenvalues of A that can be stabilized by two inputs up to linear transformations in the input space; and so on so forth.

B. Optimal complementary sensitivity

Consider the complementary sensitivity function as in (3). Assume temporarily that $l = m$ and $T = R = I$, i.e., the encoder and decoder are simply trivial, then

$$\boldsymbol{T}(s) = F(sI - A - BF)^{-1}B.$$

The following lemma gives a solution to $\mathcal{H}_2$ optimal $\boldsymbol{T}(s)$.

Lemma 2 ([4]): There holds

$$\inf_{F: A+BF \text{ is stable}} \|\boldsymbol{T}(s)\|_2 = [2H(A)]^{\frac{1}{2}}.$$

Moreover, when A has no eigenvalues on the imaginary axis, the infimum can be achieved by the optimal state feedback gain $F = -B'X$, where X is the stabilizing solution to the algebraic Riccati equation

$$A'X + XA - XBB'X = 0.$$

C. Majorization

For $x, y \in \mathbb{R}^n$, we denote by $x^\uparrow$ and $y^\uparrow$ the rearranged versions of x and y so that their elements are arranged in a non-decreasing order. We say that x is majorized by y, denoted by $x \preccurlyeq y$, if

$$\begin{aligned} x_1^\uparrow &\geq y_1^\uparrow \\ x_1^\uparrow + x_2^\uparrow &\geq y_1^\uparrow + y_2^\uparrow \\ &\vdots \\ x_1^\uparrow + x_2^\uparrow + \cdots + x_{n-1}^\uparrow &\geq y_1^\uparrow + y_2^\uparrow + \cdots + y_{n-1}^\uparrow \\ x_1^\uparrow + x_2^\uparrow + \cdots + x_n^\uparrow &= y_1^\uparrow + y_2^\uparrow + \cdots + y_n^\uparrow. \end{aligned} \tag{6}$$

The physical interpretation of majorization is quite interesting. It orders the level of fluctuations when the averages are the same. In other words, $x \preccurlyeq y$ says that the elements of x are more even or, less spread out, than the elements of y.

Now, if the last equality in (6) is changed to an inequality $\geq$, then x is said to be weakly majorized by y from above, denoted by $x \preccurlyeq^w y$. Furthermore, if all the inequalities $\geq$ in (6), including the last equality, are changed to strict inequalities $>$, then x is said to be strictly weakly majorized by y from above, denoted by $x \prec^w y$. Note that when two vectors are compared via majorization or weak majorizations, the order of the elements in the vectors is irrelevant.

The following lemma characterizes the relation between majorization and weak majorization.

Lemma 3 ([14]): $x \preccurlyeq^w y$ ($x \prec^w y$, respectively), if and only if there exists z such that $x \geq z$ ($x > z$, respectively), and $z \preccurlyeq y$.

Another useful lemma is given below.

Lemma 4 ([14]): There exists a real symmetric matrix X with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and diagonal elements $d_1, d_2, \ldots, d_n$, if and only if

$$\begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix}' \preccurlyeq \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}'.$$

When the majorization condition in Lemma 4 is satisfied, efficient algorithms for finding the desired symmetric matrix X have also been developed in the literature. See for example [3].

IV. Main Result

The main result of this paper is presented in the following theorem. The necessity proof is omitted due to page limit. The sufficiency proof is constructive and gives a systematic way to perform the coding/control co-design.

Theorem 1: $[A|B]$ is stabilizable via MIMO communication over AWGN subchannels, if and only if

$$\begin{aligned} &\begin{bmatrix} \mathfrak{C}_1 & \mathfrak{C}_2 & \cdots & \mathfrak{C}_l \end{bmatrix}' \\ &\prec^w \begin{bmatrix} H(A_1) & H(A_2) & \cdots & H(A_k) & 0 & \cdots & 0 \end{bmatrix}', \end{aligned} \tag{7}$$

where $H(A_i), i = 1, 2, \ldots, k$, are the topological entropies of the cyclic subsystems $[A_i|b_i]$ as in (5).

Proof of Sufficiency: For brevity, assume that all the eigenvalues of A lie in the open right half complex plane. This assumption can be removed following the same arguments as in [2], [5], [17], [23].

