NETWORKED ROBUST STABILITY FOR LTV SYSTEMS WITH SIMULTANEOUS UNCERTAINTIES IN PLANT, CONTROLLER AND COMMUNICATION CHANNELS[*]

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Abstract. In this paper, we study the robust stability of a networked control system (NCS) under the framework of infinite-dimensional discrete-time linear time-varying (LTV) systems. The NCS consists of a pair of uncertain plant and controller, as well as an uncertain bilateral communication channel in between. The uncertainties in the plant and controller are measured by the gap metric. The communication channel between the plant and controller is described by a cascade of two-port networks whose transmission matrices are subject to norm bounded additive uncertainties. Such an uncertain two-port network can model distortions and interferences occurring during control and measurement signal transmissions. The causality of the LTV subsystems is characterized by using nest algebras. A necessary and sufficient condition for the robust stability of the NCS, with the causality of all system components explicitly considered, is established in the form of an arcsine inequality, which generalizes a similar result for linear time-invariant NCSs.

Key words. networked control system, robust stability, two-port network, gap metric, linear time-varying system

AMS subject classifications. 93B28, 93C05, 93C25, 93D09, 93D25

1. Introduction.

Robust stability of feedback systems has attracted a considerable amount of attention over the past few decades. In networked control systems (NCSs), due to the presence of distortions and interferences in the signal transmission, the uncertainties exist not only in modeling the plants and controllers but also in the communication channels in between. Hence the study of robust stability of such NCSs poses new challenges. In this paper, we study robust stability of NCSs under the framework of discrete-time linear time-varying (LTV) systems. The uncertainties in the plant and controller are measured by the gap metric. The bilateral communication channel between the plant and controller is described as a cascade of two-port networks whose transmission matrices are subject to norm bounded additive uncertainties. The causality of the LTV subsystems is characterized by using nest algebras.

The gap metric was initially introduced to control literature for the study of robust control of linear time-invariant (LTI) systems by Zames and El-Sakkary [41]. It was shown a few years later by Georgiou [21] that the gap metric is computable exactly in terms of standard "two-block" H_∞ optimization problems. Based on this computation result, a rather comprehensive analysis and synthesis theory was developed by Georgiou and Smith in [22]. The LTI gap metric and its variants, as well as the associate robust control theory, have also been extensively studied in the last three decades [21, 22, 25, 32, 33, 35, 36, 37]. In terms of simultaneous uncertainties measured by the gap [33], pointwise gap [32] and ν-gap [36], the tight robust stability conditions have been obtained, respectively.

[*]Submitted to the editors DATE.

Funding: This work is supported by the Research Grants Council of Hong Kong SAR under project (GRF 16201115) and Guangdong Science and Technology Department, China (No. 2019B010117002).

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The extension of LTI robust control theory to LTV systems is also underway. With the development of H_∞ control theory, significant insights have been obtained by considering its time-varying analogue, a control theory in the framework of the nest algebra of causal bounded operators on an appropriate complex Hilbert space of input-output signals [16]. Such a theory for LTV systems generalizes the H_∞ control theory in the sense that the systems are considered as linear operators on the Hilbert signal spaces. In the context of LTV robust control theory, the gap metric has also played an important role [10, 11, 14, 16]. Feintuch [13] generalized the two-block H_∞ optimization method for the computation of the gap in [21] to the LTV case. This was achieved by introducing the time-varying gap metric [13, 16], which is different from the standard gap metric for LTV systems. A sufficient condition and a necessary condition have been obtained in [16] for robust stability of LTV systems under plant uncertainty measured by the directed time-varying gap, respectively. These two conditions are different in the time-varying case. A more general geometric framework for robust stabilization of feedback systems using operator-theoretic methods has been developed in [5, 19]. Specifically, a necessary and sufficient condition for robust stability under simultaneous gap-metric uncertainties of the plant and the controller was presented in [19], which is a generalization of the arcsine condition of [33] to the time-varying case, but the causality of systems is not considered.

In the continuous-time context, a time-varying generalization of Vinnicombe's ν-gap was presented in [3, 4, 29] for causal linear systems. Accordingly, a time-invariant ν-gap robust stability result extends with respect to a definition of closed-loop stability. It is shown that the generalized ν-gap metric and an adaptation of Feintuch's time-varying gap metric give rise to the same topology and thus qualitatively equivalent robust stability results [3], in which the development also corrects various aspects of the results in [4] and [29].

Networked control systems (NCSs) are feedback control loops closed via a real-time shared media network [38]. The difference between the NCS and the standard feedback system lies in the presence of a communication network, which is deployed to exchange information, between the plant and controller. In networked environments, the bidirectional control signals are transmitted through imperfect communication channels for most practical systems. Due to the presence of channel distortions and interferences, it is necessary to consider the channel uncertainties when investigating the feedback stability. In this paper, a two-port NCS model is developed under the framework of discrete-time LTV systems. by extending the standard closed-loop system (Fig. 1) to the feedback system with cascaded two-port connections (Fig. 3). Such an NCS model is motivated by the application scenario of stabilizing a feedback system, where the plant and controller cannot communicate directly and the signals can only pass through the communication network consisting of a sequence of relays, such as, satellite networks [1], wireless sensor networks [2] and so on. Furthermore, each communication channel between two neighbouring relays can be viewed as a subsystem that involves not only multiplicative distortions on the transmitted signal itself, but also additive interferences induced by the signal in the opposite direction. Such a phenomenon is usually encountered in a bidirectional wireless network subject to communication error caused by channel loss, fading or some malicious attacks.

Two-port networks first appeared in electrical circuit theory [6, 7], and were later borrowed to represent LTI systems in chain-scattering formalism [28]. Recently, a two-port approach was taken in [20] to model the communication channel in a networked feedback system. More specifically, the robust stability of the networked feedback system was investigated under the framework of H_∞ control. Later in [39], a concise necessary and sufficient robust stability condition was obtained for the continuous-time LTI networked control systems with the uncertain communication channels described by cascaded two-port networks. Furthermore, in this study, the robust stability of cascaded two-port NCSs is investigated in the framework of discrete-time causal LTV systems. In particular, we model a discrete-time LTV system as a (possibly unbounded) linear operator described by a block lower-triangular infinite-dimensional complex matrix due to the causality of the system. The system is said to be stable if the operator is bounded in norm. Particularly, the uncertainty in a two-port channel is described by a stable LTV system additive to the transmission matrix of the two-port network. Regarding norm bounded uncertainties in the communication channels as well as standard gap bounded uncertainties in the plant and controller, we present a necessary and sufficient condition for robust stability of the cascaded two-port NCS in the form of an arcsine inequality, which generalizes of the main results in [39] to the LTV case.

The rest of the paper is organized as follows. In Section 2, we introduce the main definitions, terminology, some auxiliary propositions, and the NCS model to be studied in this paper. In Section 3, we first examine the robust stability of a special case with only one uncertain two-port network in the communication channel via the small gain theorem, then present the robust stability result for a general LTV NCS with simultaneous uncertainties. Last in Section 4, we conclude with a summary of the contributions of this paper.

2. Preliminaries. In this section, general definitions and the mathematical background used throughout the paper are introduced. Denote by $\mathbb{C}$ the set of complex numbers, and by $\mathbb{C}^n$ the space of n dimensional complex vectors. Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces and consider a linear operator $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{Y}$, where $\mathcal{D}(A) = \{x \in \mathcal{X} : Ax \in \mathcal{Y}\}$ is the domain of A. The range and kernel of A are defined to be $\mathcal{R}(A) := \{Ax : x \in \mathcal{D}(A)\}$ and $\mathcal{K}(A) := \{x \in \mathcal{D}(A) : Ax = 0\}$, respectively. The operator A is said to be bounded if there exists a positive constant c such that $\|Ax\| \le c\|x\|$ for all $x \in \mathcal{D}(A)$. Let $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ denote the Banach space of all bounded linear operators $A : \mathcal{X} \to \mathcal{Y}$ endowed with the operator norm

$$\|A\| := \sup_{x\in\mathcal{X},\|x\|=1} \|Ax\|,$$

and let $\tau(A) := \inf_{x\in\mathcal{X},\|x\|=1} \|Ax\|$ and $\mathcal{B}(\mathcal{X}) := \mathcal{B}(\mathcal{X}, \mathcal{X})$. For $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, denote by $A^* \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ the Hilbert adjoint of A. An operator $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called an isometry if $A^*A = I$. Furthermore, $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called a unitary operator if $A^*A = AA^* = I$. Finally, for a subspace $\mathcal{M}$ of $\mathcal{X}$, $\mathcal{M}^\perp$ is the orthogonal complement of $\mathcal{M}$, and $\Pi_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$. The restriction of A to $\mathcal{M} \subset \mathcal{X}$ is $A|_{\mathcal{M}}$, which is from $\mathcal{M}$ to $\mathcal{Y}$. For $z \in \mathcal{X}, y \in \mathcal{Y}$, we denote by $y \otimes z$ a rank-one operator defined by $(y \otimes z)x := \langle x, z\rangle y,\ \forall x \in \mathcal{X}$, where $\langle \cdot, \cdot\rangle$ denotes the inner product on $\mathcal{X}$.

2.1. LTV systems. In this paper, we model a linear system as a (possibly unbounded) linear operator mapping between signal spaces. A typical choice for the input and output spaces is the complex separable Hilbert space

$$h_2^n = \left\{ (x_0, x_1, \ldots, x_k, x_{k+1}, \ldots) : x_i \in \mathbb{C}^n, \sum_{i=0}^{\infty} \|x_i\|^2_{\mathbb{C}^n} < \infty \right\},$$

with the inner product and norm in the following form:

$$\langle x, y\rangle = \sum_{i=0}^{\infty}\langle x_i, y_i\rangle_{\mathbb{C}^n}, \quad \|x\| = \left(\sum_{i=0}^{\infty}\|x_i\|_{\mathbb{C}^n}^2\right)^{\frac{1}{2}}.$$

Here $\|\cdot\|_{\mathbb{C}^n}$ and $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ denote the standard Euclidean norm and inner product on $\mathbb{C}^n$, respectively. Denote by $h^n := \{(x_0, x_1, \ldots, x_k, x_{k+1}, \ldots) : x_i \in \mathbb{C}^n\}$ the set of all time sequences, which is the extended space of h_2^n.

