Networked Robust Stabilization with Simultaneous Uncertainties in Plant, Controller and Communication Channels

Di Zhao and Li Qiu

***Abstract*— In this paper, we study the robust stabilization of a networked control system (NCS) with the communication channels described by cascade two-port networks. Simultaneous uncertainties are assumed to be in the plant, controller and two-port communication channels. The cascade two-port connections apply to the scenario where signals in the NCS are required to pass through bidirectional communication channels separated by several relays. Distortions and interferences are taken into account at each stage during the communication. In terms of robustness, we consider $\mathcal{H}_\infty$ norm bounded uncertainties in the transmission matrices of the communication channels as well as the gap metric uncertainties in the plant and controller. A necessary and sufficient condition for the robust stability of the NCS is given in the form of an "arcsin" inequality, taking advantages of the properties of the canonical angles between subspaces defined on system graphs. With the analysis result, the synthesis problem can be solved through a tractable $\mathcal{H}_\infty$ optimization.**

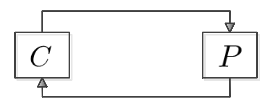

Fig. 1: A Standard Closed-Loop System

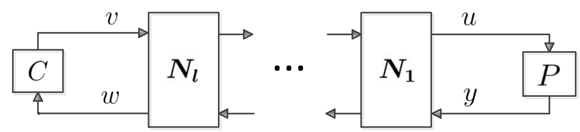

Fig. 2: An NCS with Two-Port Connections

I. Introduction

Robust control and, in particular, robust stabilization problems have been shown to be critical in the analysis and design of control systems with inaccurate or partially known models and communication uncertainties.

In order to address robust stabilization problems, appropriate distances defined on linear time-invariant (LTI) systems are fundamental to characterize the uncertainties. A natural and mathematically tractable method to model uncertain dynamics is through the gap metric and its variations, among which the gap [1]–[3], the pointwise gap [4] and the ν-gap [5], [6] have been intensively studied. The work [5] shows that the ν-gap is superior to the others for describing the largest set of systems given the same robust stability condition. We adopt the gap metric in this study to measure the uncertainties in the plant and controller for the sake of simplicity, and most of the results still hold with similar arguments for the ν-gap and the pointwise gap. As long as the uncertainties can be quantified, problems of robust stabilization can be formulated. The robust stabilization problem of a standard closed-loop system (see Fig. 1) has been well studied and neatly solved in the last decades [3]–[9]. Considering the uncertainties both in the plant and the controller, the works [4], [9] strengthen the stability condition by introducing the angular gap metric, which is the "arcsin" of the gap (or the pointwise gap). It is seen that the gap metric and its variations play an important role in robust stabilization. Furthermore, these metrics can be used to measure the uncertainties of a feedback system with a more general setup, bringing about some concise analytic results.

Practically in most systems, the control signals are transmitted through imperfect communication channels. As the quality of control heavily relies on the communication channels, it is meaningful to consider an robust stabilization problem of a networked control system (NCS) with channel uncertainties. An NCS differs from a standard closed-loop system as the information exchanged between the plant and controller is through communication networks [10]. The communication channels in an NCS can be modeled differently so as to reveal actual situations. In our study, we give a two-port NCS model by extending the standard closed-loop system (Fig. 1) to the feedback system with cascade two-port connections (Fig. 2). Based on the architecture of the two-port NCS, we measure the dynamic uncertainties in the plant and controller with the gap metric and measure those in the two-port networks with $\mathcal{H}_\infty$-norm bound on the perturbations to transmission matrices. Our formulation of robust stabilization problem is mainly motivated by the application scenario on stabilizing a feedback system where the plant and the controller cannot communicate directly and the signals can only pass through communication networks with several relays, as in, for example, satellite networks [11], wireless sensor networks [12] and so on. Moreover, each sub-system between two neighbouring relays, representing a communication channel, may involve not only the multiplicative distortion on the transmitted signal itself but the additive interference caused by the signal in the reverse direction, which is usually encountered in a bidirectional wireless

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The authors are with the Department of Electronic and Computer Engineering, the Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong dzhaoaa@connect.ust.hk, eeqiu@ece.ust.hk

network subject to channel fading or under malicious attacks [13]. So far, we have presented the architecture of our two-port NCS model and formulated the robust stabilization problem by considering uncertainties in different parts of the NCS with different measures. The uncertain two-port networks play a leading role in our NCS model.