To show the sufficiency, we will seek a state feedback gain F together with an encoder matrix T and a decoder matrix R such that the NCS is stabilized and the constraints (4) are satisfied. Without loss of generality, assume that $[A|B]$ is in the cyclic decomposition form (5), where each cyclic subsystem $[A_i|b_i], i = 1, 2, \ldots, k$, is stabilizable with state dimension n_i. For each $[A_i|b_i]$, we can design a stabilizing state feedback gain f_i such that $\|\boldsymbol{T}_i(s)\|_2^2 = 2H(A_i)$, where

$$\boldsymbol{T}_i(s) = f_i(sI - A_i - b_i f_i)^{-1} b_i. \tag{8}$$

The existence of such f_i is guaranteed by Lemma 2. Let $f = \operatorname{diag}\{f_1, f_2, \ldots, f_k\}$ and design $F = \begin{bmatrix} f \\ 0_{(m-k)\times n} \end{bmatrix}$. It is easy to verify that F is stabilizing, i.e., $A + BF$ is stable. Regarding the encoder/decoder pair, let

$$T = N^{\frac{1}{2}} U D^{-1}, \text{ and } R = D U' N^{-\frac{1}{2}}, \tag{9}$$

where $U \in \mathbb{R}^{l\times m}$ is an isometry to be designed and $D = \operatorname{diag}\{1, \epsilon, \ldots, \epsilon^{m-1}\}$ with ϵ being a small positive number. Also set $S = \operatorname{diag}\{I_{n_1}, \epsilon I_{n_2}, \ldots, \epsilon^{k-1} I_{n_k}\}$. Then

$$\begin{aligned} \boldsymbol{T}(s) &= TF(sI - A - BF)^{-1}BR \\ &= N^{\frac{1}{2}} U \bar{F}(sI - \bar{A} - \bar{B}\bar{F})^{-1} \bar{B} U' N^{-\frac{1}{2}}, \end{aligned}$$

where

$$\bar{F} = D^{-1}FS = F,$$

$$\bar{A} = S^{-1}AS = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_k \end{bmatrix},$$

$$\bar{B} = S^{-1}BD = \begin{bmatrix} b_1 & o(\epsilon) & \cdots & o(\epsilon) & o(\epsilon) \\ 0 & b_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & o(\epsilon) & o(\epsilon) \\ 0 & \cdots & 0 & b_k & o(\epsilon) \end{bmatrix},$$

and $\frac{o(\epsilon)}{\epsilon}$ approaches to a finite constant as $\epsilon \to 0$. It follows that

$$\begin{aligned} &\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty} N^{-\frac{1}{2}}\boldsymbol{T}(j\omega)N\boldsymbol{T}(j\omega)^* N^{-\frac{1}{2}} d\omega \\ &= U\left(\mathrm{diag}\left\{\frac{\|\boldsymbol{T}_1(s)\|_2^2}{2}, \ldots, \frac{\|\boldsymbol{T}_k(s)\|_2^2}{2}, 0, \ldots, 0\right\} + o(\epsilon)\right)U' \\ &= U\left(\mathrm{diag}\{H(A_1), \ldots, H(A_k), 0, \ldots, 0\} + o(\epsilon)\right)U'. \end{aligned} \tag{10}$$

When the relation (7) holds, by Lemma 3, there exists a vector $\begin{bmatrix}\gamma_1 & \gamma_2 & \ldots & \gamma_l\end{bmatrix}'$ such that

$$\begin{bmatrix}\mathfrak{C}_1 & \mathfrak{C}_2 & \ldots & \mathfrak{C}_l\end{bmatrix}' > \begin{bmatrix}\gamma_1 & \gamma_2 & \ldots & \gamma_l\end{bmatrix}', \tag{11}$$

and

$$\begin{aligned} &\begin{bmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_l\end{bmatrix}' \\ &\quad \preccurlyeq \begin{bmatrix}H(A_1) & H(A_2) & \cdots & H(A_k) & 0 & \cdots & 0\end{bmatrix}'. \end{aligned} \tag{12}$$

Further, in view of (12) and Lemma 4, an isometry U can be constructed such that

$$\{U\left(\mathrm{diag}\{H(A_1), \ldots, H(A_k), 0, \ldots, 0\}\right)U'\}_{ii} = \gamma_i, \tag{13}$$

for $i = 1, 2, \ldots, l$. Finally, putting (10), (11), and (13) together, we can verify that the constraints (4) are satisfied when ϵ is sufficiently small. This completes the proof. ■

Let us now revisit the coding/control co-design from a more intuitive perspective. In order to stabilize the NCS, each control input requires certain communication resource for transmission. As such, the control inputs can be regarded as the demand side for communication resource, while the SISO subchannels in the transceiver can be regarded as the supply side. The supply capabilities of the subchannels are characterized by their capacities and are fixed a priori. The challenge lies in the fact that the demand and supply may not match in general. To resolve such demand/supply imbalance, the coding mechanism plays a crucial role by shaping the demands judiciously so as to match the supplies. For comparison, the channel/controller co-design utilized in [17], [5], [23] does the exact opposite, i.e., tailoring the supplies so as to match the demands. It is worth mentioning that demand shaping is a quite general principle in economics. It has led to many successful stories in engineering fields as well such as power systems, transportation, and data networks, etc.