For each integer $k \geq 0$, E_k denotes the standard truncation projection from h_2^n or h^n onto the subspace $\mathcal{N}_k = \{(x_0, x_1, \ldots, x_k, 0, \ldots) : x_i \in \mathbb{C}^n\}$; that is,

$$(E_k x)_i := \begin{cases} x_i, & i \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

Define $\|x\|_k := \|E_k x\|$ for each $k \geq 0$ for $x \in h^n$. Then $\{\|\cdot\|_k : k \geq 0\}$ is a separating family of semi-norms on h^n and defines on h^n a metrizable topology, called the resolution topology on h^n [16, Chapter 5]. The extended space h^n is the completion of h_2^n with respect to this topology. The set $\{E_k : 0 \leq k < \infty\}$ is used to introduce the physical definition of causality for linear systems.

Definition 2.1 ([16, Chapter 5]). *Let $P : h^n \to h^m$ be a linear operator.*

(i) *P is causal if, for each $k \geq 0, E_k P = E_k P E_k$.*

(ii) *P is a linear time-varying (LTV) system if P is a causal linear operator that is continuous with respect to the resolution topology.*

We denote by $\mathcal{L}^{n,m}$ the set of all LTV systems from h^n to h^m. For $P \in \mathcal{L}^{n,m}$, it follows from [16, Theorem 5.2.6] that P can be described as a block lower-triangular complex infinite matrix (not necessarily a bounded operator). As a result, $y = Px$ can be expressed by

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} P_{00} & & & \\ P_{10} & P_{11} & & \\ P_{20} & P_{21} & P_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix},$$

where P_{ij} is a $m \times n$ matrix. It was shown in [15] that P is a closed operator, i.e., $\mathcal{G}_P := \left\{ \begin{bmatrix} x \\ Px \end{bmatrix} : x \in \mathcal{D}(P) \right\}$ is a closed subspace of $h_2^{n+m} := h_2^n \oplus h_2^m$. This subspace is called the graph of P.

A system $P \in \mathcal{L}^{n,m}$ is stable if its restriction to h_2^n is a bounded operator. Since $P \in \mathcal{L}^{n,m}$ is a closed operator, it follows from the closed graph theorem [26] that P is stable if and only if $Ph_2^n \subset h_2^m$. In the case when $n = m$, the set of all stable LTV systems on h_2^n, denoted by $\mathcal{S}^{n,n}$, is a weakly closed algebra containing the identity, where n is any positive integer. Indeed, $\mathcal{S}^{n,n}$ is a nest algebra [9] determined by the complete nest $\{F_k h_2^n : -1 \leq k \leq \infty\}$ on h_2^n, where $F_k := I - E_k$, $F_\infty := 0$ and $F_{-1} := I$. In the sequel, the spatial dimensions n and m are often dropped for notational convenience. Throughout this paper, for $P \in \mathcal{L}$ or $\mathcal{S}$, let P_{kk} be the kth main-diagonal block of P and

$$P(k) := P|_{F_k\mathcal{X}} = \begin{bmatrix} P_{kk} & & \\ P_{k+1\ k} & P_{k+1\ k+1} & \\ \vdots & \vdots & \ddots \end{bmatrix},$$

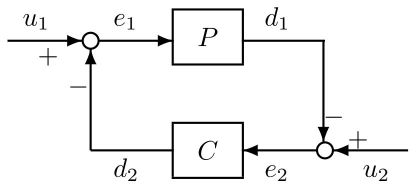

FIG. 1. *Standard closed-loop system.*

where $\mathcal{X} = h$ or h_2.

The invertibility property of elements in $\mathcal{L}$ and $\mathcal{S}$ has been shown to be critical for the study of feedback systems. Invertibility in $\mathcal{L}$ is a purely algebraic property: P is invertible in $\mathcal{L}$ if and only if it has no singular elements on its main diagonal. In other words, P is invertible in $\mathcal{L}$ if and only if P_{kk} is invertible for each $k \geq 0$. While invertibility in $\mathcal{S}$ is a topological property: P is invertible in $\mathcal{S}$ if and only if P is invertible in $\mathcal{L}$, and $\|(E_k P E_k|_{E_k h_2})^{-1}\|$ is uniformly bounded on $E_k h_2$. We will say that P is invertible if P is invertible in $\mathcal{L}$. The system P is stably invertible if P is invertible in $\mathcal{S}$; that is, P has a bounded causal inverse.

2.2. Feedback systems. The closed-loop system in Fig. 1 is denoted as $P\#C$, where $P \in \mathcal{L}$ represents the plant and $C \in \mathcal{L}$ the controller. The closed-loop system $P\#C$ is said to be well-posed if the internal signal $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ can be expressed as a causal function of any external input $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. This is equivalent to requiring that $\begin{bmatrix} I & C \\ P & I \end{bmatrix}$ is invertible, and its inverse is given by the four-block operator

$$H(P, C) = \begin{bmatrix} (I - CP)^{-1} & -C(I - PC)^{-1} \\ -P(I - CP)^{-1} & (I - PC)^{-1} \end{bmatrix}.$$

In order for $H(P, C)$ to exist, $I - PC$ and $I - CP$ have to be invertible. Hence, $P\#C$ is well-posed if and only if $I - PC$ is invertible. Clearly, a sufficient condition for the well-posedness is that P or C has all zeros on its main diagonal, i.e., it is strictly causal.

DEFINITION 2.2. *The closed-loop system $P\#C$ is stable if*

$$\begin{bmatrix} I & C \\ P & I \end{bmatrix} : \mathcal{D}(P) \oplus \mathcal{D}(C) \to h_2$$

has a bounded causal inverse defined on h_2; that is, $H(P, C) \in \mathcal{S}$. A system P is said to be stabilizable if there exists a controller C such that $P\#C$ is stable.

The stability of feedback systems is closely related to the existence of coprime factorizations. We introduce the right and left coprime factorizations for LTV systems in the following.

DEFINITION 2.3 ([15]). *Let $P \in \mathcal{L}$.*

(i) *$P = NM^{-1}$ is a right coprime factorization of P if M and N are causal, bounded operators, and $\begin{bmatrix} M \\ N \end{bmatrix}$ has a causal, bounded left inverse.*

The right coprime factorization is normalized if $M^*M + N^*N = I$.

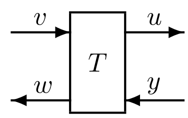

FIG. 2. *A single two-port network*

(ii) $P = \tilde{M}^{-1}\tilde{N}$ *is a left coprime factorization of* P *if* $\tilde{M}$ *and* $\tilde{N}$ *are causal, bounded operators, and* $[-\tilde{N}\ \tilde{M}]$ *has a causal, bounded right inverse.*

The left coprime factorization is normalized if $\tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I$.

The following result can be found in [16].

LEMMA 2.4. *Let* NM^{-1} *be a right coprime factorization of* $P \in \mathcal{L}$, VU^{-1} *and* $\tilde{U}^{-1}\tilde{V}$ *be right and left coprime factorizations of* $C \in \mathcal{L}$, *respectively. The following statements are equivalent:*

(i) $P\#C$ *is stable.*

(ii) $\tilde{U}M + \tilde{V}N$ *is stably invertible.*

(iii) $\begin{bmatrix} M & V \\ N & U \end{bmatrix}$ *is stably invertible.*

In the discrete-time time-varying case, a system is stabilizable if and only if it has right and left coprime factorizations [12]. Moreover, these factorizations can always be normalized [16]. The equivalence between the existences of a right and a left coprime factorization was obtained in [31]. These results can be summarized in the following theorem.

THEOREM 2.5. *Let* $P \in \mathcal{L}$. *The following statements are equivalent:*

(i) P *is stabilizable.*

(ii) P *has a (normalized) right coprime factorization.*

(iii) P *has a (normalized) left coprime factorization.*

2.3. Two-port networks as communication channels. The use of two-port networks in electrical circuits theory [6], [7] as a model of communication channels is adopted from [20] and [39]. In this subsection, we present the time-varying analogue of networked control systems (NCSs) involving cascaded two-port connections. The network T in Fig. 2 has two ports, where v and w compose one port and u, y compose the other. In general, the downlink transmission from v to u and the uplink transmission from y to w share the two-port network T. In this study, we will focus on the transmission representation of T. Define the transmission matrix T, and the descriptions of the communication channel as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \text{ and } \begin{bmatrix} v \\ w \end{bmatrix} = T \begin{bmatrix} u \\ y \end{bmatrix}.$$

Here, the symbol T denotes both the two-port network and its transmission representation for notational simplicity. In the case that the communication is ideal, i.e., the channel has no distortions or interferences, the transmission matrix is $T = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. When the bidirectional channel admits both distortions and interferences, we model the transmission matrix in the following form:

$$T = I + \Delta = \begin{bmatrix} I + \Delta_{\div} & \Delta_{-} \\ \Delta_{+} & I + \Delta_{\times} \end{bmatrix},$$

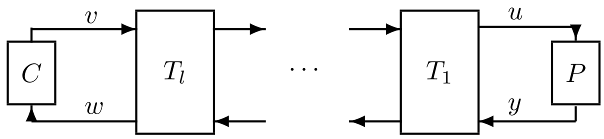

FIG. 3. *An NCS with two-port connections*

where $\Delta = \begin{bmatrix} \Delta_{\div} & \Delta_{-} \\ \Delta_{+} & \Delta_{\times} \end{bmatrix} \in \mathcal{S}$ with $\|\Delta\| < r,\ r \in (0, 1]$. The diagonal terms $\Delta\div, \Delta_\times$ are used to model the transmission distortion. The off-diagonal terms Δ_-, Δ_+ are used to model the channel interference. The four-block operator matrix Δ is called the uncertainty quartet. A more detailed analysis of the network uncertainty Δ can be found in [20] and [39].

In the following, we introduce the two-port network into the standard feedback system $P\#C$, where $P, C \in \mathcal{L}$. Assume that P and C admit the right coprime factorzations $P = NM^{-1}$ and $C = VU^{-1}$, respectively. In Fig. 3, the plant P and controller C communicate with each other through a two-port network. Considering the input and output of P, we obtain that $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} u = \begin{bmatrix} M \\ N \end{bmatrix} M^{-1}u$, for any $u \in h_2$ such that $M^{-1}u \in h_2$. Or, $\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} x$ for any $x \in h_2$.

