On Spectral Properties of Signed Laplacians for Undirected Graphs

Wei Chen, Dan Wang, Ji Liu, Tamer Başar, and Li Qiu

Abstract— **Recently, signed weighted graphs have appeared in broad applications, ranging from social networks to biological networks, from distributed control systems to electric power systems. This paper studies the spectral properties of the signed Laplacians associated with undirected signed graphs. We first revisit and provide a new dimension of understanding on the positive semidefiniteness of signed Laplacians via n-port network theory. We then go beyond positive semidefiniteness and characterize the inertia of a signed Laplacian via the notion of the conductance matrix.**

I. Introduction

A signed weighted graph refers to a group of nodes linked through signed weighted edges. Such signed weighted graphs arise in many applications, ranging from social networks [1]–[4] to biological networks [5], [6], from distributed control and computation [7]–[13] to electric grid [14]. For example, in distributed control and optimization, negative edge weights may come from faulty communication processes among the agents or adversarial attacks on the network. For another example, in the study of small-disturbance angle stability of power systems [14], negative edge weights may occur due to the critical transmission lines across which the phase angle differences are greater than 90 degrees.

Studying dynamics over a signed weighted graph often requires the analysis of an associated signed Laplacian. In this paper, we focus on the undirected signed weighted graphs for which the associated signed Laplacians are real symmetric matrices. Below is a brief review of some pertinent works on symmetric signed Laplacians.

A first issue frequently discussed in the literature is the semidefiniteness of signed Laplacians. Exploring conditions under which the signed Laplacians are positive semidefinite is of great importance to many applications. For instance, positive semidefiniteness of a signed Laplacian is crucial for the convergence of a linear consensus process in the presence of negative weights. In [8], signed Laplacians with only one negative weight were studied. It was shown that such a signed Laplacian is positive semidefinite if, and only if, the effective resistance over the negatively weighted edge is nonnegative. The result was extended therein to signed Laplacians with multiple negative weights under the restriction that different negatively weighted edges cannot be contained in the same cycle. The same results were then reestablished in [9] using geometrical and passivity-based approaches. Recently, the work in [15] extended the study to general signed Laplacians containing multiple negative weights with no restrictions on the positions of the negatively weighted edges.

When a signed Laplacian is not positive semidefinite, its inertia is of more interest. In [14], the authors connected the type of unstable equilibrium points in power systems with the inertia of certain signed Laplacians. In [6], the authors discussed how the inertia of a signed Laplacian varies as the signed weights vary in magnitude. In [10], the influence of the structure of signed graphs on the inertia of the associated signed Laplacians was investigated.

Motivated by the above, we study in this paper the spectral properties of signed Laplacians for undirected graphs. We first obtain a necessary and sufficient condition of low complexity under which a signed Laplacian, with no restrictions on the negatively weighted edges, is positive semidefinite and has a simple zero eigenvalue. We then provide a characterization of the inertia of a signed Laplacian via the notion of the conductance matrix. The main tool we utilize is the n-port network in circuit theory.

The rest of the paper is organized as follows. The signed Laplacians and associated resistive networks are introduced in Section II. Some preliminary knowledge on n-port network theory is given in Section III. Positive semidefiniteness of signed Laplacians is studied in Section IV. The inertia of signed Laplacians is characterized in Section V. Finally, some concluding remarks follow in Section VI.

Notation: Denote by $\mathbf{1}$ a vector with all elements being 1, where the dimension is to be understood from the context. Given a real symmetric matrix S, we write $S \geq 0$ if S is positive semidefinite, and $S > 0$ if S is positive definite. Denote the inertia of a real symmetric matrix S by $\pi(S) = \{\pi_-(S), \pi_0(S), \pi_+(S)\}$, where $\pi_-(S)$, $\pi_0(S)$, and $\pi_+(S)$ are respectively the numbers of negative, zero, and positive eigenvalues with multiplicity counted. For a real symmetric matrix S partitioned as $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, denote by $S/_{22} = S_{11} - S_{12}S_{22}^{\dagger}S_{21}$ the (generalized) Schur complement of S_{22} in S, where $S_{22}^{\dagger}$ means the Moore-Penrose pseudoinverse of S_{22}. Similarly, $S/_{11} = S_{22} - S_{21}S_{11}^{\dagger}S_{12}$.

