

On Semidefiniteness of Signed Laplacians with Application to Microgrids[★]

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Abstract: The paper investigates the positive semidefiniteness of signed Laplacians. It is noted that a symmetric signed Laplacian defines a unique resistive electrical network, wherein the negative weights correspond to negative resistances. As such, the positive semidefiniteness of the signed Laplacians is equivalent to the passivity of the associated resistive networks. By utilizing n -port circuit theory, we obtain several equivalent conditions for signed Laplacians to be positive semidefinite with a simple zero eigenvalue. These conditions characterize a set of negative weights that maintain the semidefiniteness of the Laplacian. The results are used to analyze the small-disturbance angle stability of microgrids as an application.

1. INTRODUCTION

An undirected signed weighted graph refers to a group of nodes linked via undirected signed weighted edges. Such a graph is associated with a signed Laplacian matrix. It is well known that when all the edge weights are positive, the associated Laplacian is positive semi-definite. In addition, the Laplacian has a simple zero eigenvalue if, and only if, the underlying graph is connected, or equivalently, has a spanning tree. In the presence of negative edge weights, we are curious to find out the graph-theoretic meaning for a signed Laplacian to be positive semidefinite with a simple zero eigenvalue. Will this lead to a new dimension of understanding of weighted graphs? These are the questions explored in this paper.

Signed weighted graphs appear in many different applications. One strong motivation for this work comes from the recent developments in distributed computation and control among a group of interactive agents via local interactions. See, for instance, Cortés et al. (2004); Lin et al. (2007); Hu and Evans (2004); Krick et al. (2009); Dörfler et al. (2013); Altafini (2013), just to name a few. In a realistic network, negative edge weights may arise from some faulty processes occurring in distributed computation or communication among agents. For example, sign errors may be present in some communication channels. If that is the case, the actual weights used in the updates of distributed algorithms can be negative, yielding Lapla-

cians with negative weights. Another possible occurrence of Laplacians with negative weights comes from adversarial attacks on a network. For example, in a continuous-time linear consensus network (Ren and Beard, 2005), an external attacker may intentionally hack the communication link between some pairs of neighboring agents by flipping the signs of the values transmitted through the link, with the purpose of preventing the agents from reaching a consensus. In both cases above, negative weights appear in the associated Laplacian matrices and the positive semidefiniteness of the Laplacians plays a salient role in the analysis.

Another motivation for this work comes from the study of small-disturbance angle stability of microgrids. It has been shown in Song et al. (2015) that the local stability of an equilibrium point in the microgrid dynamics boils down to the spectral properties of a signed Laplacian. The presence of negative weights comes from the so-called critical lines across which the phase angle difference is greater than $\pi/2$.

Apart from the engineering field, signed Laplacians also appear in many other areas, for instance, neural networks and social networks. One can refer to Bronski and Deville (2014); Altafini (2013) for more elaborations. All in all, there is ample motivation to study the properties of signed Laplacians.

We examine in this paper the positive semidefiniteness of signed Laplacians. Specifically, we are interested in characterizing the conditions under which the signed Laplacians are positive semidefinite with a simple zero eigenvalue. This would be an interesting and relevant exploration since it has been proven crucial in many different contexts,

[★] This work was supported in parts by the Research Grants Council of Hong Kong Special Administrative Region, China, under the Theme-Based Research Scheme T23-701/14-N, the Knut and Alice Wallenberg Foundation, the Swedish Research Council, and a MURI Grant at the University of Illinois.

e.g., the consensus dynamics under attacks (Khanafar and Başar, 2016) and angle stability analysis of microgrids, etc.

In general, signed Laplacians may exhibit negative eigenvalues and/or multiple zero eigenvalues, even when the underlying graphs are connected. In a recent paper, Zelazo and Bürger (2014), signed Laplacians with only one negative weight have been studied. It has been shown that such a signed Laplacian is positive semidefinite if, and only if, the effective resistance over the negatively weighted edge is nonnegative. The result has been extended therein to signed Laplacians with multiple negative weights, but with the restriction that the negatively weighted edges are isolated in different cycles in the underlying graphs. Later, the same results were reestablished in Chen et al. (2016b) using geometrical and passivity-based approaches, leading to a significant simplification of the proof and more transparent physical interpretations in terms of circuit theory. Notwithstanding this, necessary and sufficient conditions for general signed Laplacians with negative weights to be positive semidefinite were lacking until recently, where we have provided one such condition in Chen et al. (2016a).

The contribution of this paper is multifold. We provide a series of graph-theoretical conditions under which a signed Laplacian, without any restrictions on the negatively weighted edges in the underlying graph, is positive semidefinite and has a simple zero eigenvalue. These conditions also characterize a set of negative weights that maintain the semidefiniteness of the Laplacian. The conditions are given in terms of certain effective resistance matrices and can be physically interpreted via the passivity of resistive multiport networks.