Consider the transmission representation of the two-port networks $\{T_i\}_{i=1}^l$. If the i-th stage of the network admits an uncertainty $\Delta_i \in \mathcal{S}$, then the transmission matrix is given by $T_i = I + \Delta_i$. For each integer $i \in (0, l)$, we can associate the first i stages of the cascaded two-port networks with the plant P, and the remaining $l - i$ stages with the controller C. It follows from similar derivations as in [39] that signals satisfy the following relations:

$$\begin{bmatrix} u_i \\ y_i \end{bmatrix} = T_i T_{i-1} \cdots T_1 \begin{bmatrix} u \\ y \end{bmatrix} = (I + \Delta_i)(I + \Delta_{i-1}) \cdots (I + \Delta_1) \begin{bmatrix} u \\ y \end{bmatrix},$$
$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = T_{i+1}^{-1} T_{i+2}^{-1} \cdots T_l^{-1} \begin{bmatrix} v \\ w \end{bmatrix} = (I + \Delta_{i+1})^{-1}(I + \Delta_{i+2})^{-1} \cdots (I + \Delta_l)^{-1} \begin{bmatrix} v \\ w \end{bmatrix}.$$

Regarding these relations, we view P together with $\{T_j\}_{j=1}^i$ as a perturbed plant P_i' with uncertainties $\{\Delta_j\}_{j=1}^i$. Then $P_i' = N_i M_i^{-1}$ can be determined by its graph:

$$\text{(2.1)} \qquad \mathcal{G}_{P_i'} = \begin{bmatrix} M_i \\ N_i \end{bmatrix} h_2 = (I + \Delta_i)(I + \Delta_{i-1}) \cdots (I + \Delta_1)\mathcal{G}_P.$$

Similarly, we view C together with $\{T_j\}_{j=i+1}^l$ as a perturbed controller C_i' with uncertainties $\{\Delta_j\}_{j=i+1}^l$. Then $C_i' = V_i U_i^{-1}$ can be determined by its inverse graph:

$$\text{(2.2)} \qquad \mathcal{G}'_{C_i'} = \begin{bmatrix} V_i \\ U_i \end{bmatrix} h_2 = (I + \Delta_{i+1})^{-1}(I + \Delta_{i+2})^{-1} \cdots (I + \Delta_l)^{-1}\mathcal{G}'_C,$$

where the inverse graph $\mathcal{G}'_C$ of $C = VU^{-1}$ is defined as $\mathcal{G}'_C = \begin{bmatrix} V \\ U \end{bmatrix} h_2$.

For convenience, we regard $i = 0$ as the situation where all the two-port networks are grouped with C, and $i = l$ as the situation where all the two-port networks are grouped with P, i.e., $P_0' = P$ and $C_l' = C$. In addition, since $\Delta_i \in \mathcal{S}$ and $\|\Delta_i\| < 1$, it follows that $I+\Delta_i$ is stably invertible. Then (M_i, N_i) and (V_i, U_i) are right coprime, respectively. In order to keep the perturbed plants P_i' and controllers C_i' well-defined, we add a mild condition on Δ_i, so that M_i and U_i are invertible. In the following, we extend the definition on the stability of the two-port NCS in [39] to the time-varying case.

DEFINITION 2.6. *The NCS in Fig. 3 is said to be stable if the perturbed closed-loop system $P_i' \# C_i'$ is stable for $i=0,1,\ldots,l$.*

2.4. The gap metric for LTV systems. We briefly introduce, in this subsection, some key concepts and main properties of the gap metric for LTV systems. Let $\mathcal{X}$ and $\mathcal{Y}$ be two closed subspaces of a Hilbert space $\mathcal{H}$, and let $\Pi_{\mathcal{X}}$ and $\Pi_{\mathcal{Y}}$ be the orthogonal projections on $\mathcal{X}$ and $\mathcal{Y}$, respectively. The gap (or aperture) between the two subspaces is the metric defined as

$$\gamma(\mathcal{X},\mathcal{Y}) := \|\Pi_{\mathcal{X}} - \Pi_{\mathcal{Y}}\|$$

(see [26] and [27]). It is shown in [27, p. 205] and [16] that $\gamma(\mathcal{X},\mathcal{Y}) = \max\{\vec{\gamma}(\mathcal{X},\mathcal{Y}), \vec{\gamma}(\mathcal{Y},\mathcal{X})\}$, where $\vec{\gamma}(\mathcal{X},\mathcal{Y}) := \|(I-\Pi_{\mathcal{Y}})\Pi_{\mathcal{X}}\|$ is the directed gap. This equation can be written in the equivalent form: $\vec{\gamma}(\mathcal{X},\mathcal{Y}) = \sup_{x\in\mathcal{X},\|x\|=1} \operatorname{dist}(x,\mathcal{Y})$, where $\operatorname{dist}(x,\mathcal{Y}) := \inf_{y\in\mathcal{Y}} \|x-y\| = \|(I-\Pi_{\mathcal{Y}})x\|$.

PROPOSITION 2.7 ([16] and [27]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be two closed subspaces of a Hilbert space $\mathcal{H}$. Then $\Pi_{\mathcal{Y}}$ maps $\mathcal{X}$ one-to-one onto $\mathcal{Y}$ if and only if $\gamma(\mathcal{X},\mathcal{Y})<1$. Moreover, if $\gamma(\mathcal{X},\mathcal{Y})<1$, then $\gamma(\mathcal{X},\mathcal{Y}) = \vec{\gamma}(\mathcal{X},\mathcal{Y}) = \vec{\gamma}(\mathcal{Y},\mathcal{X})$.*

The gap between LTV systems P_1 and $P_2 \in \mathcal{L}$ is defined to be the gap between their respective graphs as follows:

$$\delta(P_1,P_2) := \gamma\left(\mathcal{G}_{P_1},\mathcal{G}_{P_2}\right).$$

The gap ball centered at $P\in\mathcal{L}$ with radius $r\in(0,1]$ is then given by

$$\mathcal{B}(P,r) := \left\{P'\in\mathcal{L} : \delta(P',P) < r\right\}.$$

The next result shows that the gap between two stabilizable systems is not less than the gap between their respective restrictions to the truncation subspaces.

PROPOSITION 2.8. *Assume that $P_1, P_2 \in \mathcal{L}$ are stabilizable. Then for $k\geq 0$,*

$$\delta\left((P_1)_{kk},(P_2)_{kk}\right) \leq \delta(P_1,P_2), \quad \delta(P_1(k),P_2(k)) \leq \delta(P_1,P_2)$$

Proof. We prove the first inequality below. The proof of the second can be shown similarly. Let $\delta(P_1,P_2)=r$. Then $r\in[0,1]$. Clearly, the case $r=0$ or 1 is trivial. Thus $0<r<1$ is assumed. Let $P_1 = N_1 M_1^{-1}$ be a normalized right coprime factorization. Then it follows from [16, Corollary 10.1.4 and Theorem 10.4.1] that there exist causal, bounded operators $\overline{\Delta}_1, \overline{\Delta}_2$ with $\left\|\begin{bmatrix}\overline{\Delta}_1\\ \overline{\Delta}_2\end{bmatrix}\right\| \leq r$ such that $(N_1+\overline{\Delta}_2)(M_1+\overline{\Delta}_1)^{-1}$ is a right coprime factorization of P_2. For each $k\geq 0$, it is easy to see that $(P_1)_{kk} = (N_1)_{kk}(M_1)_{kk}^{-1}$ and $(P_2)_{kk} = \left((N_1)_{kk}+(\overline{\Delta}_2)_{kk}\right)\left((M_1)_{kk}+(\overline{\Delta}_1)_{kk}\right)^{-1}$ are right coprime factorizations of $(P_1)_{kk}$ and $(P_2)_{kk}$, respectively. Moreover, $\left\|\begin{bmatrix}(\overline{\Delta}_1)_{kk}\\ (\overline{\Delta}_2)_{kk}\end{bmatrix}\right\| \leq r$. Therefore, we obtain $\delta\left((P_1)_{kk},(P_2)_{kk}\right) \leq r = \delta(P_1,P_2)$. ◻

Based on the uncertainty quartets in equations (2.1) and (2.2), two special uncertainty neighborhoods are as follows.

DEFINITION 2.9. *Assume that $P \in \mathcal{L}$ and $P = NM^{-1}$ is a right coprime factorization. For $r \in (0,1]$, define*

$$\mathcal{N}_1(P,r) := \left\{ P' = N'(M')^{-1} : \begin{bmatrix} M' \\ N' \end{bmatrix} = (I+\Delta)\begin{bmatrix} M \\ N \end{bmatrix}, \right.$$
$$\left. \Delta \in \mathcal{S},\ \|\Delta\| < r,\ M' \textit{is invertible} \right\};$$
$$\mathcal{N}_2(P,r) := \left\{ P' = N'(M')^{-1} : \begin{bmatrix} M' \\ N' \end{bmatrix} = (I+\Delta)^{-1}\begin{bmatrix} M \\ N \end{bmatrix}, \right.$$
$$\left. \Delta \in \mathcal{S},\ \|\Delta\| < r,\ M' \textit{is invertible} \right\}.$$

In the time-invariant case, the above neighborhoods of a linear time-invariant system G are introduced in [23] and [24]. From [24], we know for $r \in (0,1]$,

$$\mathcal{N}_1(G,r) \cup \mathcal{N}_2(G,r) \subset \mathcal{B}(G,r). \tag{2.3}$$

In what follows, we extend relation (2.3) to the time-varying case.

PROPOSITION 2.10. *Let $P \in \mathcal{L}$ and $r \in (0,1]$. Then*

$$\mathcal{N}_1(P,r) \cup \mathcal{N}_2(P,r) \subset \mathcal{B}(P,r).$$

Proof. If $P' \in \mathcal{N}_1(P,r)$, then $\mathcal{G}_{P'} = (I+\Delta)\mathcal{G}_P$. From the definition of the directed gap, it follows that

$$\vec{\gamma}(\mathcal{G}_P,\mathcal{G}_{P'}) = \sup_{0\neq x\in\mathcal{G}_P}\ \inf_{0\neq y\in\mathcal{G}_{P'}} \frac{\|y-x\|}{\|x\|} = \sup_{0\neq x\in\mathcal{G}_P}\ \inf_{0\neq x_1\in\mathcal{G}_P} \frac{\|(I+\Delta)x_1 - x\|}{\|x\|} < r.$$

Since NM^{-1} is a right coprime factorization of P, then, by [16, Theorem 6.3.8], there exists stably invertible $Q \in \mathcal{S}$ such that $\begin{bmatrix} MQ \\ NQ \end{bmatrix}$ is an isometry. Thus, the orthogonal projection on $\mathcal{G}_P$ is given by $\Pi_{\mathcal{G}_P} = \begin{bmatrix} M \\ N \end{bmatrix} QQ^*[M^* \ N^*]$. This shows

$$\Pi_{\mathcal{G}_P}\begin{bmatrix} M' \\ N' \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} Q \left(I + Q^*[M^* \ N^*]\Delta \begin{bmatrix} M \\ N \end{bmatrix} Q \right) Q^{-1}.$$

Note that $\left\| Q^*[M^* \ N^*]\Delta \begin{bmatrix} M \\ N \end{bmatrix} Q \right\| \leq \|\Delta\| < 1$ implies that $I + Q^*[M^* \ N^*]\Delta \begin{bmatrix} M \\ N \end{bmatrix} Q$ is invertible in $\mathcal{B}(h_2)$. Thus, $\Pi_{\mathcal{G}_P}$ maps $\mathcal{G}_{P'}$ one-to-one onto $\mathcal{G}_P$. By Proposition 2.7, we have $\gamma(\mathcal{G}_{P'},\mathcal{G}_P) = \vec{\gamma}(\mathcal{G}_{P'},\mathcal{G}_P) = \vec{\gamma}(\mathcal{G}_P,\mathcal{G}_{P'}) < r$. This proves $\mathcal{N}_1(P,r) \subset \mathcal{B}(P,r)$.

