

# Phase Analysis of MIMO LTI Systems

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**Abstract**—In this paper, we introduce a definition of phase response for a class of multi-input multi-output (MIMO) linear time-invariant (LTI) systems, the frequency responses of which are sectorial at all frequencies. This phase concept generalizes the notions of positive realness and negative imaginarity. We also define the half-sectorial systems and provide a time-domain interpretation. As a starting point in an endeavour to develop a comprehensive phase theory for MIMO systems, we establish a small phase theorem for feedback stability, which complements the well-known small gain theorem. In addition, we derive a sectored real lemma for phase-bounded systems as a natural counterpart of the bounded real lemma.

**Index Terms**—MIMO systems, phase response, small phase theorem, sectored real lemma, half-sectorial systems

## I. INTRODUCTION

In the classical frequency domain analysis of single-input-single-output (SISO) systems, the magnitude (gain) response and phase response go hand in hand. In particular, the Bode magnitude plot and phase plot are always drawn shoulder to shoulder. The combined Bode plot of a loop transfer function provides a significant amount of useful information about the closed-loop stability and performance. The gain and phase crossover frequencies of a loop transfer function give salient information on the gain and phase margins of the feedback system. The famous Bode gain-phase integral relation binds the gain and phase together. In frequency domain controller synthesis, phase also plays an important role. Loop-shaping design techniques, such as lead and lag compensation, are rooted in the phase stabilization ideas.

The inception of MIMO systems theory sees extension and thriving of the magnitude concept, but not equal flourishing in the phase concept. While the small gain theorem is widely known in the field of robust control, much less attention has been paid to the development of a small phase theorem. Moreover, the magnitude plot of a MIMO frequency response has been inbuilt to the computing environment MATLAB, a

useful phase plot has not been available in practice. Several preliminary works on phase information of MIMO systems include [2], [6], [8], [17], [22], [24]. The references [2], [6], [8] extended the Bode gain-phase integral relation for SISO systems to MIMO systems. The reference [24] proposed a definition of phases for MIMO LTI systems based on matrix polar decomposition and formulated a small phase theorem. However, the condition therein depends on both phase and gain information, which somewhat deviates from the purpose of finding a phase counterpart to the small gain theorem. An alternative way to extract the phasic information of a MIMO LTI system was proposed in [22] by examining the numerical range of the frequency response. The idea of using numerical range as a tool for robust stability studies has received more attention recently. The reference [17] suggested to use phase spread as a performance measure of uncertain systems whose numerical ranges are contained in a cone sector. In this paper, we define a vector-valued phase response of a MIMO system based on numerical range. The phase response fully captures the phasic information of a MIMO system, as opposed to the phase spread used in [17].

An important line of research with a phasic point of view is on positive real (passive) and negative imaginary systems. Roughly speaking, one can think of positive real systems as those whose phases lie within  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  and negative imaginary systems as those whose phases over positive frequencies lie within  $[-\pi, 0]$ . Research on positive real systems can be traced back to more than half a century ago and has led to a rich theory through efforts of generations of researchers. See books [1], [4], [5], [7] and the survey paper [16] for a review. Over the past two decades, negative imaginary systems [18], [23] and counter-clockwise dynamics [3] have attracted much attention. The abundant studies on these systems, concerning feedback stability, performance and beyond, provide valuable insights in developing a general phase theory for MIMO LTI systems.

One main reason accounting for the underdevelopment of MIMO phases is the following. While the gains of a complex matrix are well described by its singular values, a universally accepted definition of matrix phases has been lacking over a long period. Very recently, we initiated to adopt the canonical angles introduced in [9] as the phases of a sectorial complex matrix whose numerical range does not contain the origin [27]. We studied various properties of matrix phases, some of which are briefly reviewed later. This paves the ground for conducting a systematic study of phase analysis and design for MIMO LTI systems.