Two-port network is not a new concept and has been studied over decades for different purposes. Historically, the two-port network was originally introduced in the electrical circuit theory [14]. Later, the two-port network was borrowed to represent LTI systems in book [15], where an LTI system is characterized by a chain-scattering representation, which is essentially a two-port network. Recently, some approaches based on the two-port network to modeling the communication in a networked feedback system is studied in [16], where uncertain two-port connections are used to introduce channel uncertainties, based on which we propose our cascade two-port communication model.

One of the contributions of our study is a clean result on analysing the stability of a feedback system with multiple sources of uncertainties. As we know, a general approach to handling the robust stabilization problem with structured uncertainties is to solve a μ-synthesis problem related to the structured singular value, which is difficult when there are multiple sources of uncertainties in the model [17]. Surprisingly, by generalizing the "$\arcsin$" theorem [9] for a standard closed-loop system, we are able to give a concise necessary and sufficient robust stability condition for the two-port NCS. Furthermore, as the stability margin given by the stability condition of the two-port NCS coincides with that of the standard closed-loop system, the synthesis problem can be solved with the same approaches, i.e., an $\mathcal{H}_\infty$ optimization.

Notice that there are previous works on robust stabilization of NCSs with special architectures and various uncertainty descriptions. For example, the work [18] considers a plant with parametric uncertainties over networks subject to packet loss, the work [19] considers a plant with polytopic uncertainties in its coefficients over a communication channel subject to fadings and so on. The differences of our work from the previous ones are that we model the dynamic uncertainties not only in the plant but in the controller and that our channel model characterizes bi-directional communication involving both distortions and interferences.

This paper is organized as follows. In Section II, we introduce the notation system and preliminaries of the robust control problem related to the standard closed-loop system. In Section III, the physical meaning of a two-port network is discussed, then a two-port NCS is modeled and a stability criterion for the NCS is given. In Section IV, we present our main results, where a necessary and sufficient robust stability condition is given for the NCS. Last, we summarize our contribution and discuss future work in Section V. Most of the proofs of the lemmas and theorems are omitted due to the space limitation.

II. Preliminary Results

A. Notation

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be the real or complex field and $\mathbb{F}^n$ be the linear space of n-dimensional vectors over the field $\mathbb{F}$. For matrix $A \in \mathbb{F}^{m\times n}$, its conjugate transpose is denoted by A^* and its k-th singular value is denoted by the symbol $\sigma_k(A)$, $k = 1, 2, \ldots, \min\{m, n\}$, in a nonincreasing order. The spectral norm of A is defined as $\|A\| = \sigma_1(A)$, and the range of A is $\mathcal{R}(A)$.

We assume that all the systems in this paper are continuous-time LTI systems represented by its transfer function and the symbol s of the transfer functions may be omitted for briefness. $\mathcal{L}_2$ ($\mathcal{L}_\infty$, respectively) and $\mathcal{H}_2$ ($\mathcal{H}_\infty$, respectively) denote the standard Lebesgue and Hardy 2-spaces (∞-spaces, respectively). $\mathcal{RL}_\infty$ ($\mathcal{RH}_\infty$, respectively) consists of all the real rational members of $\mathcal{L}_\infty$ ($\mathcal{H}_\infty$, respectively). $\mathcal{P}$ denotes the field of real rational transfer functions. For transfer function $P(s) \in \mathcal{P}^{m\times n}$, its conjugate is denoted as $P^\sim(s) = P^T(-s)$.

Two transfer matrices M and N in $\mathcal{RH}_\infty$ are right coprime if there exist transfer matrices X_r and Y_r in $\mathcal{RH}_\infty$ such that

$$X_r M + Y_r N = I.$$

Similarly, two transfer matrices $\tilde{M}$ and $\tilde{N}$ in $\mathcal{RH}_\infty$ are left coprime if there exist transfer matrices X_l and Y_l in $\mathcal{RH}_\infty$ such that

$$\tilde{M} X_l + \tilde{N} Y_l = I.$$

It is known [8] that P admits right and left coprime factorizations:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N},$$

where $M, N, \tilde{M}, \tilde{N} \in \mathcal{RH}_\infty$.