By Definition 2.9, we have $P' \in \mathcal{N}_2(P,r) \Leftrightarrow P \in \mathcal{N}_1(P',r)$. Since $P' \in \mathcal{B}(P,r) \Leftrightarrow P \in \mathcal{B}(P',r)$, it follows that $\mathcal{N}_2(P,r) \subset \mathcal{B}(P,r)$. This completes the proof. ∎

3. Main results: networked robust stability. In this section, we are interested in the robust stability conditions for the NCS shown in Fig. 3 when the plant, controller and communication channels are subject to simultaneous perturbations. First, the situation where a single two-port network is perturbed is considered. Then the general case of the networked robust stability in the face of simultaneous perturbations to the plant, controller and communication channels is investigated.

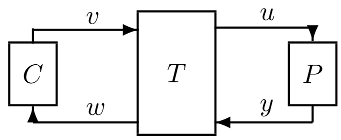

Fig. 4. *Two-port NCS with one stage of two-port network*

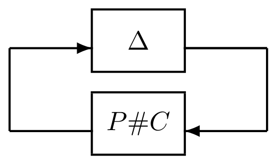

Fig. 5. *Standard closed-loop system equivalent to one-stage two-port NCS*

3.1. One-stage two-port NCS. In this subsection, the robust stability result for the one-stage two-port NCS is established when norm-bounded perturbations to the network alone are considered. Before proceeding to the NCS, we introduce the following operator associated with a standard feedback system, which plays a crucial role in robust stability analysis [16]. Given a well-posed feedback system $P\#C$, and with a little abuse of notation, we let

$$P\#C := \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} [I \ \ -C].$$

Observe that $P\#C = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} H(P,C) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$. Therefore, the stability of $P\#C$ is equivalent to the boundedness of $P\#C$. When $P\#C$ is stable, the value $\|P\#C\|^{-1}$ is often called the robust stability margin.

Following the derivation in [20], we equivalently transform into that in Fig. 5 to form a standard closed-loop system $(P\#C)\#\Delta$. Therefore, suppose that the nominal system $P\#C$ is stable, then the one-stage two-port NCS is stable if and only if $(P\#C)\#\Delta$ is stable. The robust stability of this system can be analyzed through the following asymptotic small-gain result.

Lemma 3.1. *Let $A \in \mathcal{S}$ and $r \in (0, 1]$. Then $I - \Delta A$ is stably invertible for all $\Delta \in \mathcal{S}$ with $\|\Delta\| < r$ if and only if*

$$r \leq \min \left\{ \frac{1}{\sup_{k \geq 0} \|A_{kk}\|}, \ \frac{1}{\inf_{j \geq 0} \|A(j)\|} \right\}. \tag{3.1}$$

Proof. If $r \leq \dfrac{1}{\sup_{k\geq 0} \|A_{kk}\|}$, then for all $k \geq 0$, we have $r \leq \dfrac{1}{\|A_{kk}\|}$. Thus, $\|\Delta_{kk} A_{kk}\| < 1$. By small-gain theorem, we obtain that 1 is not an eigenvalue of $\Delta_{kk} A_{kk}$ for each $k \geq 0$. Thus, $I - \Delta A$ is invertible. Conversely, assume that $I - \Delta A$ is invertible for all $\Delta \in \mathcal{S}$ with $\|\Delta\| < r$. For all matrices $\tilde{\Delta}_{kk}$ with $\|\tilde{\Delta}_{kk}\| < r$, construct a block diagonal operator Δ such that $\Delta_{ii} := \tilde{\Delta}_{kk}$ for $i = k$, and $\Delta_{ii} := 0$ otherwise. Clearly, $\Delta \in \mathcal{S}$ and $\|\Delta\| < r$. By hypothesis, $I - \Delta A$ is invertible. Then for each $k \geq 0$, $(I - \Delta A)_{kk} = I_n - \Delta_{kk} A_{kk}$ is invertible for all matrices $\tilde{\Delta}_{kk}$ with

$\|\tilde{\Delta}_{kk}\| < r$, where I_n is the identity matrix. Hence, it follows from [40, Theorem 8.1] that $r \le \dfrac{1}{\|A_{kk}\|}$ for each $k \ge 0$, which shows that $r \le \dfrac{1}{\sup\limits_{k\ge 0}\|A_{kk}\|}$. Finally, similarly to the proof of [17, Theorem 4.2], we know that $(I - \Delta A)^{-1} \in \mathcal{S}$ for all $\Delta \in \mathcal{S}$ with $\|\Delta\| < r$ if and only if $r \le \dfrac{1}{\inf\limits_{j\ge 0}\|A(j)\|}$. This completes the proof. □

It is worth noting that the first term $\dfrac{1}{\sup\limits_{k\ge 0}\|A_{kk}\|}$ in inequality (3.1) is equal to infinity under the hypothesis in [17, Theorem 4.2]. An application of Lemma 3.1 gives rise to a necessary and sufficient condition for robust stability of the one-stage two-port NCS.

THEOREM 3.2. *Let $P\#C$ be stable and $r \in (0, 1]$. Then the two-port NCS in Fig. 4 is stable for all $\Delta \in \mathcal{S}$ with $\|\Delta\| < r$ if and only if*

$$r \le \min\left\{\frac{1}{\sup\limits_{k\ge 0}\|(P\#C)_{kk}\|},\ \frac{1}{\inf\limits_{j\ge 0}\|(P\#C)(j)\|}\right\}. \tag{3.2}$$

Remark 3.3. The first bound on the right side of inequality (3.2) ensures that $(P\#C)\#\Delta$ is well-posed. When $(P\#C)\#\Delta$ is well-posed, the second bound ensures that $(P\#C)\#\Delta$ is stable.

3.2. Multiple-stage two-port NCS. The main result of this paper concerning the robust stability of the NCS is stated as follows, which extends the result of Zhao and Qiu [39] to the time-varying case.

THEOREM 3.4. *Let $P\#C$ be stable and r_p, r_c, $r_i \in (0, 1]$. Then the NCS in Fig. 3 is stable for all $P' \in \mathcal{B}(P, r_p)$, $C' \in \mathcal{B}(C, r_c)$ and $\Delta_i \in \mathcal{S}$ with $\|\Delta_i\| < r_i$, $i = 1, 2, \ldots, l$, if and only if*

$$\begin{aligned}&\arcsin r_p + \arcsin r_c + \sum_{i=1}^{l}\arcsin r_i \le \\ &\min\left\{\arcsin\frac{1}{\sup\limits_{k\ge 0}\|(P\#C)_{kk}\|},\ \arcsin\frac{1}{\inf\limits_{j\ge 0}\|(P\#C)(j)\|}\right\}.\end{aligned} \tag{3.3}$$

Remark 3.5. In condition (3.3), the following inequality:

$$\arcsin r_p + \arcsin r_c + \sum_{i=1}^{l}\arcsin r_i \le \arcsin\frac{1}{\sup\limits_{k\ge 0}\|(P\#C)_{kk}\|} \tag{3.4}$$

guarantees that the NCS in Fig. 3 is well-posed, which will be discussed in following subsections. If the well-posedness of the NCS is satisfied, then condition (3.3) can be rewritten as

$$\arcsin r_p + \arcsin r_c + \sum_{i=1}^{l}\arcsin r_i \le \arcsin\frac{1}{\inf\limits_{j\ge 0}\|(P\#C)(j)\|}. \tag{3.5}$$

Naturally, we can view the value $\dfrac{1}{\inf\limits_{j\geq 0}\|(P\#C)(j)\|}$ as the stability margin of the NCS in Fig. 3 in the time-varying case. The larger the margin is, the more uncertainties the NCS can tolerate.

Theorem 3.4 reduces to Theorem 3.2 when $r_p = 0, r_c = 0$ and $r_i = 0$ for each integer $i \in [2, l]$. As an important special case of Theorem 3.4, the following result gives a necessary and sufficient condition for robust stability of LTV systems when only the plant is subject to uncertainty. We state this as a corollary.

COROLLARY 3.6. *Let $P\#C$ be stable and $r_p \in (0, 1]$. Then $P'\#C$ is stable for all $P' \in \mathcal{B}(P, r_p)$ if and only if*

$$r_p \leq \min\left\{\frac{1}{\sup\limits_{k\geq 0}\|(P\#C)_{kk}\|},\ \frac{1}{\inf\limits_{j\geq 0}\|(P\#C)(j)\|}\right\}.$$

Proof. The proof follows directly from Theorem 3.4 by letting $r_c = 0$ and $r_i = 0,\ i = 1, 2, \ldots, l$. ∎

The following result is an immediate consequence of Theorem 3.4 when the transmission matrices of the two-port channels have no uncertainties, i.e., $r_i = 0, 1 \leq i \leq l$.

COROLLARY 3.7. *Let $P\#C$ be stable and $r_p, r_c \in (0, 1]$. Then $P'\#C'$ is stable for all $P' \in \mathcal{B}(P, r_p)$ and $C' \in \mathcal{B}(C, r_c)$ if and only if*

$$\arcsin r_p + \arcsin r_c \leq \min\left\{\arcsin\frac{1}{\sup\limits_{k\geq 0}\|(P\#C)_{kk}\|}, \arcsin\frac{1}{\inf\limits_{j\geq 0}\|(P\#C)(j)\|}\right\}.$$

Remark 3.8. We remark that some works, for instance [16] and [19], have given similar robust stability conditions for LTV systems. In [16], Feintuch derived a sufficient condition and a necessary condition for the robust stability under directed time-varying gap perturbations of the plant, respectively. These two conditions are different in the time-varying case. In our study, we obtain a necessary and sufficient condition for the robust stability of LTV systems for the case when the plant is subject to the standard gap metric uncertainty. In [19], necessary and sufficient conditions have been obtained for the feedback robust stability based on the linear operator theory, but the causality of systems is not considered. Nevertheless, our models for systems and uncertainties incorporate the causality issue. In addition, the uniform boundedness condition is in fact necessary in [19], but is not required in our main results.