This work was partially supported by the Research Grants Council of Hong Kong Special Administrative Region, China, under the Theme-Based Research Scheme T23-701/14-N, and a MURI Grant at the University of Illinois.

W. Chen was with the Department of Electrical Engineering and Computer Sciences, University of California at Berkeley, Berkeley, CA 94720, USA. He is now with the Department of Electronic & Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China. wchenust@gmail.com

D. Wang and L. Qiu are with the Department of Electronic & Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China. dwangah@connect.ust.hk, eeqiu@ust.hk

J. Liu is with the Department of Electrical and Computer Engineering, Stony Brook University, Stony Brook, NY 11794, USA. ji.liu@stonybrook.edu

T. Başar is with the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA. basar1@illinois.edu

II. Signed Laplacians and Resistive Networks

Consider an undirected graph $\mathbb{G}=(\mathcal{V},\mathcal{E})$ which consists of a set of nodes $\mathcal{V}=\{1,2,\dots,n\}$ and a set of edges $\mathcal{E}=\{e_1,e_2,\dots,e_m\}$. We use (i,j) to denote the edge connecting node i and node j, and associate with each edge $(i,j)\in\mathcal{E}$ a nonzero real-valued weight a_{ij} that can be either positive or negative. If there is no edge connecting node i and node j, a_{ij} is understood to be zero. Such a graph is called a *signed weighted graph*. For brevity, the signed weighted graphs are also referred to as signed graphs in this paper.

An undirected graph $\mathbb{G}=(\mathcal{V},\mathcal{E})$ has a spanning tree $\mathbb{T}$, i.e., a spanning subgraph[1] which itself is a tree, if and only if $\mathbb{G}$ is connected. For a disconnected graph, a spanning forest $\mathbb{F}$ is considered instead, which is a spanning subgraph containing a spanning tree in each connected component of the graph. A spanning tree can be regarded as a special case of a spanning forest. Therefore, hereinafter we shall use $\mathbb{F}$ to represent a spanning tree or a spanning forest, depending on whether the underlying graph $\mathbb{G}$ is connected or not.

For a signed graph, the associated *signed Laplacian matrix* $L=[l_{ij}]\in\mathbb{R}^{n\times n}$ is defined by

$$l_{ij}=\begin{cases}-a_{ij}, & i\neq j,\\ \sum_{j=1,j\neq i}^{n} a_{ij}, & i=j.\end{cases}$$

At first sight, the way a signed Laplacian matrix is defined looks no different from the conventional one, except that the conventional Laplacians only have positive weights. Indeed, some properties known for the conventional Laplacian matrices remain in the presence of negative weights. For instance, a signed Laplacian L is clearly a symmetric matrix, and thus has real eigenvalues. Also, L has a zero eigenvalue with a corresponding eigenvector being $\mathbf{1}\in\mathbb{R}^n$.

However, due to the presence of negative weights, signed Laplacians exhibit some fundamental differences from the conventional Laplacians. First, a signed Laplacian may not be positive semidefinite as opposed to the conventional Laplacian. Second, while the multiplicity of zero eigenvalue of a conventional Laplacian is equal to the number of connected components in the underlying graph, this is in general not true for a signed Laplacian.

Motivated by the aforementioned similarities and differences, we wish to address two questions regarding the signed Laplacian L:

(1) Is it possible to characterize the set of negative weights with which the signed Laplacian is still positive semidefinite and has a simple zero eigenvalue?
(2) Is there a simple way of low complexity to characterize the inertia of a signed Laplacian?

The connection between a signed Laplacian and an associated resistive network plays an important role in answering the above questions.