We note that the problem considered here is also related to the literature on the problem of bounding the number of negative and zero eigenvalues of signed Laplacians (Bronski and Deville, 2014).

Notation: We use $\mathbf{1}$ to denote the vector with all entries equal to 1, while the size of the vector is to be understood from the context. Denote by u_i the vector with the i th entry equal to 1 and other entries equal to 0. We define $u_{ij} = u_i - u_j$. The transpose of a matrix A is denoted by A' . The corank of A is denoted by $\text{corank}(A)$. The spectrum and spectral radius of a square matrix A are denoted by $\sigma(A)$ and $\rho(A)$, respectively. For a symmetric matrix S , we write $S \geq 0$ if S is positive semidefinite, and $S > 0$ if S is positive definite.

The rest of the paper is organized as follows. In Section 2, signed Laplacians are introduced. Some preliminaries on effective resistance matrices are given in Section 3. The main results of the paper are presented in Section 4. An application in microgrids is studied in Section 5. The paper ends with some concluding remarks in Section 6.

2. SIGNED LAPLACIANS

Consider an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ which consists of a set of nodes $\mathcal{V} = \{1, 2, \dots, n\}$ and a set of edges $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$. We use (i, j) to denote the edge connecting node i and node j , and associate with each edge $(i, j) \in \mathcal{E}$ a nonzero real-valued weight a_{ij} that can be either positive or negative. If there is no edge connecting node i and node j , a_{ij} is understood to be zero. Such

a graph is called a *signed weighted graph*. For brevity, hereinafter the signed weighted graphs are also referred to as signed graphs.

Denote by \mathcal{E}_+ (\mathcal{E}_- , respectively) the subset of \mathcal{E} containing all edges with positive weights (negative weights, respectively). Denote by $\mathbb{G}_+ = (\mathcal{V}, \mathcal{E}_+)$ ($\mathbb{G}_- = (\mathcal{V}, \mathcal{E}_-)$, respectively) the spanning subgraph¹ of \mathbb{G} whose edge set is \mathcal{E}_+ (\mathcal{E}_- , respectively). A spanning tree \mathbb{T} of an undirected graph \mathbb{G} is a spanning subgraph that is a tree. A spanning tree exists if, and only if, the underlying graph is connected. If the graph is not connected, a spanning forest \mathbb{F} is considered instead, which is a spanning subgraph containing a spanning tree in each connected component of the graph. A spanning tree can be regarded as a special case of a spanning forest. Therefore, hereinafter we shall use \mathbb{F} to represent a spanning tree or a spanning forest of a graph \mathbb{G} , depending on whether the graph is connected or not.

For a signed graph, the associated *signed Laplacian matrix* $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ is defined by

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{j=1, j \neq i}^n a_{ij}, & i = j, \end{cases}$$

with $a_{ij} = a_{ji}$, $i \neq j$. Clearly, L is symmetric, and thus has real eigenvalues. Also, L has a zero eigenvalue with a corresponding eigenvector being $\mathbf{1} \in \mathbb{R}^n$.

When all the edges are positively weighted, L reduces to the conventional Laplacian matrix, to which a substantial literature is dedicated (Merris, 1994). However, when some edges are negatively weighted, the corresponding signed Laplacian exhibits quite a few significant differences from the conventional one. Firstly, L is no longer an M-matrix² when negative weights are present. Consequently, many well-studied properties of M-matrices do not hold. Secondly, L is not necessarily a positive semidefinite matrix as opposed to the conventional Laplacians. Thirdly, while the multiplicity of the zero eigenvalue of a conventional Laplacian is equal to the number of connected components in the underlying graph, this is in general not true for a signed Laplacian. All these differences necessitate the development of a theory for signed Laplacian matrices.

Accordingly, we study in this paper the spectral properties of signed Laplacians with both positive and negative weights, and specifically, positive semidefiniteness. It is well known that a conventional Laplacian matrix is always positive semidefinite, and it has a simple zero eigenvalue if, and only if, the underlying graph is connected (Fiedler, 1973). This is no longer the case for signed Laplacians. The following simple example demonstrates that a signed Laplacian may have negative eigenvalues and multiple zero eigenvalues even when the graph is connected. Consider a complete graph with three nodes. Let $a_{12} = -1$, $a_{13} = 2$, and $a_{23} = 2$. Then, it can be verified that L has a zero eigenvalue of multiplicity two (and the third one is positive). Now let $a_{12} = -2$, and a_{13} , a_{23} as before; then L has one negative, one zero, and one positive eigenvalue.

¹ A spanning subgraph of \mathbb{G} is a graph which contains the same set of nodes as \mathbb{G} and whose edge set is a subset of that of \mathbb{G} .