In the rest of this paper, we will give the proof of Theorem 3.4. The proof of the sufficiency is a generalization of the idea introduced in [16] and [39] to the time-varying case. The key point is the proof of the necessity, which makes use of the one-vector interpolation problem for nest algebras [30].

3.3. Sufficiency of the robust stability condition. In this subsection, we will prove the sufficiency part of Theorem 3.4. The proof is closely related to the fact that $\arcsin \delta(P_1, P_2)$ is a metric for $P_1, P_2 \in \mathcal{L}$, called the angular metric [33]. We first briefly review the minimal angle between subspaces in a Hilbert space $\mathcal{H}$.

Given two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{H}$, the minimal angle between $\mathcal{X}$ and $\mathcal{Y}$ is defined as $\theta_{min}(\mathcal{X},\mathcal{Y}) := \inf\{\theta(x,y) : 0 \neq x \in \mathcal{X}, 0 \neq y \in \mathcal{Y}\}$, where $\theta(x,y) := \arccos \frac{|\langle x,y\rangle|}{\|x\|\|y\|}$ is the angle between two nonzero vectors $x, y \in \mathcal{H}$. When $P\#C$ is stable, $\theta_{min}(\mathcal{G}_P, \mathcal{G}'_C) = \arcsin \|P\#C\|^{-1}$ (see [19]).

We are now ready to show the sufficiency part of the proof for Theorem 3.4.

Proof. Assume that condition (3.3) holds. We first prove that P' is stabilizable for all $P' \in \mathcal{B}(P, r_p)$. If there exists $P' \in \mathcal{B}(P, r_p)$ such that P' is not stabilizable, then, by [16, Theorem 6.1.3], we have that the operator $\Pi_{\mathcal{Y}^\perp}|_{\mathcal{X}'}$ is not invertible, where $\mathcal{X}' := \mathcal{G}_{P'}$ and $\mathcal{Y} := \mathcal{G}'_C$. Then one of the following two possibilities occurs:

(i) $\tau\left(\Pi_{\mathcal{Y}^\perp}|_{\mathcal{X}'}\right)$ is not bounded below; (ii) $\Pi_{\mathcal{X}'}\Pi_{\mathcal{Y}^\perp}$ is not injective.

In case (i), for all $\varepsilon > 0$, there exists a unit vector $x' \in \mathcal{X}'$ such that $\|\Pi_{\mathcal{Y}^\perp} x'\| < \varepsilon$. Setting $y := \Pi_{\mathcal{Y}} x' \in \mathcal{Y}$, we obtain that $\theta(x', y) = \arccos \frac{|\langle x', y\rangle|}{\|x'\|\|y\|} = \arccos\left(\frac{1 - \|\Pi_{\mathcal{Y}^\perp} x'\|^2}{\|y\|}\right) < \arcsin \varepsilon$. Since $\delta(P', P) < r_p$, we can choose $\bar{r}_p \in (0, r_p)$ such that $\delta(P', P) \leq \bar{r}_p$. This implies $\|(I - \Pi_{\mathcal{G}_P})x'\| \leq \bar{r}_p$. Let $x = \Pi_{\mathcal{G}_P} x' \in \mathcal{G}_P$. Then $\theta(x', x) \leq \arcsin \bar{r}_p$. Since P is stabilizable, it follows from Theorem 2.5 that P admits normalized right and left coprime factorizations $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$. Clearly, $\mathcal{G}_P = \mathcal{R}\left(\begin{bmatrix} M \\ N \end{bmatrix}\right) = \mathcal{K}([-\tilde{N}\ \ \tilde{M}])$. Then, we can write $x = \begin{bmatrix} M \\ N \end{bmatrix} u$ for some $u \in h_2$. Let $x_j = \begin{bmatrix} M \\ N \end{bmatrix} E_j u$. It is easily seen that $x_j \in (\mathcal{G}_{P(j)})^\perp$ and $\lim_{j\to\infty} \theta(x_j, x) = \lim_{j\to\infty} \arccos \frac{\|E_j u\|}{\|u\|} = 0$, where the last equality follows from that $\{E_j\}$ converges to I in the strong operator topology. Thus, there exists $j_1 > 0$ such that $\theta(x_j, x) < \varepsilon$ for all $j \geq j_1$. Similarly, we can find $y_j \in (\mathcal{G}'_{C(j)})^\perp$ such that $\theta(y_j, y) < \varepsilon$ for all $j \geq j_2$. Consequently, for all $j \geq \max\{j_1, j_2\}$,

$$\begin{aligned} &\arcsin \bar{r}_p + \arcsin \varepsilon + 2\varepsilon > \theta(x', x) + \theta(x', y) + \theta(x_j, x) + \theta(y_j, y) \geq \theta(x_j, y_j) \\ &\geq \theta_{min}\left((\mathcal{G}_{P(j)})^\perp, (\mathcal{G}'_{C(j)})^\perp\right) = \arcsin \|P(j)\#C(j)\|^{-1} = \arcsin \|(P\#C)(j)\|^{-1}, \end{aligned}$$

where the last equality follows from the fact that $(P\#C)(j) = P(j)\#C(j)$ for each $j \geq 0$. Since the above inequality holds for all $\varepsilon > 0$, we get $r_p > \bar{r}_p \geq \frac{1}{\inf_{j\geq 0} \|(P\#C)(j)\|}$, which leads to a contradiction to condition (3.3).

In case (ii) we proceed similarly. Since $\Pi_{\mathcal{X}'}\Pi_{\mathcal{Y}^\perp}$ is not injective, there exists a nonzero vector $z \in \mathcal{Y}^\perp \cap (\mathcal{X}')^\perp$. Define $w = \Pi_{\mathcal{G}_P^\perp} z$. Note that $\gamma\left((\mathcal{X}')^\perp, \mathcal{G}_P^\perp\right) = \gamma\left(\mathcal{X}', \mathcal{G}_P\right) = \delta(P', P) \leq \bar{r}_p$ implies that $\theta(w, z) \leq \arcsin \bar{r}_p$. Noting $w \in \mathcal{G}_P^\perp = (\mathcal{K}[-\tilde{N}\ \ \tilde{M}])^\perp = \mathcal{R}\left(\begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix}\right)$, we obtain that $w = \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} v$ for some $v \in h_2$. We set $w_j = \begin{bmatrix} -\tilde{N}^* \\ \tilde{M}^* \end{bmatrix} E_j v$. It is easy to verify that $w_j \in \mathcal{G}_P^\perp \subset (\mathcal{G}_{P(j)})^\perp$ and $\theta(w_j, w) < \varepsilon$ for all $j \geq j_3$. Also, there exists $z_j \in (\mathcal{G}'_{C(j)})^\perp$ such that $\theta(z_j, z) < \varepsilon$ for $j \geq j_4$.

Therefore, for all $j \geq \max\{j_3, j_4\}$,

$$\arcsin \bar{r}_p + 2\varepsilon > \theta(w,z) + \theta(w_j, w) + \theta(z_j, z) \geq \theta(w_j, z_j) \geq \theta_{min}\left((\mathcal{G}_{P(j)})^{\perp}, (\mathcal{G}'_{C(j)})^{\perp}\right)$$
$$= \arcsin \|(P\#C)(j)\|^{-1}.$$

Hence, $r_p > \bar{r}_p \geq \dfrac{1}{\inf_{j\geq 0} \|(P\#C)(j)\|}$, which also violates condition (3.3).

The stabilizability of $C' \in \mathcal{B}(C, r_c)$ can be shown similarly. By Theorem 2.5, it follows that P' and C' have right coprime factorizations $P' = N'(M')^{-1}$ and $C' = V'(U')^{-1}$, respectively. Denote $\begin{bmatrix} M_i \\ N_i \end{bmatrix} = (I+\Delta_i)\cdots(I+\Delta_1)\begin{bmatrix} M' \\ N' \end{bmatrix}$ and $\begin{bmatrix} V_i \\ U_i \end{bmatrix} = (I+\Delta_{i+1})^{-1}\cdots(I+\Delta_l)^{-1}\begin{bmatrix} V' \\ U' \end{bmatrix}$. Then the ith perturbed plant $P'_i = N_i M_i^{-1}$ is well-defined and so is the perturbed controller $C'_i = V_i U_i^{-1}$, where $P'_0 = P'$ and $C'_l = C'$. To complete the proof, we need to prove that the perturbed closed-loop system $P'_i\#C'_i$ is stable for $i = 0, 1, \ldots, l$. We first show the well-posedness of $P'_i\#C'_i$. Since $P\#C$ is stable, it follows that $I - PC$ is invertible; that is, $I_n - (PC)_{kk}$ is invertible for each $k \geq 0$. It follows from Proposition 2.8 that $P'_{kk} \in \mathcal{B}(P_{kk}, r_p)$ and $C'_{kk} \in \mathcal{B}(C_{kk}, r_c)$. Moreover, $\|(\Delta_i)_{kk}\| < r_i$. Note that $(P\#C)_{kk} = P_{kk}\#C_{kk}$. Then, by hypothesis (3.4) and [39, Theorem 2], we know that for all $k \geq 0$, $(I - P'_iC'_i)_{kk} = I_n - (P'_i)_{kk}(C'_i)_{kk}$ is invertible for each $k \geq 0$. Immediately, $I - P'_iC'_i$ is invertible. Therefore, $P'_i\#C'_i$ is well-posed.