Consider a connected signed graph $\mathbb{G}$. One can associate with each edge a resistor of (possibly negative) resistance $r_k=1/w_k$, where w_k represents the weight on edge e_k. While such an association with resistive networks is completely natural, it gives the signed Laplacian L an important physical interpreation, i.e., L captures the linear relationship between the vector of voltage potential at each node and the vector of current flow into each node. To be specific, let $\boldsymbol{c}\in\mathbb{R}^n$ be a vector whose entries denote the amount of current injected to each node by external independent sources. Assume that the sum of the entries of $\boldsymbol{c}$ is zero, i.e., $\boldsymbol{c}'\mathbf{1}=0$, meaning that there is no current accumulation in the electrical network. Denote by $\boldsymbol{u}\in\mathbb{R}^n$ the resulting voltage potential at the nodes. Then, $\boldsymbol{c}$ and $\boldsymbol{u}$ are related via the current balance equation

$$\boldsymbol{c}=L\boldsymbol{u}.$$

This relation can be readily verified by applying Ohm's law and Kirchhoff's current law; see [16] for details. From circuit theory, a resistive network is passive if $\boldsymbol{u}'\boldsymbol{c}\geq 0$, and strictly passive if $\boldsymbol{u}'\boldsymbol{c}>0$ for all $\boldsymbol{c}$.

Before proceeding, we introduce a useful factorization of L. Denote the weight matrix by $W=\operatorname{diag}\{w_1,w_2,\dots,w_m\}$, where $w_k=a_{ij}$, for $(i,j)=e_k$. Also, assign an (arbitrary) orientation to each edge $e_k\in\mathcal{E}$. Then, the oriented incidence matrix $D=[d_{ik}]\in\mathbb{R}^{n\times m}$ is defined as:

$$d_{ik}=\begin{cases}1, & \text{if } i \text{ is the head of } e_k,\\ -1, & \text{if } i \text{ is the tail of } e_k,\\ 0, & \text{otherwise.}\end{cases}$$

A signed Laplacian L admits the factorization $L=DWD'$.

III. Preliminaries

Some preliminary results on n-port (multiport) networks are given below. See [17] and references therein for details.

As depicted in Fig. 1, an n-port network is an electrical network with its external terminals being grouped into n pairs such that for every pair of terminals, the current flowing into one terminal is equal to the current flowing out of the other. Such pairs of terminals are called ports of the network. Note that the $2n$ external terminals are counted with multiplicity, i.e., two distinct ports may share a common terminal. The external behavior of an n-port network is completely determined by port voltages $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_n$, and port currents $\boldsymbol{i}_1,\boldsymbol{i}_2,\dots,\boldsymbol{i}_n$. Let $\boldsymbol{v}=\begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n\end{bmatrix}'$ and $\boldsymbol{i}=\begin{bmatrix}\boldsymbol{i}_1 & \boldsymbol{i}_2 & \dots & \boldsymbol{i}_n\end{bmatrix}'$.

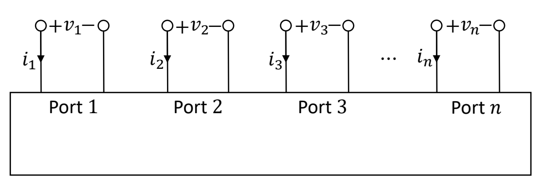

Fig. 1. An n-port network

Now consider an n-port resistive network containing both positive and negative resistances. Such an n-port network can be characterized either by expressing port voltages in terms

[1] A spanning subgraph of $\mathbb{G}$ is a graph which contains the same set of nodes as $\mathbb{G}$ and whose edge set is a subset of that of $\mathbb{G}$.

of port currents as in $\boldsymbol{v} = Z\boldsymbol{i}$, or by expressing port currents in terms of port voltages as in $\boldsymbol{i} = Y\boldsymbol{v}$. The two matrices Z and Y are symmetric and are called resistance matrix and conductance matrix of the n-port network, respectively. Each diagonal element of matrix Z (Y, respectively) represents the resistance (conductance, respectively) over a corresponding port, while each off-diagonal element of Z (Y, respectively) represents the mutual resistance (mutual conductance, respectively) between two ports. An n-port resistive network is strictly passive if and only if $\boldsymbol{v}'\boldsymbol{i} > 0$ for all nonzero $\boldsymbol{i} \in \mathbb{R}^n$. Therefore, both resistance matrix and conductance matrix of a strictly passive n-port network are positive definite. When both Z and Y are finite and nonsingular, we have $Z = Y^{-1}$.