² A square matrix M is said to be an M-matrix if it can be expressed as $M = sI - B$, where I is the identity matrix, B is nonnegative, and $s \geq \rho(B)$.

We wish to understand under what conditions a signed Laplacian matrix is positive semidefinite and has a simple zero eigenvalue. As will be shown later, a signed Laplacian matrix defines a resistive multiport network, where the negative weights correspond to negative resistances. In this context, the objective is also to explore when a multiport network with active components remains passive.

Before proceeding, let us introduce a useful factorization. Let $W = \text{diag}\{w_1, w_2, \dots, w_m\}$ denote an $m \times m$ diagonal matrix with diagonal elements given by the edge weights, i.e.,

$$w_k = a_{ij}, \text{ for } (i, j) = e_k.$$

Also, assign an (arbitrary) orientation to each edge of the graph, i.e., for each edge $e_k \in \mathcal{E}$, denote one endpoint as the head and the other as the tail. Then, the oriented incidence matrix $D = [d_{ik}] \in \mathbb{R}^{n \times m}$ is defined as

$$d_{ik} = \begin{cases} 1, & \text{if } i \text{ is the head of } e_k, \\ -1, & \text{if } i \text{ is the tail of } e_k, \\ 0, & \text{otherwise.} \end{cases}$$

An important property of the incidence matrix is $D'\mathbf{1} = 0$. Now, with the weight matrix W and the incidence matrix D , the signed Laplacian matrix L can be factorized as

$$L = DWD'. \quad (1)$$

It is worth noting that while the incidence matrix depends on the choice orientations, the signed Laplacian matrix L does not. To see this, suppose the orientation of edge e_k is changed and the orientations of other edges remain the same. Denote the resulting incidence matrix by \tilde{D} . Then, $\tilde{D} = DS$, where S is a diagonal matrix whose k th diagonal entry is -1 and other diagonal entries are 1 . Therefore, $\tilde{D}W\tilde{D}' = DSWSD' = DWS^2D' = DWD'$.

3. PRELIMINARIES ON EFFECTIVE RESISTANCE MATRIX

Consider an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$. Associate with each edge a resistor of (possibly negative) resistance value $r_k = 1/w_k$, where w_k is the weight on edge e_k . In other words, the weight w_k is the conductance of the corresponding resistor.

Let $\mathbf{c} \in \mathbb{R}^n$ be a vector whose entries denote the amount of current injected into each node by external independent sources. Assume that the sum of the entries of \mathbf{c} is equal to zero, i.e., $\mathbf{c}'\mathbf{1} = 0$, meaning that there is no current accumulation in the electrical network. Denote by $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{i} \in \mathbb{R}^m$ the vector of voltages at all nodes and the vector of currents through all edges, respectively. Then, Kirchhoff's current law asserts that the difference between the outgoing current and the incoming current through the edges adjacent to a given node equals to the external current injection at that node, i.e., $D\mathbf{i} = \mathbf{c}$. On the other hand, Ohm's law asserts that the current across each edge is given by the voltage difference divided by the resistance, i.e., $WD'\mathbf{v} = \mathbf{i}$. Combining these two equalities, we have

$$DWD'\mathbf{v} = L\mathbf{v} = \mathbf{c}.$$

When the Laplacian L has a simple zero eigenvalue, we can solve the above equation to yield

$$\mathbf{v} = L^\dagger \mathbf{c} + \alpha \mathbf{1}, \quad (2)$$

where L^\dagger is the Moore-Penrose pseudoinverse of L and α is an arbitrary real number. The electric power of the network is given by $\mathbf{v}'\mathbf{c}$. The electrical network is said to be passive (Desoer and Kuh, 1969) if $\mathbf{v}'\mathbf{c} \geq 0$, and strictly passive if $\mathbf{v}'\mathbf{c} > 0$.

Let $\mathbf{c} = u_{ij}$. This means that a unit of current is injected into node i and extracted from node j . In light of the voltage formula (2), the voltage difference between these two nodes is given by $u'_{ij}L^\dagger u_{ij}$. This quantity is called the *effective resistance* across the pair (i, j) , and we denote it by

$$r_{\text{eff}}(i, j) = u'_{ij}L^\dagger u_{ij}.$$

When all the edge weights are positive, it has been shown that the effective resistance serves as a distance function in the node set of a weighted graph (Klein and Randić, 1993).