With the input of a possibly unstable system P as u and the output as y, the graph of P is defined as

$$\mathcal{G}_P = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} :\ u \in \mathcal{H}_2,\ y = Pu \in \mathcal{H}_2 \right\}.$$

Following some simple argument, we obtain that

$$\mathcal{G}_P = \begin{bmatrix} M \\ N \end{bmatrix} \mathcal{H}_2.$$

B. Robust Stability in Gap Metric

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a Hilbert space $\mathcal{H}$ and let $\Pi_\mathcal{X}$ ($\Pi_\mathcal{Y}$, respectively) be the orthogonal projection on $\mathcal{X}$ ($\mathcal{Y}$, respectively). The gap between the two subspaces is defined as

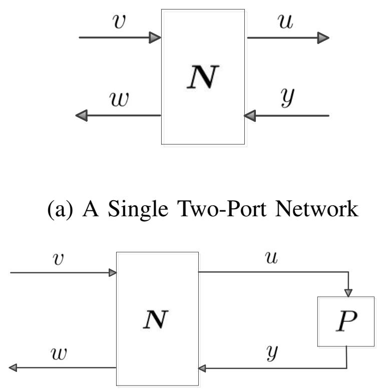
$$\gamma(\mathcal{X}, \mathcal{Y}) = \|\Pi_\mathcal{X} - \Pi_\mathcal{Y}\|.$$

The gap between LTI systems P_1 and $P_2 \in \mathcal{P}$ is defined on their graphs, i.e.,

$$\delta(P_1, P_2) = \gamma(\mathcal{G}_{P_1}, \mathcal{G}_{P_2}).$$

Denote the corresponding gap ball as

$$\mathcal{B}(P_0, r) = \{P : \delta(P_0, P) \le r\},$$



(a) A Single Two-Port Network

(b) One Stage Two-Port Connection

Fig. 3: Two-Port Networks: an Illustration

where $r \geq 0$.

The standard closed-loop system in Fig. 1 is denoted as $[P, C]$, where $P \in \mathcal{P}^{m \times p}$ represents the plant and $C \in \mathcal{P}^{p \times m}$ the controller.

Under the mild condition that $[P,\ C]$ is well-posed, i.e., $I - CP$ has full normal rank, the well-known "Gang of Four" transfer matrix [20] can be represented as

$$\mathit{GoF}(P, C) = \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} I & -C \end{bmatrix}.$$

The closed-loop system $[P,\ C]$ is said to be stable if $\mathit{GoF}(P, C)$ is stable, or $\mathit{GoF}(P, C) \in \mathcal{RH}_\infty$. If $[P,\ C]$ is stable, the stability margin is given as

$$b_{P,C} = \|\mathit{GoF}(P, C)\|_\infty^{-1}.$$

In light of the gap metric for describing system uncertainties, an important robust stability result, namely, the "arcsin" theorem, was obtained in [9].

Theorem 1. *Assume the nominal system $[P_0, C_0]$ is stable. Then for some $r_p, r_c \in [0, 1)$, $[P, C]$ is stable for all $P \in \mathcal{B}(P_0, r_p)$ and $C \in \mathcal{B}(C_0, r_c)$ if and only if*

$$\arcsin r_p + \arcsin r_c < \arcsin b_{P_0,C_0}. \tag{1}$$

Theorem 1 precisely quantifies the largest simultaneous uncertainties in the plant and the controller that the feedback system can tolerate while the stability is maintainted. The stability margin b_{P_0,C_0} is the objective function in the synthesis problem for an optimal robust controller.

III. A Cascade Two-Port Model for Networked Control Systems

A. Two-Port Networks as Communication Channels

The two-port network was firstly investigated in electrical circuit theories. The network $\boldsymbol{N}$ in Fig. 3a has two external ports[1], with one port composed of v, w and the other of u, y, which is the reason we call it a two-port network. A two-port network $\boldsymbol{N}$ has various representations, from which we choose the transmission type representation to model the network as a communication channel. Define the transmission matrix T satisfying that

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \text{ and } \begin{bmatrix} v \\ w \end{bmatrix} = T \begin{bmatrix} u \\ y \end{bmatrix}. \tag{2}$$

When the communication channel is perfect, i.e., communication without distortion or interference, the transmission matrix is

$$T = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix}.$$

If the bidirectional channel admits both distortions and interferences, we can assume the transmission matrix to be

$$T = I + \Delta = \begin{bmatrix} I_p + \Delta_\div & \Delta_- \\ \Delta_+ & I_m + \Delta_\times \end{bmatrix},$$

where $\Delta = \begin{bmatrix} \Delta_\div & \Delta_- \\ \Delta_+ & \Delta_\times \end{bmatrix} \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty \leq r$ and $r \in [0, 1)$. The four-block matrix Δ will be called the uncertainty quartet.