It remains to show that $P'_i\#C'_i$ is stable. Clearly, the sequence $\{\|P(j)\#C(j)\|\}_{j=1}^{\infty}$ is non-increasing in j. Then $\inf_{j\geq 0} \|(P\#C)(j)\| = \lim_{j\to\infty} \|P(j)\#C(j)\|$. This implies that $\arcsin r_p + \arcsin r_c + \sum_{i=1}^{l} \arcsin r_i \leq \lim_{j\to\infty} \arcsin \dfrac{1}{\|P(j)\#C(j)\|}$. It follows from Definition 2.9 and Proposition 2.10 that $P'_i \in \mathcal{N}_1\left(P'_{i-1}, r_i\right) \subset \mathcal{B}\left(P'_{i-1}, r_i\right)$, $C'_i \in \mathcal{N}_2\left(C'_{i+1}, r_{i+1}\right) \subset \mathcal{B}\left(C'_{i+1}, r_{i+1}\right)$. By the triangular inequality of the angular metric [33, Proposition 1], we have for each $j \geq 0$,

$$\arcsin \delta\left(P'_i(j), P'(j)\right) \leq \sum_{k=1}^{i} \arcsin \delta\left(P'_k(j), P'_{k-1}(j)\right) \leq \sum_{k=1}^{i} \arcsin \delta\left(P'_k, P'_{k-1}\right),$$
$$\arcsin \delta\left(C'_i(j), C'(j)\right) \leq \sum_{k=i+1}^{l} \arcsin \delta\left(C'_k(j), C'_{k-1}(j)\right) \leq \sum_{k=i+1}^{l} \arcsin \delta\left(C'_k, C'_{k-1}\right).$$

Again from Proposition 2.8, we know that $P'(j) \in \mathcal{B}\left(P(j), r_p\right)$ and $C'(j) \in \mathcal{B}\left(C(j), r_c\right)$. Applying the triangular inequality again gives

$$\arcsin \delta\left(P'_i(j), P(j)\right) < \arcsin r_p + \sum_{k=1}^{i} \arcsin \delta\left(P'_k, P'_{k-1}\right),$$
$$\arcsin \delta\left(C'_i(j), C(j)\right) < \arcsin r_c + \sum_{k=i+1}^{l} \arcsin \delta\left(C'_k, C'_{k-1}\right).$$

This implies that

$$\lim_{j\to\infty} \arcsin \delta\left(P_i'(j), P(j)\right) \le \arcsin r_p + \sum_{k=1}^{i} \arcsin \delta\left(P_k', P_{k-1}'\right),$$

$$\lim_{j\to\infty} \arcsin \delta\left(C_i'(j), C(j)\right) \le \arcsin r_c + \sum_{k=i+1}^{l} \arcsin \delta\left(C_k', C_{k-1}'\right).$$

Thus, we have

$$\begin{aligned}
&\lim_{j\to\infty} \left(\arcsin \delta\left(P_i'(j), P(j)\right) + \arcsin \delta\left(C_i'(j), C(j)\right)\right) \\
&\le \arcsin r_p + \arcsin r_c + \sum_{k=1}^{i} \arcsin \delta\left(P_k', P_{k-1}'\right) + \sum_{k=i+1}^{l} \arcsin \delta\left(C_k', C_{k-1}'\right) \\
&< \arcsin r_p + \arcsin r_c + \sum_{i=1}^{l} \arcsin r_i \le \lim_{j\to\infty} \arcsin \frac{1}{\|P(j) \# C(j)\|}.
\end{aligned}$$

This means there exists $j_0 > 0$ such that

$$\arcsin \delta\left(P_i'(j_0), P(j_0)\right) + \arcsin \delta\left(C_i'(j_0), C(j_0)\right) < \arcsin \frac{1}{\|P(j_0) \# C(j_0)\|}.$$

By [19, Theorem 4], we know that the closed-loop system $P_i'(j_0) \# C_i'(j_0)$ is stable.

Now it is easy to see that $N_i M_i^{-1}$ and $V_i U_i^{-1}$ is a right coprime factorizations of P_i' and C_i', respectively. According to Theorem 2.5, C_i' has a left coprime factorization $C_i' = \tilde{U}_i^{-1} \tilde{V}_i$. Let $W_i := \tilde{U}_i M_i - \tilde{V}_i N_i$. Then W_i is invertible because $P_i' \# C_i'$ is well-posed. It can be easily verified that $N_i(j_0) M_i^{-1}(j_0)$ is a right coprime factorization of $P_i'(j_0)$, and $\tilde{U}_i^{-1}(j_0) \tilde{V}_i(j_0)$ is a left coprime factorization of $C_i'(j_0)$. Since $P_i'(j_0) \# C_i'(j_0)$ is stable, it follows from Lemma 2.4 that $W_i(j_0)$ is stably invertible. We partition W_i into $W_i = \begin{bmatrix} E_{j_0} W_i E_{j_0}|_{E_{j_0} h_2} & 0 \\ F_{j_0} W_i E_{j_0}|_{E_{j_0} h_2} & W_i(j_0) \end{bmatrix} =: \begin{bmatrix} W_{i1} & 0 \\ W_{i2} & W_{i3} \end{bmatrix}$. Consequently, $W_i^{-1} = \begin{bmatrix} W_{i1}^{-1} & 0 \\ -W_{i3}^{-1} W_{i2} W_{i1}^{-1} & W_{i3}^{-1} \end{bmatrix}$ is causal and bounded; that is, $\tilde{U}_i M_i - \tilde{V}_i N_i$ is stably invertible. Again, from Lemma 2.4, we obtain that $P_i' \# C_i'$ is stable for $i = 0, 1, \ldots, l$. Therefore, the NCS in Fig. 3 is stable. This finishes the proof for the sufficiency part. ∎

3.4. Necessity of the robust stability condition. The necessity part of Theorem 3.4 will be proved by using the contrapositive argument. First, assuming that condition (3.4) fails, we will employ the idea in the proof of necessity part of [39, Theorem 2] to show that there exists $i \in \{0, 1, \ldots, l\}$ such that $P_i' \# C_i'$ is not well-posed. Finally, given condition (3.5) violated, we will construct a series of uncertainty quartets $\{\Delta_i\}_{i=1}^{l} \subset \mathcal{S}$, a perturbed plant P' and a perturbed controller C', which destabilize the NCS. The stability of a feedback system is determined by the minimum angle between the graphs of the plant and controller. In order to construct Δ_i, we aim to rotate a specific vector in the subspace $\mathcal{G}_{P(j)}$ for some j with cascaded operators in the form of $I + \Delta_i$. Then the uncertainty quartets $\Delta_i \in \mathcal{S}$ for $1 \le i \le l$ will be completely constructed through one-vector interpolation problem for nest algebras. As a result, we first briefly review the direct rotations of subspaces in a Hilbert space $\mathcal{H}$. The background and notation follow from [8].

Given two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a Hilbert space $\mathcal{H}$, It is shown in [8] that if $\|\Pi_{\mathcal{X}} - \Pi_{\mathcal{Y}}\| < 1$, then there exists a unitary operator U such that $U\Pi_{\mathcal{X}} = \Pi_{\mathcal{Y}}U$, namely, $\mathcal{X}$ can be transformed to $\mathcal{Y}$ by U. Define the following isometries: $X_1 : \mathcal{K}(X_1)^\perp \to \mathcal{H}$ and $X_2 : \mathcal{K}(X_2)^\perp \to \mathcal{H}$ with $X_1\left(\mathcal{K}(X_1)^\perp\right) = \mathcal{X}$ and $X_2\left(\mathcal{K}(X_2)^\perp\right) = \mathcal{X}^\perp$. Then $X_1X_1^* = \Pi_{\mathcal{X}}, X_2X_2^* = \Pi_{\mathcal{X}^\perp}$ and $[X_1\ X_2]^{-1} = \begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix}$. We can write $U = [X_1\ X_2]\begin{bmatrix} X_1^*UX_1 & X_1^*UX_2 \\ X_2^*UX_1 & X_2^*UX_2 \end{bmatrix}\begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} =: X\begin{bmatrix} C_0 & -S_1 \\ S_0 & C_1 \end{bmatrix}X^*$, where $X := [X_1\ X_2]$. Let $\Theta = \arccos(C_0C_0^*)^{\frac{1}{2}}$ be the continuous functional calculus for $(C_0C_0^*)^{\frac{1}{2}}$ [16, Chapter 2]. Then $\theta_{min}(\mathcal{X},\mathcal{Y})$ is the minimum singular value of Θ [8].

Definition 3.9 ([8, Definition 3.1]). *A unitary solution* $U = X\begin{bmatrix} C_0 & -S_1 \\ S_0 & C_1 \end{bmatrix}X^*$ *of* $U\Pi_{\mathcal{X}} = \Pi_{\mathcal{Y}}U$ *is called a direct rotation from* $\mathcal{X}$ *to* $\mathcal{Y}$ *if it satisfies that* $C_0 \geq 0,\ C_1 \geq 0$ *and* $S_1 = S_0^*$.

As shown in [8], among all unitary transformations mapping $\mathcal{X}$ to $\mathcal{Y}$, the direct rotation is the "most economic" in some sense.

Proposition 3.10 ([8, Proposition 3.2]). *A direct rotation exists if and only if* $\dim \mathcal{X}\cap\mathcal{Y}^\perp = \dim \mathcal{X}^\perp \cap \mathcal{Y}$.

Now, assume that $\dim \mathcal{X}\cap\mathcal{Y}^\perp = \dim \mathcal{X}^\perp\cap\mathcal{Y}$. Following the derivation in [8], we obtain the direct rotation from $\mathcal{X}$ to $\mathcal{Y}$ as $U = X\exp\left(\begin{bmatrix} 0 & -A \\ A^* & 0 \end{bmatrix}\right)X^*$, where the minimum singular value of A is $\theta_{min}(\mathcal{X},\mathcal{Y})$. For $\lambda \in [0,1]$, let

$$\mathcal{Z} = X\exp\left(\begin{bmatrix} 0 & -\lambda A \\ \lambda A^* & 0 \end{bmatrix}\right)X^*\mathcal{X}.$$

Then a direct rotation from $\mathcal{X}$ to $\mathcal{Z}$ is $X\exp\left(\begin{bmatrix} 0 & -\lambda A \\ \lambda A^* & 0 \end{bmatrix}\right)X^*$, and it can be seen in [34] that $X\exp\left(\begin{bmatrix} 0 & -(1-\lambda)A \\ (1-\lambda)A^* & 0 \end{bmatrix}\right)X^*$ is a direct rotation from $\mathcal{Z}$ to $\mathcal{Y}$. Consequently, we get $\theta_{min}(\mathcal{X},\mathcal{Z}) = \lambda\theta_{min}(\mathcal{X},\mathcal{Y})$ and $\theta_{min}(\mathcal{Z},\mathcal{Y}) = (1-\lambda)\theta_{min}(\mathcal{X},\mathcal{Y})$. This implies that

$$\theta_{min}(\mathcal{X},\mathcal{Y}) = \theta_{min}(\mathcal{X},\mathcal{Z}) + \theta_{min}(\mathcal{Z},\mathcal{Y}). \tag{3.6}$$

Notably, in the proof of the necessity part of Theorem 3.4, we will make use of the direct rotations of one-dimensional subspaces in Hilbert space.

The uncertainty quartets $\Delta_i \in \mathcal{S}$ for $1 \leq i \leq l$ will be completely constructed through the following one-vector interpolation problem for nest algebras [30].

Lemma 3.11. *Let* $x, y \in h_2$. *There exists* $A \in \mathcal{S}$ *such that* $Ax = y$ *if and only if there exists a constant* c *such that for each* $k \geq 0$, $\|E_ky\| \leq c\|E_kx\|$. *If such an* A *exists, it can be chosen so that* $\|A\| \leq c$.

The stability of feedback systems can be characterized in terms of the minimal angle between the graphs of the plant and controller [16, Chapter 9]. We state this as a proposition.