In many cases, it is also of interest to consider the voltage difference across a node pair (i, j) when a unit of current is injected into and extracted from another node pair (k, l) . Such a quantity is called the *mutual effective resistance* between the two node pairs:

$$r_{\text{mut}}((i, j), (k, l)) = u'_{ij}L^\dagger u_{kl}.$$

Since L^\dagger is symmetric, we have

$$r_{\text{mut}}((i, j), (k, l)) = r_{\text{mut}}((k, l), (i, j)).$$

In the context of multiport circuit theory (Anderson and Vongpanitlerd, 1973), every pair of nodes in the network can be regarded as one port. Consider an m -port network whose ports correspond to the adjacent node pairs. Then, both the effective resistance across the ports and mutual effective resistance between different ports can be captured by an *effective resistance matrix* $\Gamma = [\gamma_{kl}] \in \mathbb{R}^{m \times m}$ given by

$$\Gamma = D'L^\dagger D.$$

Clearly, Γ is a symmetric matrix. The diagonal entries of Γ correspond to the effective resistances and the off-diagonal entries correspond to the mutual effective resistances.

4. MAIN RESULTS

In this section, a series of results revealing the connection between the spectral properties of the signed Laplacians and the corresponding graph-theoretical interpretations are given. The proofs are left out because of page limitations, and can be found in the longer version of the paper available from the authors.

Consider a signed graph \mathbb{G} equipped with the signed Laplacian L . The graph \mathbb{G} can be expressed as $\mathbb{G} = \mathbb{F} \cup \mathbb{C}$, where $\mathbb{F} = (\mathcal{V}, \mathcal{E}_{\mathbb{F}})$ denotes a spanning tree (spanning forest, respectively) of \mathbb{G} when \mathbb{G} is connected (disconnected, respectively), and \mathbb{C} denotes another spanning subgraph of \mathbb{G} containing the remaining edges, i.e., $\mathbb{C} = (\mathcal{V}, \mathcal{E} \setminus \mathcal{E}_{\mathbb{F}})$. With a proper labeling of the edges, the incidence matrix D can be rewritten as $D = [D_{\mathbb{F}} \ D_{\mathbb{C}}]$. Then, the effective resistance matrix Γ admits the form

$$\Gamma = D'L^\dagger D = \begin{bmatrix} D'_{\mathbb{F}}L^\dagger D_{\mathbb{F}} & D'_{\mathbb{F}}L^\dagger D_{\mathbb{C}} \\ D'_{\mathbb{C}}L^\dagger D_{\mathbb{F}} & D'_{\mathbb{C}}L^\dagger D_{\mathbb{C}} \end{bmatrix}.$$

Let $\Gamma_{\mathbb{F}} = D'_{\mathbb{F}}L^\dagger D_{\mathbb{F}}$. The following theorem constitutes the first step in our series of results.

Theorem 1. A signed Laplacian L is positive semidefinite and of $\text{corank}(L) = 1$ if, and only if, the underlying signed graph \mathbb{G} has a spanning tree \mathbb{F} , and $\Gamma_{\mathbb{F}} > 0$.

Note that $\text{corank}(L) = 1$ is equivalent to L has a simple zero eigenvalue. Theorem 1 holds for an arbitrary choice of spanning tree, which may or may not contain negatively weighted edges. For a chosen spanning tree \mathbb{F} , one can view the resistive network associated with \mathbb{G} as an $(n-1)$ -port network whose ports correspond to the edges of \mathbb{F} . The resistance matrix of such an $(n-1)$ -port network is exactly given by $\Gamma_{\mathbb{F}}$. By the multiport circuit theory (Anderson and Vongpanitlerd, 1973), $\Gamma_{\mathbb{F}} > 0$ means that the $(n-1)$ -port network is strictly passive. In this regard, Theorem 1 can be physically interpreted as follows:

An $n \times n$ signed Laplacian L is positive semidefinite and has a simple zero eigenvalue if, and only if, the underlying signed graph \mathbb{G} has a spanning tree and the corresponding $(n-1)$ -port network is strictly passive.

Remark 1. When all the weights are positive, the inequality $\Gamma_{\mathbb{F}} > 0$ holds automatically if a spanning tree \mathbb{F} exists. As such, in that case, Theorem 1 reduces to the well-known result on classical Laplacian matrices.

In many applications, it may well happen that the number of negatively weighted edges is relatively small compared to the size of the graph. Hence, examining the resistance matrix of an $(n-1)$ -port network may contain much redundancy. This has been evidenced in Zelazo and Bürger (2014) which revealed that when there is one single negatively weighted edge (i, j) , the signed Laplacian is positive semidefinite and has $\text{corank} 1$ if, and only if, the graph is connected and $r_{\text{eff}}(i, j) > 0$, i.e., only a 1-port network needs to be examined. In the sequel, we will show how to perform such a reduction of redundancy and, thus, lower the computational complexity in the general case.