B. Analysis of the Uncertainty Quartet

As shown in Fig. 3b, we connect one stage of the two-port network to the plant and denote the transmission matrix of $\boldsymbol{N}$ as $T = I + \Delta$. It is shown in [15] that combining Equation (2) with $y = Pu$, we can determine an equivalent plant P^e with input v and output w from the linear fractional transformation (LFT). It follows that

$$\begin{aligned} P^e &= \text{LFT}\left(\begin{bmatrix} I_p + \Delta_\div & \Delta_- \\ \Delta_+ & I_m + \Delta_\times \end{bmatrix},\ P \right) \\ &= [(I_m + \Delta_\times)P + \Delta_+][I_p + \Delta_\div + \Delta_- P]^{-1} \end{aligned} \tag{3}$$

and Fig. 4 is the diagram showing how the plant is affected by the uncertainties. Looking through the diagram, we assign each of the members in the uncertainty quartet a detailed explanation [21], namely, the uncertainty of inverse multiplication ($\div$), inverse addition ($-$), addition ($+$) and multiplication ($\times$). The diagonal terms $\Delta_\div, \Delta_\times$ and the off-diagonal terms Δ_-, Δ_+ model two types of perturbations. The diagonal terms represent multiplicative linear distortions of the transmitted signals, mostly due to signal attenuations in the fading channel. The off-diagonal represent additive interference from the reverse signals, which occurs mostly in bidirectional wireless channels and also can be caused by malicious attacks.

In order to keep the system well-posed, we add a mild condition on the channel uncertainty Δ, so that the transfer matrix $I_p + \Delta_\div + \Delta_- P$ has full normal rank. One way to achieve that is to assume Δ to be strictly proper.

Remark 1. *It is worth noting that describing an uncertain system using an LFT is not new in robust control. Traditionally, an uncertain system takes the form of LFT(G, Δ), a fixed LFT of an uncertain component Δ. Nevertheless, in our study, an uncertain system takes the form of LFT$(I + \Delta, P)$,*

[1]Something on the notation we hope to clarify. Note that $\boldsymbol{N}$ denotes certain two-port network, N usually denotes a coprime factor of a transfer function and $\mathcal{N}(\cdot, \cdot)$ denotes certain neighborhood of systems.

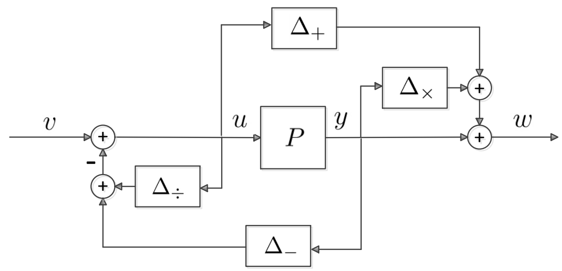


Fig. 4: Plant with the Uncertainty Quartet

which is an uncertain LFT of a possibly known plant P. It is the uncertainty quartet that brings in the main difference.

C. Graph Analysis on Cascade Two-Port NCSs

Although we know an equivalent plant can be derived through LFT$(I+\Delta, P)$, we introduce a more intuitive and simple way to produce the same input-output relations of equivalent plants or controllers by analysing their graphs. As illustrated in Fig. 2, the plant $P = NM^{-1}$ and controller $C = VU^{-1}$ communicate with each other through cascade two-port networks. Considering the input and output of P, we can find all the elements in the graph of P as

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} x,$$

for every $x \in \mathcal{H}_2$.

Consider the transmission type representation of the two-port networks $\{\boldsymbol{N}_k\}_{k=1}^{l}$. If the k-th stage of the network admits an uncertainty $\Delta_k \in \mathcal{RH}_\infty$, then the transmission matrix is given as $T_k = I + \Delta_k$. Signals in Fig. 5 admit the following relations:

$$\begin{bmatrix} u_k \\ y_k \end{bmatrix} = \left(\prod_{j=1}^{k} T_{k+1-j}\right) \begin{bmatrix} u \\ y \end{bmatrix} = \left(\prod_{j=1}^{k} (I + \Delta_{k+1-j})\right) \begin{bmatrix} u \\ y \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = \left(\prod_{j=k+1}^{l} T_j\right) \begin{bmatrix} v \\ w \end{bmatrix} = \left(\prod_{j=k+1}^{l} (I + \Delta_j)^{-1}\right) \begin{bmatrix} v \\ w \end{bmatrix}.$$