Consider an n-port network $\boldsymbol{A}$. Let $\boldsymbol{C}$ be an $(n-r)$-port network obtained by shorting the first r ports of network $\boldsymbol{A}$. We partition the resistance matrix of network $\boldsymbol{A}$ into the form $Z^a = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$, where $Z_{11} \in \mathbb{R}^{r\times r}$. Then, the resistance matrix of $\boldsymbol{C}$ is given by $Z^c = Z_{22} - Z_{21}Z_{11}^{\dagger}Z_{12}$, which is simply the Schur complement applied to the resistance matrix. Similarly, we can partition the conductance matrix of $\boldsymbol{A}$ into the form $Y^a = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, where $Y_{11} \in \mathbb{R}^{r\times r}$. Then, the conductance matrix of $\boldsymbol{C}$ is given by $Y^c = Y_{22}$.

Now consider two n-port networks $\boldsymbol{A}$ and $\boldsymbol{B}$. Let $\boldsymbol{C}$ be an n-port network obtained by a parallel connection of $\boldsymbol{A}$ and $\boldsymbol{B}$. Then, $Z^c = Z^a(Z^a + Z^b)^{\dagger}Z^b$ and $Y^c = Y^a + Y^b$.

IV. Semidefiniteness of Signed Lapalcians

The resistive network associated with a connected signed graph $\mathbb{G}$ of n nodes can be regarded as an $(n-1)$-port network, where the ports correspond to the edges of a spanning tree $\mathbb{F}$. It is easy to see that L is positive semidefinite with a simple zero eigenvalue if, and only if, such an $(n-1)$-port network is strictly passive. However, considering $n-1$ ports may contain much redundancy, especially when the number of negatively weighted edges is small. How can we eliminate the redundancy? Is it possible to consider only those ports corresponding to the negatively weighted edges?

To solve this riddle, denote by $\mathbb{G}_+$ and $\mathbb{G}_-$ the spanning subgraph of $\mathbb{G}$ with only all positively weighted edges and the spanning subgraph of $\mathbb{G}$ with only all negatively weighted edges, respectively. We express $\mathbb{G}$ as $\mathbb{G} = \mathbb{F}_- \cup \mathbb{C}_- \cup \mathbb{G}_+$, where $\mathbb{F}_-$ is a spanning forest of $\mathbb{G}_-$, and $\mathbb{C}_-$ is a spanning subgraph of $\mathbb{G}_-$ containing the remaining edges in $\mathbb{G}_-$. With a proper re-labeling of the edges, the incidence matrix D can be rewritten as $D = \begin{bmatrix} D_{\mathbb{F}_-} & D_{\mathbb{C}_-} & D_{\mathbb{G}_+} \end{bmatrix}$. Suppose that $\mathbb{G}_-$ has m_- edges and $\mathbb{F}_-$ has m_1 edges.

If we only care about the external behavior of the resistive network at those ports corresponding to $\mathbb{F}_-$, we have an m_1-port network at hand. In many real applications, it is often the case that m_1 is much smaller than $n-1$. The resistance matrix and conductance matrix of this m_1-port network are given by $Z_{\mathbb{F}_-} = D'_{\mathbb{F}_-}L^{\dagger}D_{\mathbb{F}_-}$ and $Y_{\mathbb{F}_-} = \left(D'_{\mathbb{F}_-}L^{\dagger}D_{\mathbb{F}_-}\right)^{\dagger}$, respectively [18]. This m_1-port network is strictly passive if, and only if, $Z_{\mathbb{F}_-} > 0$, or equivalently, $Y_{\mathbb{F}_-} > 0$. Then, the following question arises: Does the strict passivity of such an m_1-port network imply semidefiniteness of L?

The answer is in the affirmative. See the following theorem which has been stated and proved in the authors' earlier work [15]. Here, we give an alternative proof which is more informative and has a nice physical interpretation via shorted connection of a multiport network.

Theorem 1: $L \geq 0$ and has a simple zero eigenvalue if, and only if, $\mathbb{G}_+$ is connected and $Z_{\mathbb{F}_-} > 0$, or equivalently, $Y_{\mathbb{F}_-} > 0$.

Proof: The necessity proof is quite straightforward and is omitted for brevity.