Let $\mathbb{G}_+ = (\mathcal{V}, \mathcal{E}_+)$ and $\mathbb{G}_- = (\mathcal{V}, \mathcal{E}_-)$ be as defined before. We now express the signed graph \mathbb{G} as the union of three subgraphs:

$$\mathbb{G} = \mathbb{F}_- \cup \mathbb{C}_- \cup \mathbb{G}_+,$$

where $\mathbb{F}_- = (\mathcal{V}, \mathcal{E}_{\mathbb{F}_-})$ is a spanning forest of \mathbb{G}_- and \mathbb{C}_- is a spanning subgraph of \mathbb{G}_- containing the remaining edges of \mathbb{G}_- . With a proper labeling of the edges, the incidence matrix D can be rewritten into $D = [D_{\mathbb{F}_-} \ D_{\mathbb{C}_-} \ D_{\mathbb{G}_+}]$. Then the effective resistance matrix Γ admits the form

$$\Gamma = D' L^\dagger D = \begin{bmatrix} D_{\mathbb{F}_-}' L^\dagger D_{\mathbb{F}_-} & D_{\mathbb{F}_-}' L^\dagger D_{\mathbb{C}_-} & D_{\mathbb{F}_-}' L^\dagger D_{\mathbb{G}_+} \\ D_{\mathbb{C}_-}' L^\dagger D_{\mathbb{F}_-} & D_{\mathbb{C}_-}' L^\dagger D_{\mathbb{C}_-} & D_{\mathbb{C}_-}' L^\dagger D_{\mathbb{G}_+} \\ D_{\mathbb{G}_+}' L^\dagger D_{\mathbb{F}_-} & D_{\mathbb{G}_+}' L^\dagger D_{\mathbb{C}_-} & D_{\mathbb{G}_+}' L^\dagger D_{\mathbb{G}_+} \end{bmatrix}.$$

Let $\Gamma_{\mathbb{F}_-} = D_{\mathbb{F}_-}' L^\dagger D_{\mathbb{F}_-}$. Since \mathbb{F}_- is a spanning forest of \mathbb{G}_- , it follows that $D_{\mathbb{F}_-}$ has full column rank (see Theorem 2.5 in Grossman et al. (1995)).

Theorem 2. (Chen et al. (2016a)). A signed Laplacian L is positive semidefinite and of $\text{corank}(L) = 1$ if, and only if, the underlying signed graph \mathbb{G} is connected, and $\Gamma_{\mathbb{F}_-} > 0$.

It should be clear that the choice of a spanning forest \mathbb{F}_- in \mathbb{G}_- is not unique. Nevertheless, Theorem 2 holds for any choice of \mathbb{F}_- .

Suppose \mathbb{F}_- contains m_1 edges. Then, one can view the resistive network associated with \mathbb{G} as an m_1 -port network whose ports correspond to the edges of \mathbb{F}_- . The

resistance matrix of such an m_1 -port network is given by $\Gamma_{\mathbb{F}_-}$. Clearly, $m_1 \leq n-1$. As a matter of fact, in many applications, m_1 may be much smaller than $n-1$. Hence, Theorem 2 reduces the redundancy in checking the passivity of an $(n-1)$ -port network as in Theorem 1.

From Theorem 2, the following corollary can be deduced, whose proof can be found in Chen et al. (2016a).

Corollary 1. If \mathbb{G} does not have any cycle containing two negatively weighted edges, then L is positive semidefinite and of $\text{corank}(L) = 1$ if, and only if, \mathbb{G} is connected, and $r_{\text{eff}}(i, j) > 0$ for all $(i, j) \in \mathcal{E}_-$.

Note that in Theorem 2, the interaction between negative weights and positive weights is reflected only implicitly in the positive definiteness of $\Gamma_{\mathbb{F}_-}$. Then, a further question arises: Is it possible to come up with a condition that explicitly separates the impact of the negative and positive weights? Having such a condition is important, especially in applications concerning the fragility of networks under perturbations.

It turns out the answer to the above question is in the affirmative. The key is to exploit the parallel connection of multiport networks and the induced matrix operation. Specifically, one can view the m_1 -port network with ports corresponding to the edges of \mathbb{F}_- as a parallel connection of an m_1 -port network with only all positive resistances and another m_1 -port network with only all negative resistances. Denote the signed Laplacians corresponding to \mathbb{G}_+ and \mathbb{G}_- by L_+ and L_- , respectively. Then, the resistance matrices of the m_1 -port network with positive resistances and the m_1 -port network with negative resistances are given by

$$\begin{aligned} \Gamma_{\mathbb{F}_-}^+ &= D_{\mathbb{F}_-}' L_+^\dagger D_{\mathbb{F}_-}, \\ \Gamma_{\mathbb{F}_-}^- &= D_{\mathbb{F}_-}' L_-^\dagger D_{\mathbb{F}_-}, \end{aligned}$$

respectively.

The following theorem reveals explicitly how the negative and positive weights influence the positive semidefiniteness of L .