If we view P together with $\{\boldsymbol{N}_j\}_{j=1}^{k}$ as an equivalent plant P_k^e with uncertainties $\{\Delta_j\}_{j=1}^{k}$, $P_k^e = N_k M_k^{-1}$ can be identified by its graph:

$$\mathcal{G}_{P_k^e} = \begin{bmatrix} M_k \\ N_k \end{bmatrix} \mathcal{H}_2 = \left(\prod_{j=1}^{k} (I + \Delta_{k+1-j})\right) \mathcal{G}_P. \tag{4}$$

Similarly, if we view C together with $\{\boldsymbol{N}_j\}_{j=k+1}^{l}$ as an equivalent controller C_k^e with uncertainties $\{\Delta_j\}_{j=k+1}^{l}$, $C_k^e = V_k U_k^{-1}$ can be identified by its inverse graph:

$$\mathcal{G}'_{C_k^e} = \begin{bmatrix} V_k \\ U_k \end{bmatrix} \mathcal{H}_2 = \left(\prod_{j=k+1}^{l} (I + \Delta_j)^{-1}\right) \mathcal{G}'_C. \tag{5}$$

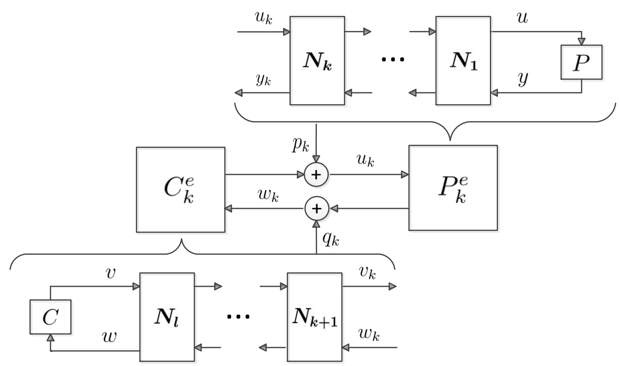


Fig. 5: Equivalent Plant and Controller

where the inverse graph $\mathcal{G}'_C$ is defined as

$$\mathcal{G}'_C = \begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H}_2.$$

For convenience, we regard $k = 0$ as the situation where P is isolated from two-port networks and $k = l$ the situation where C is isolated.

Since $\Delta_k \in \mathcal{RH}_\infty$ and $\|\Delta_k\|_\infty < 1$, we have $I + \Delta_k$ and $(I + \Delta_k)^{-1} \in \mathcal{RH}_\infty$. Hence (M_k, N_k) and (U_k, V_k) must be right coprime, respectively. As the systems are well-posed here, M_k and U_k have full normal rank. Therefore, the equivalent plants and controllers P_k^e and C_k^e are well-defined by their graphs.

D. Two-Port NCS: A Stability Criterion

The stability of the NCS is defined as follows:

Definition 1. *See Fig. 5. The NCS is said to be stable if we inject signals p_k and $q_k \in \mathcal{H}_2$ at the k-th stage for each $k = 0, 1, \ldots, l$, then the signals on all ports, namely, $u_0, \ldots, u_l$, $y_0, \ldots, y_l$, $v_0, \ldots, v_l$ and $w_0, \ldots, w_l$ will be energy-bounded, i.e., they belong to $\mathcal{H}_2$.*

The following lemma simplifies the procedures to determine the stability of the two-port NCS.

Lemma 1. *See Fig. 5. If we inject signals p_k and $q_k \in \mathcal{H}_2$ at the k-th stage for each $k = 0, 1, \ldots, l$, the signals on all ports are energy-bounded if and only if u_k and w_k are energy-bounded, i.e., u_k and $w_k \in \mathcal{H}_2$.*

Note that the system in Fig. 5 is a standard closed-loop system with P_k^e as plant, C_k^e as controller. Therefore, the NCS is stable if and only if the equivalent closed-loop system $[P_k^e, C_k^e]$ in Fig. 5 is stable given the excitation at the k-th stage for $k = 0, 1, \ldots, l$.

IV. Robust Stability for Two-port NCSs

Although it seems very difficult to establish a concise condition for the robust stability of the two-port NCS with structured uncertainties from multiple sources, it turns out the robust stability can be guaranteed by an "$\arcsin$" inequality, which is also shown to be necessary.