In this paper, we first define the phase responses of MIMO LTI systems whose frequency responses are sectorial at all

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frequencies. Such phase concept agrees with and generalizes the notions of positive realness and negative imaginarity. We then develop a small phase theorem for negative feedback interconnections of phase bounded systems, complementing the well known small gain theorem. We derive a sectorial real lemma, which gives state space conditions for phase-bounded systems in terms of linear matrix inequalities (LMIs). This serves as a counterpart of bounded real lemma. In addition, we pay special attention to the class of half-sectorial systems which exhibit a nice time-domain interpretation. We absorb much nutrition from the existing studies on positive real systems, negative imaginary systems, KYP lemma, generalized KYP lemma, integral quadratic constraints (IQCs), etc. along the way.

The rest of the paper is organized as follows. A review of matrix phases is presented in Section II. The phase responses of MIMO LTI systems are defined in Section III, followed by the discussions on half-sectorial systems in Section IV. A small phase theorem is presented in Section V. State-space conditions are derived for phase bounded systems in Section VI. The paper is concluded in Section VII. The notation used in this paper is more or less standard and will be made clear as we proceed. The proofs are omitted due to page limit.

## II. PHASES OF A COMPLEX MATRIX

A nonzero complex scalar  $c$  can be represented in the polar form as  $c = \sigma e^{i\phi}$  with  $\sigma > 0$  and  $\phi$  taking values in a half open  $2\pi$ -interval, typically  $[0, 2\pi)$  or  $(-\pi, \pi]$ . Here  $\sigma = |c|$  is called the modulus or the magnitude and  $\phi = \angle c$  is called the argument or the phase. The polar form is particularly useful when multiplying two complex numbers. We simply have  $|ab| = |a||b|$  and  $\angle(ab) = \angle a + \angle b \pmod{2\pi}$ .

It is well understood that an  $n \times n$  complex matrix  $C$  has  $n$  magnitudes, served by the  $n$  singular values

$$\sigma(C) = [\sigma_1(C) \quad \sigma_2(C) \quad \cdots \quad \sigma_n(C)]$$

with  $\bar{\sigma}(C) = \sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C) = \underline{\sigma}(C)$  [11]. The magnitudes of a matrix possess plentiful nice properties, among which the following majorization inequality regarding the magnitudes of matrix products are of particular interest to the control community.

Given  $x, y \in \mathbb{R}^n$ , we denote by  $x^\downarrow$  and  $y^\downarrow$  the rearranged versions of  $x$  and  $y$  so that their elements are sorted in a non-increasing order. Then,  $x$  is said to be *majorized* by  $y$  [20], denoted by  $x \prec y$ , if

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow, \quad k = 1, \dots, n-1, \quad \text{and} \quad \sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow.$$

When  $x$  and  $y$  are nonnegative,  $x$  is said to be *log-majorized* by  $y$ , denoted by  $x \prec_{\log} y$ , if

$$\prod_{i=1}^k x_i^\downarrow \leq \prod_{i=1}^k y_i^\downarrow, \quad k = 1, \dots, n-1, \quad \text{and} \quad \prod_{i=1}^n x_i^\downarrow = \prod_{i=1}^n y_i^\downarrow.$$

The magnitudes of matrix product satisfy [20]

$$\sigma(AB) \prec_{\log} \sigma(A) \odot \sigma(B), \quad (1)$$

where  $\odot$  denotes the Hadamard product, i.e., the elementwise product.

In contrast to the magnitudes of a complex matrix  $C$ , how to define the phases of  $C$  appears to be an unsettled issue. An early attempt [24] defined the phases of  $C$  as the phases of the eigenvalues of the unitary part of its polar decomposition. This definition was motivated by the seeming generalization of the polar form of a scalar to the polar decomposition of a matrix. However, phases defined this way do not have certain desired properties.

Very recently, we discovered a more suitable definition of matrix phases based on numerical range [27]. The numerical range, also called field of values, of a matrix  $C \in \mathbb{C}^{n \times n}$  is defined as  $W(C) = \{x^* C x : x \in \mathbb{C}^n \text{ with } \|x\| = 1\}$ , which, as a subset of  $\mathbb{C}$ , is compact and convex, and contains the spectrum of  $C$  [12].