Theorem 3. A signed Laplacian L is positive semidefinite and of $\text{corank}(L) = 1$ if, and only if, the underlying signed graph \mathbb{G} is connected, and $\Gamma_{\mathbb{F}_-}^+ < -\Gamma_{\mathbb{F}_-}^-$.

Applying Theorem 3 to the special case when \mathbb{G} has no cycles containing two negatively weighted edges yields the following corollary.

Corollary 2. If \mathbb{G} does not have any cycle containing two negatively weighted edges, then L is positive semidefinite and of $\text{corank}(L) = 1$ if, and only if, \mathbb{G} is connected, and $r_{\text{eff}}^+(i, j) < 1/|a_{ij}|$ for all $(i, j) \in \mathcal{E}_-$, where $r_{\text{eff}}^+(i, j) = u_{ij}' L_+^\dagger u_{ij}$.

Note that $r_{\text{eff}}^+(i, j)$ is the effective resistance between nodes i and j over the subgraph \mathbb{G}_+ , where $(i, j) \in \mathcal{E}_-$. This corollary is consistent with the understandings in (Zelazo and Bürger, 2014, Theorem III.3).

Remark 2. In many applications, \mathbb{G}_+ often represents the original physical graph which may suffer perturbations via negatively weighted edges. In this regard, $\Gamma_{\mathbb{F}_-}^+$ reflects the

fragility of \mathbb{G}_+ under such perturbations. The smaller $\Gamma_{\mathbb{F}_-}^+$ is, the less fragile \mathbb{G}_+ would be.

5. APPLICATION TO ANGLE STABILITY ANALYSIS IN MICROGRIDS

A microgrid is a low-voltage or medium-voltage power network that provides electricity to a local area. It usually consists of distributed generators, loads, energy storage, and control devices. Small disturbance angle stability is one of the central issues facing microgrids. In this section, we sketch how small-disturbance angle stability is connected to spectral properties of a signed Laplacian and, thus, the results presented in this paper can be applied. For more detailed discussions on the angle stability of microgrids, one can refer to Ainsworth and Grijalva (2013); Song et al. (2015).

Consider an inverter-based microgrid whose topology is described by an undirected graph $\mathbb{G} = (\mathcal{V}, \mathcal{E})$ consisting of n nodes and m edges, in which each node corresponds to a bus and each edge corresponds to a transmission line. Denote by $Y_{ij} = Y_{ji}$ the admittance of the transmission line $(i, j) \in \mathcal{E}$. We use \mathcal{V}_1 to represent the set of buses with inverter-based generators or frequency dependent loads and use \mathcal{V}_2 to represent the set of buses with only frequency independent loads. Denote the voltage magnitude and phase angle of bus i by V_i and θ_i , respectively. Then, with the aid of a singular perturbation parameter $\epsilon > 0$, the dynamics of phase angle θ_i at bus i can be described as

$$P_{Ri} = P_{Li} + (K_{Ri} + K_{Li})\dot{\theta}_i + \sum_{(i,j) \in \mathcal{E}} V_i V_j Y_{ij} \sin(\theta_i - \theta_j), \quad i \in \mathcal{V}_1,$$

$$0 = P_{Li} + \epsilon \dot{\theta}_i + \sum_{(i,j) \in \mathcal{E}} V_i V_j Y_{ij} \sin(\theta_i - \theta_j), \quad i \in \mathcal{V}_2,$$

where P_{Ri} and P_{Li} stand for the nominal active power generation and nominal load at bus i , respectively, and K_{Ri} and K_{Li} stand for the reciprocal of frequency droop gain of the generator and the frequency dependence coefficient of the load at bus i , respectively. For simplicity of analysis, we assume that the voltage magnitude at each bus is a constant (not necessarily homogenous).

We introduce additional notation so that the above set of dynamic equations can be rewritten in a compact matrix form. Let $\theta = [\theta_i] \in \mathbb{R}^n$ be the vector of bus phase angles, and let $K = \text{diag}\{K_i\} \in \mathbb{R}^{n \times n}$ be the bus total damping coefficients matrix, where

$$K_i = \begin{cases} K_{Ri} + K_{Li}, & i \in \mathcal{V}_1, \\ \epsilon, & i \in \mathcal{V}_2. \end{cases}$$

It is required that $K > 0$. Denote by $P = [P_i] \in \mathbb{R}^n$ the vector of bus injected power, where

$$P_i = \begin{cases} P_{Ri} - P_{Li}, & i \in \mathcal{V}_1, \\ -P_{Li}, & i \in \mathcal{V}_2. \end{cases}$$

Finally, the vector of active power flows through all the transmission lines is denoted by

$$P_{\text{line}}(D'\theta) = [V_i V_j Y_{ij} \sin(\theta_i - \theta_j)] \in \mathbb{R}^m, \quad \forall (i, j) \in \mathcal{E},$$

where $P_{\text{line}}(D'\theta)$ means P_{line} is a function of $D'\theta$ and D is the oriented incidence matrix of \mathbb{G} .