Proposition 3.12. *The closed-loop system* $P\#C$ *is stable if and only if*

$$\theta_{min}\left(\mathcal{G}_P, \mathcal{G}_C'\right) > 0.$$

Proof of the necessity of Theorem 3.4. We first assume that condition (3.4) does not hold. Then there exists $k_0 \geq 0$ such that

$$\arcsin r_p + \arcsin r_c + \sum_{i=1}^{l} \arcsin r_i > \arcsin \frac{1}{\|P_{k_0k_0} \# C_{k_0k_0}\|}.$$

Consider the nominal system $P_{k_0k_0} \# C_{k_0k_0}$. From the proof of [39, Theorem 2], we know that there exist matrices $\Delta_{p,k_0}, \Delta_{c,k_0}$ and Δ_{i,k_0} with $\|\Delta_{p,k_0}\| < r_p, \|\Delta_{c,k_0}\| < r_c$ and $\|\Delta_{i,k_0}\| < r_i, i = 1, 2, \ldots, l$, such that $P'_{l,k_0} \# C_{k_0k_0}$ is not well-posed. Here $P'_{l,k_0} := N_{l,k_0} M_{l,k_0}^{-1}$ is a right coprime factorization of P'_{l,k_0}, where $\begin{bmatrix} M_{l,k_0} \\ N_{l,k_0} \end{bmatrix} := (I + \Delta_{c,k_0})(I + \Delta_{l,k_0}) \cdots (I + \Delta_{1,k_0})(I + \Delta_{p,k_0}) \begin{bmatrix} M_{k_0k_0} \\ N_{k_0k_0} \end{bmatrix}$, and NM^{-1} is a right coprime factorization of P. Let VU^{-1} be a right coprime factorization of C. It is easy to check that $V_{k_0k_0} U_{k_0k_0}^{-1}$ is a right coprime factorization of $C_{k_0k_0}$. We know from Lemma 2.4 that $\begin{bmatrix} M_{l,k_0} & V_{k_0k_0} \\ N_{l,k_0} & U_{k_0k_0} \end{bmatrix}$ is not invertible.

Decompose h_2 as $E_{k_0-1}h_2 \oplus (E_{k_0} - E_{k_0-1})h_2 \oplus F_{k_0}h_2$, and define the following operators on h_2 via

$$\Delta_p := \begin{bmatrix} 0 & & \\ & \Delta_{p,k_0} & \\ & & 0 \end{bmatrix}, \ \Delta_c := \begin{bmatrix} 0 & & \\ & \Delta_{c,k_0} & \\ & & 0 \end{bmatrix} \text{ and } \Delta_i := \begin{bmatrix} 0 & & \\ & \Delta_{i,k_0} & \\ & & 0 \end{bmatrix}$$

for $i = 1, 2, \ldots, l$. Apparently, $\Delta_p, \Delta_c, \Delta_i \in \mathcal{S}$ with $\|\Delta_p\| < r_p, \|\Delta_c\| < r_c$ and $\|\Delta_i\| < r_i$. We set $\begin{bmatrix} M' \\ N' \end{bmatrix} = (I + \Delta_p) \begin{bmatrix} M \\ N \end{bmatrix}$ and $\begin{bmatrix} V' \\ U' \end{bmatrix} = (I + \Delta_c)^{-1} \begin{bmatrix} V \\ U \end{bmatrix}$. Then $P' = N'(M')^{-1} \in \mathcal{N}_1(P, r_p) \subset \mathcal{B}(P, r_p)$, and $C' = V'(U')^{-1} \in \mathcal{N}_2(C, r_c) \subset \mathcal{B}(C, r_c)$. Define $\begin{bmatrix} M_l \\ N_l \end{bmatrix} := (I + \Delta_l)(I + \Delta_{l-1}) \cdots (I + \Delta_1) \begin{bmatrix} M' \\ N' \end{bmatrix}$. Then $P'_l := N_l M_l^{-1}$ is a right coprime factorization of P'_l. It is easy to verify that $(N_l)_{k_0k_0}((M_l)_{k_0k_0})^{-1}$ and $V'_{k_0k_0}(U'_{k_0k_0})^{-1}$ are right coprime factorizations of $(P'_l)_{k_0k_0}$ and $C'_{k_0k_0}$, respectively. Furthermore, by the definitions of Δ_p, Δ_c and Δ_i, we see that $\begin{bmatrix} (M_l)_{k_0k_0} & V'_{k_0k_0} \\ (N_l)_{k_0k_0} & U'_{k_0k_0} \end{bmatrix} = (I + \Delta_{c,k_0})^{-1} \begin{bmatrix} M_{l,k_0} & V_{k_0k_0} \\ N_{l,k_0} & U_{k_0k_0} \end{bmatrix}$. Hence, the matrix in the left side of the above equality is not invertible, which shows that $(I - P'_l C')_{k_0k_0} = I_n - (P'_l C')_{k_0k_0}$ is not invertible. This violates the well-posedness of $P'_l \# C'$. Therefore, we have shown the necessity of the condition in (3.4).

In the rest, it suffices to show the necessity of the condition in (3.5). The proof proceeds by using the contrapositive argument. Suppose that condition (3.5) does not hold. Clearly, we have for all $j \geq 0$, $\arcsin \frac{1}{\|P(j) \# C(j)\|} < \sum_{i=1}^{q} \arcsin r_i$, where $q = l + 2$, $r_{l+1} := r_p$ and $r_{l+2} := r_c$. For $i = 1, \ldots, q$, we can always choose $0 < \tilde{r}_{i,j} < r_i$ such that $\arcsin \frac{1}{\|P(j) \# C(j)\|} = \sum_{i=1}^{q} \arcsin \tilde{r}_{i,j}$. By Proposition 2.10, we have $\mathcal{N}_1(P, r_p) \subset \mathcal{B}(P, r_p)$ and $\mathcal{N}_2(C, r_c) \subset \mathcal{B}(C, r_c)$. Thus, we only need to construct $\{\Delta_i\}_{i=1}^{q} \subset \mathcal{S}$ satisfying $\|\Delta_i\| < r_i$ such that $P'_q \# C$ is unstable, where $\mathcal{G}_{P'_q} = \left(\prod_{k=1}^{q} (I + \Delta_{q+1-k}) \right) \mathcal{G}_P$.

Note that $\mathcal{G}_{P(j)}$ and $\mathcal{G}'_{C(j)}$ are two closed subspaces of $F_j h_2$, and for $j \geq 0$, it holds that $\theta_{min}\left(\mathcal{G}_{P(j)}, \mathcal{G}'_{C(j)}\right) = \arcsin \frac{1}{\|P(j)\#C(j)\|} = \sum_{i=1}^{q} \arcsin \tilde{r}_{i,j}$. Now, we can choose $u_j \in \mathcal{G}_{P(j)}$ and $w_j \in \mathcal{G}'_{C(j)}$ satisfying $\theta(u_j, w_j) = \sum_{i=1}^{q} \arcsin \tilde{r}_{i,j}$. Let $\mathcal{U}_{0,j} = \operatorname{span}\{u_j\}$ and $\mathcal{W}_{0,j} = \operatorname{span}\{w_j\}$ be the one-dimensional subspaces spanned by u_j and w_j, respectively. Note that $\dim \mathcal{U}_{0,j} \cap \mathcal{W}_{0,j}^{\perp} = \dim \mathcal{U}_{0,j}^{\perp} \cap \mathcal{W}_{0,j}$. By Proposition 3.10, a direct rotation from $\mathcal{U}_{0,j}$ to $\mathcal{W}_{0,j}$ is given by $X \exp\left(\begin{bmatrix} 0 & -A \\ A^* & 0 \end{bmatrix}\right) X^*$, where the minimum singular value of A is $\theta_{min}\left(\mathcal{U}_{0,j}, \mathcal{W}_{0,j}\right) = \sum_{i=1}^{q} \arcsin \tilde{r}_{i,j}$. Denote the direct rotation operator as

$$\phi(\lambda) := X \exp\left(\begin{bmatrix} 0 & -\lambda A \\ \lambda A^* & 0 \end{bmatrix}\right) X^*, \quad \lambda \in [0,1].$$

Set $\lambda_i = \dfrac{\sum_{k=1}^{i} \arcsin \tilde{r}_{k,j}}{\sum_{k=1}^{q} \arcsin \tilde{r}_{i,j}}$ and $\lambda_q = 1$. Denote $\mathcal{U}_{i,j} = \phi(\lambda_i)\mathcal{U}_{0,j}$. It is easy to see that $\theta_{min}\left(\mathcal{U}_{i,j}, \mathcal{U}_{0,j}\right) = \lambda_i \theta_{min}\left(\mathcal{U}_{0,j}, \mathcal{W}_{0,j}\right)$ for each $i = 1, \ldots, q$, which shows $\theta_{min}\left(\mathcal{U}_{q,j}, \mathcal{U}_{0,j}\right) = \sum_{i=1}^{q} \arcsin \tilde{r}_{i,j}$. By (3.6), we get $\theta_{min}\left(\mathcal{U}_{0,j}, \mathcal{W}_{0,j}\right) = \theta_{min}\left(\mathcal{U}_{0,j}, \mathcal{U}_{q,j}\right) + \theta_{min}\left(\mathcal{U}_{q,j}, \mathcal{W}_{0,j}\right)$. Hence $\theta_{min}\left(\mathcal{U}_{q,j}, \mathcal{W}_{0,j}\right) = 0$. Furthermore, we observe that

$$\mathcal{U}_{i,j} = \phi(\lambda_i)\phi(\lambda_{i-1})^* \mathcal{U}_{i-1,j} = X \exp\left(\begin{bmatrix} 0 & (\lambda_{i-1} - \lambda_i) A \\ (\lambda_i - \lambda_{i-1}) A^* & 0 \end{bmatrix}\right) X^* \mathcal{U}_{i-1,j},$$

yielding that $\theta_{min}\left(\mathcal{U}_{i,j}, \mathcal{U}_{i-1,j}\right) = \arcsin \tilde{r}_{i,j}$ for $i = 1, \ldots, q$.