A. Main Theorem

In addition to the network uncertainties $\{\Delta_k\}_{k=1}^l$, we assume both the plant P and the controller C admit some uncertainties described by the gap balls centered at a nominal plant P_0 and a nominal controller C_0.

Theorem 2. *Assume the nominal system $[P_0, C_0]$ is stable. For $r_p,\ r_c,\ r_k \in [0,1)$, the NCS in Fig. 2 is robustly stable for all $P \in \mathcal{B}(P_0,\ r_p)$, $C \in \mathcal{B}(C_0,\ r_c)$ and $\Delta_k \in \mathcal{RH}_\infty$, $\|\Delta_k\|_\infty \le r_k,\ k = 1, 2, \ldots, l$ if and only if*

$$\arcsin r_p + \arcsin r_c + \sum_{k=1}^{l} \arcsin r_k < \arcsin b_{P_0,C_0}. \quad (6)$$

Remark 2. *The plant/controller admits the uncertainties measured by the gap, which can be extended to the ν-gap and the pointwise gap with similar arguments. Furthermore, with Theorem 2 in mind, the design of an optimal controller for robust stabilization can be attributed to solving an $\mathcal{H}_\infty$ problem with respect to the "Gang of Four" matrix.*

B. Lemmas and Proofs

Before we proceed to the brief proof of Theorem 2, we introduce more definitions and some useful lemmas on characterizing the inclusive relations of different uncertainty sets.

We already know the gap ball uncertainty is of great importance to develop the concise robust stability condition (1). Next, we introduce two neighborhoods that are equivalent to the gap ball but are expressed in a very different way. Assume that the nominal system P_0 admits the following right coprime factorizations $P_0 = N_0 M_0^{-1}$ and that $r \ge 0$. We define the following set to describe the system uncertainties [22]:

Definition 2.

$$\begin{aligned}\mathcal{N}(P_0,r) = \Bigg\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \\ M, N \in \mathcal{RH}_\infty \textit{ are coprime}, \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty \le r \Bigg\},\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{N}}(P_0,r) = \Bigg\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \\ M, N \in \mathcal{RH}_\infty \textit{ are coprime}, \Delta \in \mathcal{RL}_\infty, \|\Delta\|_\infty \le r \Bigg\}.\end{aligned}$$

From [22] we know $\mathcal{N}(P_0, r) = \widetilde{\mathcal{N}}(P_0, r)$ and we use $\mathcal{N}(P_0, r)$ as the representative. The following lemma gives the relationship between the above uncertain set and the gap ball [22].

Lemma 2. *For each $r \ge 0$, it holds that*

$$\mathcal{N}(P, r) = \mathcal{B}(P, r).$$

Remark 3. *We can define two-port neighbourhoods on the system, representing the forward transmission and the backward transmission, as*

$$\begin{aligned}\mathcal{N}_T(P_0,r) = \Bigg\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta) \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \\ M, N \textit{ are coprime}, \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty \le r \Bigg\},\end{aligned}$$

$$\begin{aligned}\widetilde{\mathcal{N}}_T(P_0,r) = \Bigg\{ P = NM^{-1} : \begin{bmatrix} M \\ N \end{bmatrix} = (I + \Delta)^{-1} \begin{bmatrix} M_0 \\ N_0 \end{bmatrix}, \\ M, N \textit{ are coprime}, \Delta \in \mathcal{RH}_\infty, \|\Delta\|_\infty \le r \Bigg\},\end{aligned}$$

where $r \ge 0$. It is clear from Definition 2 that

$$\mathcal{N}_T(P_0, r) \cup \widetilde{\mathcal{N}}_T(P_0, r) \subset \mathcal{N}(P_0, r) = \mathcal{B}(P_0, r),$$

hence the topology induced by the two-port neighbourhood is closely related to the graph topology induced by the gap. This inclusion relation will be very helpful to show the sufficiency of the robust stability condition (6).

Next based on the previously introduced lemmas, we give the proof on the sufficiency of our main theorem. The necessity is quite lengthy, involving constructions of a series of transmission matrices in the two-port networks, hence we only outline the procedures.