To show the sufficiency, we first augment $\mathbb{F}_-$ with $n-1-m_1$ edges from $\mathbb{G}_+$ to form a spanning tree $\mathbb{F}$ of $\mathbb{G}$. Then, we obtain an augmented $(n-1)$-port network $\boldsymbol{A}$. We label the ports corresponding to $\mathbb{F}_-$ as the first m_1 ports. Let the resistance matrix of $\boldsymbol{A}$ be $Z^a = \begin{bmatrix} Z^a_{11} & Z^a_{12} \\ Z^a_{21} & Z^a_{22} \end{bmatrix}$. Clearly, $Z_{\mathbb{F}_-} = Z^a_{11}$.

Then, we short the m_1 ports corresponding to $\mathbb{F}_-$, leading to a shorted connection $\boldsymbol{C}$ of $\boldsymbol{A}$, as depicted in Fig. 2. The resistance matrix of $\boldsymbol{C}$ is given by $Z^c = Z^a/_{11}$.

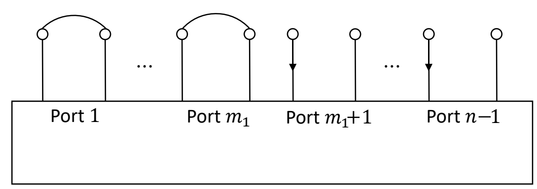

Fig. 2. Shorted connection

Since all ports corresponding to $\mathbb{F}_-$ are shorted, there is no current flowing through the negative resistors. Therefore, power is dissipated through the shorted network $\boldsymbol{C}$. If further $\mathbb{G}_+$ is connected, then $\boldsymbol{C}$ is strictly passive, yielding $Z^c = Z^a/_{11} > 0$. By the knowledge of Schur complement [19], $Z^a_{11} > 0$ together with $Z^a/_{11} > 0$ implies $Z^a > 0$ and, thus, $L \geq 0$ and has a simple zero eigenvalue. ■

When $\mathbb{G}$ does not have any cycle containing two negatively weighted edges, Theorem 1 reduces to the condition obtained in [8] and [9].

V. Inertia of Signed Laplacians

In this section, we show that when L is indefinite, $Y_{\mathbb{F}_-}$ encodes the information concerning the inertia of L.

Let D be partitioned as before, i.e, $D = \begin{bmatrix} D_{\mathbb{F}_-} & D_{\mathbb{C}_-} & D_{\mathbb{G}_+} \end{bmatrix}$. Also partition W as $W = \operatorname{diag}\{W_{\mathbb{F}_-}, W_{\mathbb{C}_-}, W_{\mathbb{G}_+}\}$. Let $D_{\mathbb{G}_-} = \begin{bmatrix} D_{\mathbb{F}_-} & D_{\mathbb{C}_-} \end{bmatrix}$ and $W_{\mathbb{G}_-} = \operatorname{diag}\{W_{\mathbb{F}_-}, W_{\mathbb{C}_-}\}$. Since $\mathbb{F}_-$ is a spanning forest of $\mathbb{G}_-$, there exists a matrix T of full row rank such that $D_{\mathbb{G}_-} = D_{\mathbb{F}_-}T$. Denote by L_+ and L_- the signed Laplacians associated with $\mathbb{G}_+$ and $\mathbb{G}_-$, respectively. Clearly, $L = L_+ + L_-$.

Recall the m_1-port network with ports corresponding to the edges of $\mathbb{F}_-$. It can be regarded as the parallel connection of a positive m_1-port network with all positive resistances and a negative m_1-port network with all negative resistances.

The conductance matrices of the positive m_1-port network and negative m_1-port network are given by

$$Y^+_{\mathbb{F}_-} = \left(D'_{\mathbb{F}_-} L^\dagger_+ D_{\mathbb{F}_-}\right)^\dagger, \quad Y^-_{\mathbb{F}_-} = \left(D'_{\mathbb{F}_-} L^\dagger_- D_{\mathbb{F}_-}\right)^\dagger.$$

Then, $Y_{\mathbb{F}_-} = Y^+_{\mathbb{F}_-} + Y^-_{\mathbb{F}_-}$. Further computation yields

$$Y^-_{\mathbb{F}_-} = \left(D'_{\mathbb{F}_-} \left(D_{\mathbb{G}_-} W_{\mathbb{G}_-} D'_{\mathbb{G}_-}\right)^\dagger D_{\mathbb{F}_-}\right)^\dagger = T W_{\mathbb{G}_-} T'.$$

For simplicity, we assume here $\mathbb{G}_+$ to be connected.