Now, the above phase angle dynamics can be rewritten in a compact matrix form:

$$K\dot{\theta} = P - DP_{\text{line}}(D'\theta).$$

With a state variable transformation

$$\tilde{\theta}_i = \theta_i - \theta_c, \quad \theta_c = \frac{\mathbf{1}'K\theta}{\mathbf{1}'K\mathbf{1}},$$

where θ_c is the center-of-damping angle, the dynamics can be further rewritten as

$$K\dot{\tilde{\theta}} = P - \frac{\mathbf{1}'P}{\mathbf{1}'K\mathbf{1}}K\mathbf{1} - DP_{\text{line}}(D'\tilde{\theta}). \quad (3)$$

The advantage of such a state transformation is that $\dot{\tilde{\theta}} = 0$ at the synchronous states, and thus the equilibrium points of system (3) are constant. Denote by $\tilde{\theta}^0$ an equilibrium point of system (3).

We wish to study the small-disturbance stability of the equilibrium point $\tilde{\theta}^0$. To this end, we linearize the system (3) around the equilibrium point $\tilde{\theta}^0$, yielding

$$\Delta\dot{\tilde{\theta}} = -K^{-1}DW(\tilde{\theta}^0)D'\Delta\tilde{\theta} = J(\tilde{\theta}^0)\Delta\tilde{\theta},$$

where

$$W(\tilde{\theta}^0) = \frac{\partial P_{\text{line}}(D'\tilde{\theta}^0)}{\partial (D'\tilde{\theta}^0)} = \text{diag}\{V_i V_j Y_{ij} \cos(\tilde{\theta}_i^0 - \tilde{\theta}_j^0)\},$$

and

$$J(\tilde{\theta}^0) = -K^{-1}DW(\tilde{\theta}^0)D'$$

is the Jacobian matrix.

The small-disturbance stability of $\tilde{\theta}^0$ is determined by the spectrum of $J(\tilde{\theta}^0)$. Notice that $J(\tilde{\theta}^0)\mathbf{1} = 0$, indicating that $J(\tilde{\theta}^0)$ has a zero eigenvalue with a corresponding eigenvector $\mathbf{1}$. In other words, this zero eigenvalue corresponds to the synchronous manifold $k\mathbf{1}$, $k \in \mathbb{R}$, and, thus, does not affect the stability of $\tilde{\theta}^0$. To be more specific, for a given equilibrium point $\tilde{\theta}^0$, $\tilde{\theta}^0 + k\mathbf{1}$ is the same equilibrium point as $\tilde{\theta}^0$ since all the angle differences across the transmission lines at $\tilde{\theta}^0$ are identical to those at $\tilde{\theta}^0 + k\mathbf{1}$. Bearing this in mind, one can claim that $\tilde{\theta}^0$ is small-disturbance stable if, and only if, $J(\tilde{\theta}^0)$ has all the eigenvalues lying in the open left half plane except a simple zero eigenvalue.

We proceed to analyze the spectral properties of $J(\tilde{\theta}^0)$. In view of (1), the Jacobian matrix can be rewritten as

$$J(\tilde{\theta}^0) = -KL(\tilde{\theta}^0),$$

where

$$L(\tilde{\theta}^0) = DW(\tilde{\theta}^0)D'.$$

In practice, there may well exist some critical lines across which the angle differences are greater than $\pi/2$. If that is the case, the entries of $W(\tilde{\theta}^0)$ corresponding to the critical lines are negative and $L(\tilde{\theta}^0)$ is a signed Laplacian. Since $K > 0$ and $\sigma(-KL(\tilde{\theta}^0)) = \sigma(-K^{\frac{1}{2}}L(\tilde{\theta}^0)K^{\frac{1}{2}})$, it follows that the eigenvalues of $J(\tilde{\theta}^0)$ are all real. In addition, by Sylvester's law of inertia, one can verify that $J(\tilde{\theta}^0)$ has exactly the same number of positive, negative, and zero eigenvalues as $-L(\tilde{\theta}^0)$. This, combined with our earlier understanding, indicates that the equilibrium point $\tilde{\theta}^0$ is small-disturbance stable if, and only if, $L(\tilde{\theta}^0)$ is positive semidefinite and has a simple zero eigenvalue.

Such an equivalence between small-disturbance stability and semidefiniteness of signed Laplacians has been shown in Song et al. (2015). Therein the authors presented a graph-theoretical characterization of the critical lines

maintaining small-disturbance stability under the assumption that $L(\tilde{\theta}^0)$ has a simple zero eigenvalue. In contrast, we remove this assumption and explicitly address the multiplicity of zero eigenvalues as part of the problem formulation. By applying the results in Section 4, three equivalent graph-theoretical characterizations for small disturbance stability are obtained via certain effective resistance matrices.