If  $0 \notin W(C)$ , then  $W(C)$  is contained in an open half complex plane due to its convexity. In this case,  $C$  is said to be a sectorial matrix. It is known that a sectorial  $C$  is congruent to a diagonal unitary matrix that is unique up to a permutation [13], [29], i.e., there exists a nonsingular matrix  $T$  and a diagonal unitary matrix  $D$  such that  $C = T^* D T$ . This factorization is called sectorial decomposition in [29]. Let  $\delta(C)$  be the field angle of  $C$ , i.e., the angle subtended by the two supporting rays of  $W(C)$  at the origin. We define the phases of  $C$ , denoted by  $\phi_1(C), \phi_2(C), \dots, \phi_n(C)$ , to be the phases of the eigenvalues of  $D$ , taking values in an interval  $(\theta, \theta + \pi)$ , where  $\theta \in [-\pi, \delta(C))$ . The phases defined in this fashion coincide with the canonical angles of  $C$  introduced in [9]. Assume without loss of generality that

$$\bar{\phi}(C) = \phi_1(C) \geq \phi_2(C) \geq \cdots \geq \phi_n(C) = \underline{\phi}(C).$$

Moreover, define  $\phi(C) = [\phi_1(C) \quad \phi_2(C) \quad \cdots \quad \phi_n(C)]$ .

A graphic interpretation of the phases is given in Fig. 1. The two angles from the positive real axis to each of the two supporting rays of  $W(C)$  are  $\bar{\phi}(C)$  and  $\underline{\phi}(C)$  respectively. The other phases of  $C$  lie in between.

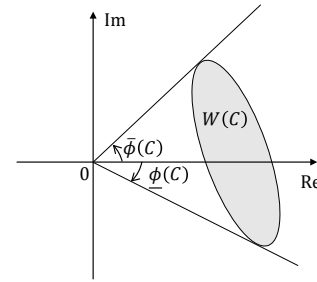


Fig. 1. Geometric interpretation of  $\bar{\phi}(C)$  and  $\underline{\phi}(C)$ .

It is noteworthy that the notion of matrix phases subsumes the well-studied strictly accretive matrices [15], i.e., matrices with positive definite Hermitian part. In particular, the phases of  $C$  lie in  $(-\pi/2, \pi/2)$  if and only if  $C$  is strictly accretive.

Given matrix  $C$ , we can check whether it is sectorial or not by plotting its numerical range. From the plot of numerical range, we can also determine a  $\pi$ -interval  $(\theta, \theta + \pi)$  in which

the phases take values. How to efficiently compute  $\phi(C)$  is an important issue. The following observation provides some insights along this direction. Suppose  $C$  is sectorial. Then it admits a sectorial decomposition  $C = T^*DT$  and thus

$$C^{-1}C^* = T^{-1}D^{-1}T^{-*}T^*D^*T = T^{-1}D^{-2}T,$$

indicating that  $C^{-1}C^*$  is similar to a diagonal unitary matrix. Hence, we can first compute  $\angle\lambda(C^{-1}C^*)$ , taking values in  $(-2\theta-2\pi, -2\theta)$ , and then let  $\phi(C) = -\frac{1}{2}\angle\lambda(C^{-1}C^*)$ . This gives one possible way to compute  $\phi(C)$ . We are currently exploring other methods, hopefully of lower complexity, for the computation of matrix phases.

The matrix phases defined above have plentiful properties, of which a comprehensive study has been conducted in [27]. First, note that the set of phase bounded matrices defined as

$$\begin{aligned} \mathcal{C}[\alpha, \beta] \\ = \{C \in \mathbb{C}^{n \times n} : C \text{ is sectorial and } \bar{\phi}(C) \leq \beta, \underline{\phi}(C) \geq \alpha\}, \end{aligned}$$

where  $0 \leq \beta - \alpha < 2\pi$ , is a cone. In addition, the following lemma can be shown.