Proof. (of Theorem 2)

We first prove the sufficient part. Given

$$\arcsin r_p + \arcsin r_c + \sum_{k=1}^{l} \arcsin r_k < \arcsin b_{P_0,C_0},$$

we need to prove the NCS is stable. As illustrated in Fig. 5, we excite the network at the k-th stage, $k = 0, 1, 2, \ldots, l$. Denote the right coprime factorizations as $P = NM^{-1}$ and $C = VU^{-1}$. Denote that

$$\begin{bmatrix} M_k \\ N_k \end{bmatrix} = \left(\prod_{j=1}^{k} (I + \Delta_{k+1-j}) \right) \begin{bmatrix} M \\ N \end{bmatrix},$$

$$\begin{bmatrix} V_k \\ U_k \end{bmatrix} = \left(\prod_{j=k+1}^{l} (I + \Delta_j)^{-1} \right) \begin{bmatrix} V \\ U \end{bmatrix}.$$

From the well-posedness of the systems, the equivalent plants $P_k^e = N_k M_k^{-1}$ are well-defined and so are the controllers $C_k^e = V_k U_k^{-1}$. We denote $P_0^e = P$ and $C_l^e = C$. Hence, from Remark 3, we know

$$\begin{aligned} P_k^e &\in \mathcal{N}_T(P_{k-1}^e, r_k) \subset \mathcal{B}(P_{k-1}^e, r_k), \\ C_k^e &\in \widetilde{\mathcal{N}}_T(C_{k+1}^e, r_{k+1}) \subset \mathcal{B}(C_{k+1}^e, r_{k+1}). \end{aligned}$$

From [9, Proposition 1], we know $\arcsin \delta(P_1, P_2)$ is a metric, called the angular gap metric, on the space $\mathcal{P}^{m \times p}$. By iteratively utilizing the triangular inequality of the angular gap, we obtain that

$$\arcsin \delta(P_k^e, P) \le \sum_{j=1}^{k} \arcsin \delta(P_j^e, P_{j-1}^e) \le \sum_{j=1}^{k} \arcsin r_j,$$

$$\arcsin \delta(C_k^e, C) \leq \sum_{j=k+1}^{l} \arcsin \delta(C_{j-1}^e, C_j^e) \leq \sum_{j=k+1}^{l} \arcsin r_j.$$

As we also have $P \in \mathcal{B}(P_0, r_p)$ and $C \in \mathcal{B}(C_0,\ r_c)$, the triangular inequality indicates that

$$\arcsin \delta(P_k^e, P_0) \leq \arcsin r_p + \sum_{j=1}^{k} \arcsin r_j,$$

$$\arcsin \delta(C_k^e, C_0) \leq \arcsin r_c + \sum_{j=k+1}^{l} \arcsin r_j.$$

From Theorem 1 and noting

$$\arcsin r_p + \arcsin r_c + \sum_{k=1}^{l} \arcsin r_k < \arcsin b_{P_0,C_0},$$

we obtain the equivalent closed-loop system $[P_k^e, C_k^e]$ in Fig. 5 is stable for all $k = 0, 1, 2, \ldots, l$. From Lemma 1, it holds that the NCS in Fig. 2 is robustly stable. This finishes the proof for sufficiency.

The necessary part is proved by contradiction. That is, with the condition (6) violated, we try to construct some P, C and $\{\Delta_k\}_{k=1}^{l}$ such that the two-port NCS is unstable. The existence of P and C simply follows from paper [9]. Concerning $\{\Delta_k\}_{k=1}^{l}$, the key idea is on iteratively rotating the"weakest" canonical angle between the graph of P and C at some specific frequency and then interpolate a series of rotation matrices at that frequency to find those transmission matrices. □

V. Conclusion and Future Work

A two-port NCS model is proposed to study the feedback control system with dynamic uncertainties. The uncertainties in the plant and controller are measured by the gap metric, while those in the communication channels are on the transmission matrices of two-port connections. A perfect channel is represented by an identity as its transmission matrix. When distortions and interferences occur in the communication, the identity matrix is perturbed by an additive dynamic uncertainty Δ, whose block elements are called the uncertainty quartet in our study. Furthermore, we give a necessary and sufficient condition for the NCS to be robustly stable, which is determined by an "arcsin" inequality. The sufficiency is mainly derived from the triangular inequality of the angular gap metric, which is the "arcsin" of the gap metric. And the necessity is mainly attributed to the tightness of the triangular inequality of the angular gap metric, rather than the gap.

A generalization of the model may work if we extend the network uncertainties to the nonlinear case, motivated by the practical channel conditions, equipments with quantizers and attack patterns of potential enemies.

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