6. CONCLUSION

In this paper, we have explored the connection between the spectral properties of a signed Laplacian and the corresponding graph-theoretical meanings. Three equivalent conditions for a signed Laplacian to be positive semidefinite with a simple zero eigenvalue have been established. These graph-theoretical conditions characterize the set of negative weights that maintain the semidefiniteness of the Laplacian via the so-called effective resistance matrix. These conditions also characterize when a resistive multiport network with active components can still remain passive. The results in this paper significantly generalize the existing ones in Zelazo and Bürger (2014) and Chen et al. (2016b), and extend those in Chen et al. (2016a).

Our future work aims at extending the results in this paper to directed signed graphs. Such an extension, however, appears to be challenging. While signed Laplacian matrices have been introduced for directed graphs as well in the literature (Young et al., 2016), how to suitably define effective resistances and physically interpret them in terms of electrical circuits remains to be investigated.

REFERENCES

- Ainsworth, N. and Grijalva, S. (2013). A structure-preserving model and sufficient condition for frequency synchronization of lossless droop inverter-based AC networks. *IEEE Trans. Power Syst.*, 28(4), 4310–4319.
- Altafini, C. (2013). Consensus problems on networks with antagonistic interactions. *IEEE Trans. Autom. Control*, 58(4), 935–946.
- Anderson, B.D.O. and Vongpanitlerd, S. (1973). *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Prentice Hall.
- Bronski, J.C. and Deville, L. (2014). Spectral theory for dynamics on graphs containing attractive and repulsive interactions. *SIAM J. Appl. Math.*, 74(1), 83–105.
- Chen, W., Liu, J., Chen, Y., Khong, S.Z., Wang, D., Başar, T., Qiu, L., and Johansson, K. (2016a). Characterizing the positive semidefiniteness of weighted Laplacians via generalized effective resistances. In *Proc. 55th IEEE Conf. Decision Control*, to appear.
- Chen, Y., Khong, S.Z., and Georgiou, T.T. (2016b). On the definiteness of graph Laplacians with negative weights: geometrical and passivity-based approaches. In *Proc. 2016 Amer. Control Conf.*, 2488–2493.
- Cortés, J., Martínez, S., Karataş, T., and Bullo, F. (2004). Coverage control for mobile sensing networks. *IEEE Trans. Robot. Autom.*, 20(2), 243–255.
- Desoer, C.A. and Kuh, E.S. (1969). *Basic Circuit Theory*. McGraw-Hill Book Company.
- Dörfler, F., Chertkov, M., and Bullo, F. (2013). Synchronization in complex oscillator networks and smart grids. *Proc. Natl. Acad. Sci.*, 110(6), 2005–2010.
- Fiedler, M. (1973). Algebraic connectivity of graphs. *Czech. Math. J.*, 23(98), 298–305.
- Grossman, J.W., Kulkarni, D.M., and Schochetman, I.E. (1995). On the minors of an incidence matrix and its Smith normal form. *Linear Algebra Appl.*, 218, 213–224.
- Hu, L. and Evans, D. (2004). Localization for mobile sensor networks. In *Proc. ACM MobiCom*, 45–57.
- Khanafer, A. and Başar, T. (2016). Robust distributed averaging: when are potential theoretic strategies optimal? *IEEE Trans. Autom. Control*, 61(7), 1767–1779.
- Klein, D.J. and Randić, M. (1993). Resistance distance. *J. Math. Chem.*, 12(1), 81–95.
- Krick, L., Broucke, M.E., and Francis, B.A. (2009). Stabilisation of infinitesimally rigid formations of multi-robot networks. *Int. J. Control*, 82(3), 423–439.
- Lin, J., Morse, A.S., and Anderson, B.D.O. (2007). The multi-agent rendezvous problem. Part 1: the synchronous case. *SIAM J. Control Optim.*, 46(6), 2096–2119.
- Merris, R. (1994). Laplacian matrices of graphs: a survey. *Linear Algebra Appl.*, 177–178, 143–176.
- Ren, W. and Beard, R.W. (2005). Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Trans. Autom. Control*, 50(5), 655–661.
- Song, Y., Hill, D.J., and Liu, T. (2015). Small-disturbance angle stability analysis of microgrids: a graph theory viewpoint. In *2015 IEEE Conf. Control Applications*, 201–206.
- Young, G., Scardovi, L., and Leonard, N. (2016). A new notion of effective resistance for directed graphs - Part I: definition and properties. *IEEE Trans. Autom. Control*, 61(7), 1727–1736.
- Zelazo, D. and Bürger, M. (2014). On the definiteness of the weighted Laplacian and its connection to effective resistance. In *Proc. 53rd IEEE Conf. Decision Control*, 2895–2900.