*Lemma 1 ([27]):* If  $\beta - \alpha < \pi$ , then  $\mathcal{C}[\alpha, \beta]$  is a convex cone.

Another important property pertinent to later developments in this paper is concerned with product of sectorial matrices. In view of the magnitude counterpart in (1), one may expect  $\phi(AB) \prec \phi(A) + \phi(B)$  to hold for sectorial matrices  $A$  and  $B$ . This, unfortunately, fails even for positive definite  $A$  and  $B$ . Notwithstanding, if we consider instead  $\lambda(AB) = [\lambda_1(AB) \ \dots \ \lambda_n(AB)]$ , i.e., the vector of eigenvalues of  $AB$ , the following weaker but useful result has been derived.

*Lemma 2 ([27]):* Let  $A, B \in \mathbb{C}^{n \times n}$  be sectorial matrices with phases in  $(\theta_1, \theta_1 + \pi)$  and  $(\theta_2, \theta_2 + \pi)$ , respectively, where  $\theta_1 \in [-\pi, \delta(A))$  and  $\theta_2 \in [-\pi, \delta(B))$ . Let  $\angle\lambda(AB)$  take values in  $(\theta_1 + \theta_2, \theta_1 + \theta_2 + 2\pi)$ . Then

$$\angle\lambda(AB) \prec \phi(A) + \phi(B).$$

The above majorization relation underlies the development of a small phase theorem, much in the spirit of (1) being the foundation of the celebrated small gain theorem. To be more specific, recall that the singularity of matrix  $I + AB$  plays an important role in the stability analysis of feedback systems. It is straightforward to see that if  $\sigma(A)$  and  $\sigma(B)$  are both sufficiently small, then  $I + AB$  is nonsingular. By contrast, one can observe that if  $\phi(A)$  and  $\phi(B)$  are both sufficiently small in magnitudes, then  $I + AB$  is nonsingular.

### III. PHASE RESPONSE OF MIMO LTI SYSTEMS

Let  $G$  be an  $m \times m$  real rational proper stable transfer matrix, i.e.,  $G \in \mathcal{RH}_\infty^{m \times m}$ . Then  $\sigma(G(j\omega))$ , the vector of singular values of  $G(j\omega)$ , is an  $\mathbb{R}^m$ -valued function of the frequency, which we call the *magnitude response* of  $G$ . The  $\mathcal{H}_\infty$  norm of  $G$ , denoted by  $\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega))$ , is of particular importance.

Suppose  $G(j\omega)$  is sectorial for all  $\omega \in \mathbb{R} \cup \{\infty\}$ . Such a system is called a frequency-wise sectorial system. Also, assume for simplicity that  $W(G(j\omega))$  does not intersect the

negative real axis for all  $\omega \in \mathbb{R} \cup \{\infty\}$ . Then  $\phi(G(j\omega))$ , the vector of phases of  $G(j\omega)$  with each element taking values in  $(-\pi, \pi)$ , is well defined as an  $\mathbb{R}^m$ -valued function of the frequency, which we call the *phase response* of  $G$ . We define the  $\mathcal{H}_\infty$  phase of  $G$ , as the counterpart to its  $\mathcal{H}_\infty$  norm, to be

$$\Phi_\infty(G) = \sup_{\omega \in \mathbb{R}} \bar{\phi}(G(j\omega)).$$

Clearly,  $\Phi_\infty(G) \leq \pi$ . It is noteworthy that the set of phase bounded systems

$$\mathcal{C}[\alpha] = \{G \in \mathcal{RH}_\infty^{m \times m} : \Phi_\infty(G) \leq \alpha\},$$

where  $\alpha \in [0, \pi)$ , is a cone.

Having defined the phase response of  $G$ , we can now plot  $\sigma(G(j\omega))$  and  $\phi(G(j\omega))$  together to complete the MIMO Bode plot of  $G$ , laying the foundation of a complete MIMO frequency-domain analysis.