Theorem 2: For a given signed Laplacian L, there holds $\pi(L) = \pi(Y_{\mathbb{F}_-}) + (0, 1, n-1-m_1)$.

Proof: First, consider a matrix $M = \begin{bmatrix} -W^{-1}_{\mathbb{G}_-} & D'_{\mathbb{G}_-} \\ D_{\mathbb{G}_-} & L_+ \end{bmatrix}$. Applying Schur complement on M yields

$$M/_{11} = L_+ + D_{\mathbb{G}_-} W_{\mathbb{G}_-} D'_{\mathbb{G}_-} = L_+ + L_- = L,$$
$$M/_{22} = -W^{-1}_{\mathbb{G}_-} - D'_{\mathbb{G}_-} L^\dagger_+ D_{\mathbb{G}_-}.$$

Note that $M_{11} > 0$, $M_{22} \geq 0$, and $\ker(M_{22}) \subset \ker(M_{12})$. By the inertia additivity formula of generalized Schur complement [20], we have

$$\begin{aligned} \pi(M) &= \pi(M_{11}) + \pi(M/_{11}) = (0, 0, m_-) + \pi(L) \\ &= \pi(M_{22}) + \pi(M/_{22}) = (0, 1, n-1) + \pi(M/_{22}). \end{aligned} \tag{1}$$

Now consider another matrix $N = \begin{bmatrix} -W^{-1}_{\mathbb{G}_-} & T' \\ T & Y^+_{\mathbb{F}_-} \end{bmatrix}$. Since $\mathbb{G}_+$ is connected, it follows that $Y^+_{\mathbb{F}_-} > 0$. Applying Schur complement on N yields

$$N/_{11} = Y^+_{\mathbb{F}_-} + T W_{\mathbb{G}_-} T' = Y^+_{\mathbb{F}_-} + Y^-_{\mathbb{F}_-} = Y_{\mathbb{F}_-},$$
$$N/_{22} = -W^{-1}_{\mathbb{G}_-} - T' {Y^+_{\mathbb{F}_-}}^{-1} T = -W^{-1}_{\mathbb{G}_-} - D'_{\mathbb{G}_-} L^\dagger_+ D_{\mathbb{G}_-} = M/_{22}.$$

Again, by the inertia additivity formula, we have

$$\begin{aligned} \pi(N) &= \pi(N_{11}) + \pi(N/_{11}) = (0, 0, m_-) + \pi(Y_{\mathbb{F}_-}) \\ &= \pi(N_{22}) + \pi(N/_{22}) = (0, 0, m_1) + \pi(N/_{22}). \end{aligned} \tag{2}$$

From (1) and (2) together with $M/_{22} = N/_{22}$, it follows that

$$\pi(M) - \pi(N) = \pi(L) - \pi(Y_{\mathbb{F}_-}) = (0, 1, n-1-m_1)$$

and, thus, $\pi(L) = \pi(Y_{\mathbb{F}_-}) + (0, 1, n-1-m_1)$. ■

The authors in [14] gave an alternative way to characterize the inertia of a signed Laplacian L, assuming that L has a simple zero eigenvalue.

VI. Conclusion

In this paper, we characterized the set of negative weights maintaining the positive semidefiniteness of signed Laplacians via n-port network theory. When a signed Laplacian is not positive semidefinite, we characterized its inertia via the notion of conductance matrix.

One future direction is to extend the discussions to directed signed graphs. The goal is to explore under what conditions the asymmetric signed Laplacians have all eigenvalues in the open right half plane except a simple zero eigenvalue. Some results on certain special cases can be found in [11], [21].

There have also been results reported for signed adjacency matrices, e.g., [22]. How to connect those results with signed Laplacians is under our current investigation.

Acknowledgements

The authors would like to thank Prof. Claudio Altafini of Linköping University, Prof. Florian Dörfler of Swiss Federal Institute of Technology, and Mr. Yue Song, Dr. Tao Liu, and Prof. David J. Hill of The University of Hong Kong for valuable discussions.

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