Let $Q_{i,j} : \mathcal{U}_{i,j}^{\perp} \to \mathcal{U}_{i-1,j}$ be the parallel projection onto $\mathcal{U}_{i-1,j}$ along $\mathcal{U}_{i,j}$ [19], Then $\|Q_{i,j}\| = \frac{1}{\tilde{r}_{i,j}}$. It is straightforward to check that there exists $v_{i,j} \in \mathcal{U}_{i,j}^{\perp}$ with $\|v_{i,j}\| = 1$, such that $\|Q_{i,j} v_{i,j}\| = \frac{1}{\tilde{r}_{i,j}} > \frac{1}{r_i}$ and $Q_{i,j} v_{i,j} = v_{i,j} + Q_{i+1,j} v_{i+1,j}$ for $i = 1, \ldots, q$, where $Q_{q+1,j} v_{q+1,j} := \lambda_j w_j$ for some $\lambda_j \in \mathbb{C}$. Since $\lim_{j \to \infty} \|E_{j+1} Q_{i,j} v_{i,j}\| > \frac{1}{r_i}$, it follows that there exists j_1 satisfying $\|E_{j_1+1} Q_{i,j_1} v_{i,j_1}\| > \frac{1}{r_i}$ for all $1 \leq i \leq q$. Therefore, for all $j \geq j_1 + 1$, we have $\frac{\|E_j v_{i,j_1}\|}{\|E_j Q_{i,j_1} v_{i,j_1}\|} \leq \frac{1}{\|E_{j_1+1} Q_{i,j_1} v_{i,j_1}\|} < r_i$. Let $c_i = \sup_{j \geq j_1+1} \frac{\|E_j v_{i,j_1}\|}{\|E_j Q_{i,j_1} v_{i,j_1}\|}$. Then $c_i < r_i$. We write $v_{i,j_1} = (v_{j_1+1}, v_{j_1+2}, \ldots)$ and $Q_{i,j_1} v_{i,j_1} = (y_{j_1+1}, y_{j_1+2}, \ldots)$. Set $v_i = (0, 0, \ldots, 0, v_{j_1+1}, v_{j_1+2}, \ldots)$, $Q_i v_i = (0, 0, \ldots, 0, y_{j_1+1}, y_{j_1+2}, \ldots) \in h_2$. Note that $E_j v_i = 0$ for $j = 1, \ldots, j_1$. Then for all $j \geq 0$, $\|E_j v_i\| \leq c_i \|E_j Q_i v_i\|$. In view of Lemma 3.11, there exists $\overline{\Delta}_i \in \mathcal{S}$ and $\|\overline{\Delta}_i\| \leq c_i < r_i$ satisfying that $\overline{\Delta}_i(Q_i v_i) = v_i$. Clearly, $\overline{\Delta}_i(j_1)(Q_{i,j_1} v_{i,j_1}) = v_{i,j_1}$. Let $\Delta_i = -\overline{\Delta}_i$. Then $\Delta_i \in \mathcal{S}$ with $\|\Delta_i\| < r_i$ such that $\left(\prod_{k=1}^{q} (I + \Delta_{q+1-k})(j_1)\right)(Q_{1,j_1} v_{1,j_1}) = \lambda_{j_1} w_{j_1}$ for some $\lambda_{j_1} \in \mathbb{C}$. Since $Q_{1,j_1} v_{1,j_1} \in$

$\mathcal{U}_{0,j_1}$ and $\lambda_{j_1} w_{j_1} \in \mathcal{W}_{0,j_1}$. Then we have $\theta_{min}\left(\prod_{k=1}^{q}(I+\Delta_{q+1-k})(j_1)\mathcal{U}_{0,j_1}, \mathcal{W}_{0,j_1}\right) = 0$. This shows $\theta_{min}\left(\prod_{k=1}^{q}(I+\Delta_{q+1-k})(j_1)\mathcal{G}_{P(j_1)}, \mathcal{G}'_{C(j_1)}\right) = 0$ because $\mathcal{U}_{0,j_1} \subset \mathcal{G}_{P(j_1)}$ and $\mathcal{W}_{0,j_1} \subset \mathcal{G}'_{C(j_1)}$. We set $\begin{bmatrix} M_i \\ N_i \end{bmatrix} = \left(\prod_{k=1}^{i}(I+\Delta_{i+1-k})\right)\begin{bmatrix} M \\ N \end{bmatrix}$ for $i = 1, \ldots, q$. In case M_q is invertible, in light of Proposition 3.12, $P'_q \# C$ is unstable, hence, the NCS is unstable. This completes the necessity part of the proof for condition (3.5). If not, we assume that M_{i-1} is invertible, but M_i is not invertible for some i. We will construct $\hat{\Delta}_i \in \mathcal{S}$ satisfying $\|\hat{\Delta}_i\| < r_i$ such that $(I+\hat{\Delta}_i)(j_1)(Q_{i,j_1} v_{i,j_1}) = Q_{i+1,j_1} v_{i+1,j_1}$ and M'_i is invertible, where $\begin{bmatrix} M'_i \\ N'_i \end{bmatrix} := (I+\hat{\Delta}_i)\begin{bmatrix} M_{i-1} \\ N_{i-1} \end{bmatrix}$.

Write $\Delta_i = \begin{bmatrix} \Delta_{i1} & \Delta_{i2} \\ \Delta_{i3} & \Delta_{i4} \end{bmatrix}$ and $Q_{i,j_1} v_{i,j_1} = \begin{bmatrix} u \\ e \end{bmatrix}$, where $u = (u_{j_1+1}, u_{j_1+2}, u_{j_1+3}, \ldots)$ and $e = (e_{j_1+1}, e_{j_1+2}, e_{j_1+3}, \ldots)$. Note that $\left\|\begin{bmatrix} u \\ e \end{bmatrix}\right\| \neq 0$. Thus at least one of u or e is not 0. Without loss of generality, assume $u \neq 0$. We consider the following two cases:

(1) $e = 0$: In this case, let $\hat{\Delta}_i = \begin{bmatrix} \Delta_{i1} & 0 \\ \Delta_{i3} & \Delta_{i4} \end{bmatrix}$. It is easy to check that $\hat{\Delta}_i \in \mathcal{S}$ with $\|\hat{\Delta}_i\| < r_i$ such that $(I+\hat{\Delta}_i)(j_1)(Q_{i,j_1} v_{i,j_1}) = Q_{i+1,j_1} v_{i+1,j_1}$ and $M'_i = M_{i-1} + \Delta_{i1} M_{i-1}$ is invertible.

(2) $e \neq 0$: In this case, since $u \neq 0$, we may assume that $u_{j_1+1} = 0, u_{j_1+2} \neq 0$ and $e_{j_1+1} \neq 0$. Define

$$V_1 := \begin{bmatrix} \varepsilon_0 I_n & & & & \\ 0 & \varepsilon_1 I_n & & & \\ 0 & -\frac{\varepsilon_2 u_{j_1+3} \otimes u_{j_1+2}}{\|u_{j_1+2}\|^2} & \varepsilon_2 I_n & & \\ 0 & -\frac{\varepsilon_3 u_{j_1+4} \otimes u_{j_1+2}}{\|u_{j_1+2}\|^2} & 0 & \varepsilon_3 I_n & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$V_2 := \begin{bmatrix} 0 & & & \\ -\frac{\varepsilon_1 u_{j_1+2} \otimes e_{j_1+1}}{\|e_{j_1+1}\|^2} & 0 & & \\ 0 & 0 & 0 & \\ \vdots & \vdots & & \ddots \end{bmatrix},$$

where V_1 and V_2 are conformal to $\Delta_{i1}(j_1)$ and $\Delta_{i2}(j_1)$, respectively, $0 < \varepsilon_k < \delta_k$ for each $k \geq 0$, and I_n is the identity matrix. If all the eigenvalues of $\left(M_i(j_1)(M_{i-1}(j_1))^{-1}\right)_{kk}$ are zero, take $\delta_k = 1$. If some eigenvalue of $\left(M_i(j_1)(M_{i-1}(j_1))^{-1}\right)_{kk}$ is nonzero, let $\delta_k = \min\{|\lambda| : \lambda$ is an eigenvalue of $\left(M_i(j_1)(M_{i-1}(j_1))^{-1}\right)_{kk}$ and $\lambda \neq 0\}$. Then $V_1 u + V_2 e = 0$. Let $\hat{\Delta}_i = \begin{bmatrix} \hat{\Delta}_{i1} & \hat{\Delta}_{i2} \\ \Delta_{i3} & \Delta_{i4} \end{bmatrix}$, where $\hat{\Delta}_{i1} := \begin{bmatrix} 0 & 0 \\ 0 & \Delta_{i1}(j_1) + V_1 \end{bmatrix}$ and $\hat{\Delta}_{i2} := \begin{bmatrix} 0 & 0 \\ 0 & \Delta_{i2}(j_1) + V_2 \end{bmatrix}$. Then it is straightforward to check that $\hat{\Delta}_i \in \mathcal{S}$ and $(I+\hat{\Delta}_i)(j_1)(Q_{i,j_1} v_{i,j_1}) = Q_{i+1,j_1} v_{i+1,j_1}$. Moreover, we can choose $\varepsilon_k > 0$ sufficiently small so that $\|\hat{\Delta}_i\| < r_i$ and $M'_i(j_1) = M_i(j_1) + V_1 M_{i-1}(j_1) + V_2 N_{i-1}(j_1)$ is invertible. We partition M'_i into

$M_i' = \begin{bmatrix} E_{j_1} M_i' E_{j_1}|_{E_{j_1} h_2} & 0 \\ F_{j_1} M_i' E_{j_1}|_{E_{j_1} h_2} & M_i'(j_1) \end{bmatrix}$. Note that $E_{j_1} M_i' E_{j_1}|_{E_{j_1} h_2} = E_{j_1} M_{i-1} E_{j_1}|_{E_{j_1} h_2}$ is invertible. Hence, M_i' is invertible. □

Remark 3.13. In the proof of the necessity of Theorem 3.4, it is required that the destabilizing perturbations of the two-port networks are causal operators. The key step to achieve this target is via solving the one-vector interpolation problem for nest algebras.

4. Conclusions. In this paper, we consider the robust stability problem for a time-varying two-port NCS. The uncertainties in the plant and controller are measured by the gap metric. The uncertainty involved in the two-port network is represented by the transmission matrix $I + \Delta$, where $\Delta \in \mathcal{S}$ is bounded by the operator norm. We obtain a necessary and sufficient condition in the form of an "arcsine" inequality, for robust stability of the NCS, which generalizes a similar result for linear time-invariant NCSs. The sufficiency is mainly derived from the triangular inequality of the angular metric. The key step in the proof of the necessity relies on the one-vector interpolation problem for nest algebras. Furthermore, as one of the important contributions of this paper, a necessary and sufficient condition for robust stability of LTV systems has been provided for the case when gap-metric perturbations to the plant alone are considered. Notably, our models for systems and uncertainties incorporate the causality issue, which is often neglected in the previous works. The optimal robust controller design problem can be directly motivated by our stability condition, and it will be taken as a future research direction based on the time-varying controller design technique in [18].

Acknowledgments. The authors would like to thank Sen Zhu and Sei Zhen Khong for interesting discussions. We wish to thank a reviewer who helped amend the proof of our main theorem. We also thank the People's Government of Pengjiang District, Jiangmen, China for partially supporting the research.

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