*Example 1:* The Bode plot of system

$$G(s) = \begin{bmatrix} \frac{1}{s^2+2s+200} & \frac{2}{s^2+2s+200} \\ \frac{2}{s^2+2s+200} & \frac{0.2s^3+0.5s^2+44.2s+24}{s^3+3s^2+202s+200} \end{bmatrix}$$

is shown in Fig. 2.

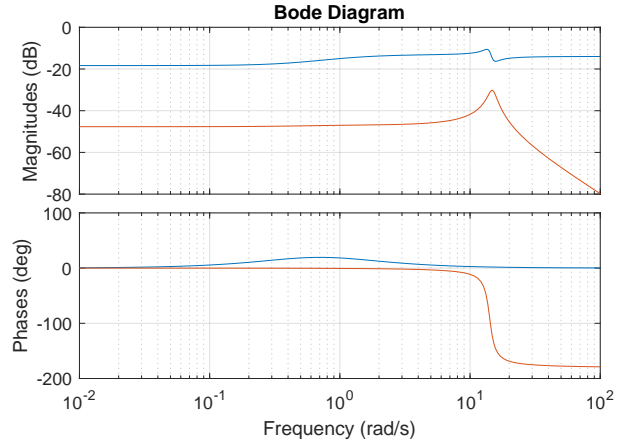


Fig. 2. MIMO Bode plot of a frequency-wise sectorial system.

Note that the well-known notions of positive real systems [1], [5], [16] and negative imaginary systems [18], [23] can be characterized using their phase responses. For simplicity, here we briefly mention the strong and strict versions of these notions. A transfer function matrix  $G \in \mathcal{RH}_\infty^{m \times m}$  is said to be strongly positive real if  $G(j\omega) + G^*(j\omega) > 0$  for all  $\omega \in [-\infty, +\infty]$  [19]. In the language of phase,  $G \in \mathcal{RH}_\infty^{m \times m}$  is strongly positive real if and only if

$$\Phi_\infty(G) < \frac{\pi}{2}.$$

On the other hand, a transfer function matrix  $G$  is said to be strictly negative imaginary if  $(G(j\omega) - G^*(j\omega))/j < 0$  for all  $\omega \in (0, \infty)$  [18]. This is equivalent to

$$[\underline{\phi}(G(j\omega)), \bar{\phi}(G(j\omega))] \subset (-\pi, 0)$$

for all  $\omega \in (0, \infty)$ . The phase concept of MIMO LTI systems gives a way to unify these concepts, together with of course the trivial SISO system phase, and more. The system shown in Fig. 2 is neither positive real nor negative imaginary but it has well-defined phase response.

#### IV. HALF-SECTORIAL SYSTEMS

Let  $G \in \mathcal{RH}_\infty^{m \times m}$ . Then,  $G(j\omega)$  is conjugate symmetric, i.e.,

$$G(-j\omega) = \overline{G(j\omega)},$$

and hence  $W(G(j\omega))$  and  $W(G(-j\omega))$  are symmetric about the real axis. This property hints that in dealing with many problems such as feedback stability, one only has to examine the frequency response for nonnegative frequency, while the other half frequency range will be automatically taken care of due to the symmetry. Following this hint, we define half-sectorial systems, and provide a time-domain interpretation for such systems.

A system  $G$  is said to be half-sectorial if

$$\text{cl. Co}\{W(G(j\omega)), \omega \geq 0\}$$

is contained in an open half plane and does not intersect the negative real axis, where cl. denotes closure and Co denotes convex hull.

Interestingly, there is a nice time-domain interpretation for half-sectorial systems. For preparation, we briefly introduce some background knowledge on signal spaces and Hilbert transform. The Hilbert transform has been used extensively in signal processing, especially in the time-frequency domain analysis. It has also been applied in the control field, mostly in gain-phase relationship and system identification, etc. We refer interested readers to [10] for more details.