What follows is an important implication from Theorem 1. Note that we initially assume that the number of SISO subchannels in the MIMO transceiver is greater than or equal to the number of data streams to be transmitted, i.e, $l \geq m$. It turns out that in some cases, it may also be possible to stabilize the NCS with a smaller number of SISO subchannels than the number of data streams. This can be inferred from the majorization type condition (7). In fact, the minimum number of SISO subchannels needed for stabilization is equal to the number of unstable cyclic subsystems $[A_i|b_i]$ yielded from the cyclic decomposition (5). This observation is consistent with earlier studies [12], [22] in the literature that investigate the minimum number of control inputs required to stabilize a linear system. In that aspect, our result strengthens those studies by indicating a fundamental limitation on the information constraints required for networked stabilization given in terms of a majorization type relation.

One can further deduce the following two corollaries from Theorem 1. The proofs are omitted here for brevity.

Corrollary 1: If the cyclic decomposition of A has only one unstable cyclic block, i.e., A_1, then $[A|B]$ is stabilizable via MIMO communication over AWGN subchannels if and only if $\mathfrak{C} > H(A)$.

Corollary 1 is consistent with the result obtained in [19]. Also note that Corollary 1 includes the single-input system as a special case since a stabilizable single-input system only has one unstable cyclic subsystem. Therefore, this corollary suggests that in stabilizing a single-input system via MIMO communication, we only require the total capacity to be greater than the topological entropy of the open-loop plant. How the individual subchannel capacities are distributed does not matter in this case.

Corrollary 2: If $\mathfrak{C}_1 = \mathfrak{C}_2 = \cdots = \mathfrak{C}_l$, then $[A|B]$ is stabilizable via MIMO communication over AWGN subchannels if and only if $\mathfrak{C} > H(A)$.

Corollary 2 somehow suggests that identical subchannels can best help each other in transmitting the signals.

V. Example

Consider the following unstable system $[A|B]$:

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $[A|B]$ is stabilizable. Moreover, it is already in the cyclic decomposition form (5) with cyclic subsystems

$$[A_1|b_1] = \left[\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \text{ and } [A_2|b_2] = [1|1].$$

It follows that $H(A_1) = 4 + 2 + 1 = 7$, and $H(A_2) = 1$.

Consider a MIMO transceiver with three subchannels. Let the noise power spectral density be $N = I$. The admissible transmission power levels in the subchannels are given by

$$P_1 = 9.1, \; P_2 = 3.1, \; \text{and } P_3 = 4.1.$$

In view of (2), the subchannel capacities are

$$\mathfrak{C}_1 = 4.55,\ \mathfrak{C}_2 = 1.55,\ \text{and}\ \mathfrak{C}_3 = 2.05.$$

One can now verify that the majorization type relation as in (7) holds and, thus, the networked stabilization can be accomplished via certain coding/control co-design. One such co-design is carried out as below.

For the controller design, we solve the $\mathcal{H}_2$ optimal $\boldsymbol{T}_i(s)$ as in (8) for each cyclic subsystem $[A_i|b_i], i = 1, 2$, yielding the optimal feedback gains $f_1 = \begin{bmatrix}-40 & 36 & -10\end{bmatrix}$ and $f_2 = -2$, respectively. Let

$$F = \operatorname{diag}\{f_1, f_2\} = \begin{bmatrix} -40 & 36 & -10 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

For the coding design, let the encoder matrix T and the decoder matrix R be as in (9) with

$$U = \begin{bmatrix} 0.7817 & 0.4714 \\ 0.4629 & 0 \\ -0.4179 & 0.8819 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

With this co-design, we observe that the closed-loop poles are exactly the mirror images of the open-loop poles with respect to the imaginary axis. This validates the stability of the closed-loop system. Moreover, further computation yields

$$\begin{aligned} \boldsymbol{E}[q_1^2] &= 9.0848 < P_1, \\ \boldsymbol{E}[q_2^2] &= 3.0299 < P_2, \\ \boldsymbol{E}[q_3^2] &= 4.0249 < P_3, \end{aligned}$$

i.e., the input power constraints (1) are satisfied. Combining these two observations, we see that the networked stabilization is accomplished via this coding/control co-design.

VI. Conclusion

This paper investigates the stabilization of an NCS wherein the communication system between the controller and the plant is modeled as a MIMO transceiver. The capacities of the subchannels in the MIMO transceiver are fixed a priori and, thus, not allocatable. Nevertheless, the encoder/decoder pair provides an additional design freedom on top of the controller, leading to a stabilization problem via coding/control co-design.

It is shown that such coding/control co-design problem is solvable, if and only if the majorization type condition (7) is satisfied. The condition (7) relates the subchannel capacities required for stabilization to the topological entropies of the cyclic subsystems of the open-loop plant via a majorization type relation. This, on the other hand, gives an application of majorization in control theory. When the relation (7) holds, a systematic procedure is also put forward to carry out the coding/control co-design.

In this paper, the subchannels in the MIMO transceiver are modeled as AWGN channels. The idea can be extended to handle other channel models as well. In the future, we wish to find more connections between communication theory and control theory. Also, more applications of majorization in control theory are to be explored.

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