Let  $\mathcal{F}$  be the usual Fourier transform on  $\mathcal{L}_2^T(-\infty, \infty)$ , the Hilbert space of complex-valued bilateral time functions

$$[\mathcal{F}x](j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt.$$

Note that  $\mathcal{F}$  is an isometry onto  $\mathcal{L}_2^\Omega(-\infty, \infty)$ , the Hilbert space of complex-valued bilateral frequency functions. If we decompose  $\mathcal{L}_2^\Omega(-\infty, \infty)$  into a positive frequency signal space and a negative frequency signal space as

$$\mathcal{L}_2^\Omega(-\infty, \infty) = \mathcal{L}_2^\Omega(0, \infty) \oplus \mathcal{L}_2^\Omega(-\infty, 0),$$

then clearly this is an orthogonal decomposition. We also naturally have the orthogonal decomposition

$$\mathcal{L}_2^T(-\infty, \infty) = \mathcal{F}^{-1}\mathcal{L}_2^\Omega(0, \infty) \oplus \mathcal{F}^{-1}\mathcal{L}_2^\Omega(-\infty, 0).$$

Let us call the first space above  $\mathcal{A}$  and hence the second space  $\mathcal{A}^\perp$ . Let  $P$  be the orthogonal projection onto  $\mathcal{L}_2^\Omega(0, \infty)$  and  $Q$  be the orthogonal projection onto  $\mathcal{A}$ . Then the commutative diagram in Fig. 3 gives a complete picture of the relationships among these spaces. Recall the Hilbert transform  $\mathcal{H} : \mathcal{L}_2^T(-\infty, \infty) \rightarrow \mathcal{L}_2^T(-\infty, \infty)$  defined as

$$[\mathcal{H}x](t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} d\tau.$$

It then turns out that  $Qx = \frac{1}{2}(x + j\mathcal{H}x)$  and  $(I - Q)x = \frac{1}{2}(x - j\mathcal{H}x)$ , the analytic part and the skew-analytic part of  $x$  respectively.

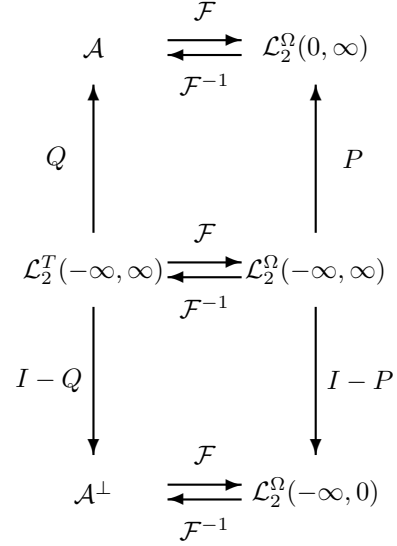


Fig. 3. A commutative diagram.

Now let  $\mathbf{G} : \mathcal{L}_2^T(-\infty, \infty) \rightarrow \mathcal{L}_2^T(-\infty, \infty)$  be the linear operator corresponding to  $G(s) \in \mathcal{RH}_\infty$ . Clearly, both  $\mathcal{A}$  and  $\mathcal{A}^\perp$  are invariant subspaces of  $\mathbf{G}$ . We define the positive frequency numerical range and negative frequency numerical range as

$$W_+(\mathbf{G}) := \{\langle Qu, \mathbf{G}u \rangle : u \in \mathcal{L}_2^T(-\infty, \infty), \|u\|_2 = 1\}$$

$$W_-(\mathbf{G}) := \{\langle (I - Q)u, \mathbf{G}u \rangle : u \in \mathcal{L}_2^T(-\infty, \infty), \|u\|_2 = 1\}$$

respectively. It can be easily seen that  $W_+(\mathbf{G})$  and  $W_-(\mathbf{G})$  are symmetric with respect to the real axis. Also, note that

$$\begin{aligned} \langle Qu, \mathbf{G}u \rangle &= \int_{-\infty}^{+\infty} [Qu]^*(t)[\mathbf{G}u](t) dt \\ &= \int_0^{+\infty} [\mathcal{F}u]^*(j\omega)G(j\omega)[\mathcal{F}u](j\omega) d\omega, \end{aligned}$$

which suggests that

$$\text{cl. } W_+(\mathbf{G}) \subset \text{cl. Co} \left\{ \frac{1}{2}W(G(j\omega)), \omega \geq 0 \right\}.$$

In fact, one can further show

$$\text{cl. } W_+(\mathbf{G}) = \text{cl. Co} \left\{ \frac{1}{2}W(G(j\omega)), \omega \geq 0 \right\},$$

and thus  $\Phi_\infty(G) = \max\{\sup_{z \in W_+(\mathbf{G})} \angle z, \sup_{z \in W_-(\mathbf{G})} \angle z\}$ . The detailed proof is omitted for brevity and will be available in a longer version of this paper.

#### V. SMALL PHASE THEOREM

Suppose  $G$  and  $H$  are  $m \times m$  real rational proper transfer function matrices. The feedback interconnection of  $G$  and  $H$ , as depicted in Fig. 4, is said to be stable if the Gang of Four matrix

$$G\#H = \begin{bmatrix} (I + HG)^{-1} & (I + HG)^{-1}H \\ G(I + HG)^{-1} & G(I + HG)^{-1}H \end{bmatrix}$$

is stable, i.e.,  $G\#H \in \mathcal{RH}_\infty^{2m \times 2m}$ .

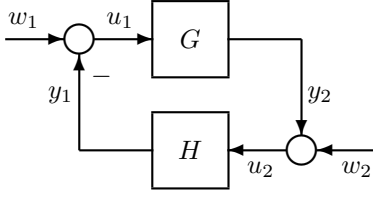


Fig. 4. A standard feedback system.

The celebrated small gain theorem [19], [30] is one of the most used results in robust control theory over the past half a century. A version of it states that for  $G, H \in \mathcal{RH}_\infty^{m \times m}$ , the feedback system  $G\#H$  is stable if

$$\bar{\sigma}(G(j\omega))\bar{\sigma}(H(j\omega)) < 1$$

for all  $\omega \in \mathbb{R} \cup \{\infty\}$ .

There was an attempt to formulate a small phase theorem by using phases defined from the matrix polar decomposition [24]. However, the condition therein involves both phase and gain information and thus deviates from the initial purpose of having a phase counterpart of the small gain theorem.

Armed with the new definition of matrix phases  $\phi(C)$ , we work out a version of the small phase theorem.

*Theorem 1 (Small phase theorem):* For frequency-wise sectorial  $G, H \in \mathcal{RH}_\infty^{m \times m}$ , the feedback system  $G\#H$  is stable if

$$\bar{\phi}(G(j\omega)) + \bar{\phi}(H(j\omega)) < \pi \quad (2)$$

for all  $\omega \in \mathbb{R} \cup \{\infty\}$ .

We wish to mention that the small phase theorem can also be established via IQCs. Specifically, when the condition (2) is satisfied, one can find a dynamic multiplier of the form

$$\Pi(s) = \begin{bmatrix} 0 & e^{j\theta(s)} \\ e^{-j\theta(s)} & 0 \end{bmatrix}$$

so that  $\Pi(s) \in \mathcal{L}_\infty$  is continuous on the imaginary axis and the following quadratic constraints

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} \geq 0, \\ \begin{bmatrix} -H(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} -H(j\omega) \\ I \end{bmatrix} < 0$$

are satisfied for all  $\omega \in \mathbb{R} \cup \{\infty\}$ . The feedback stability then follows from the result in [21]. From this perspective, the small phase theorem provides a nice phasic interpretation of the condition obtained from IQCs.

The small phase theorem generalizes a stronger version of the passivity theorem [7], [19], which states that for  $G, H \in \mathcal{RH}_\infty^{m \times m}$ , the feedback system  $G\#H$  is stable if  $G$  and  $H$  are strongly positive real.

Note that the small gain theorem provides a quantifiable tradeoff between the gains of  $G$  and  $H$ , while the above small phase theorem does the same with respect to the phases of  $G$  and  $H$ . In the literature, the notions of input feedforward

passivity index and output feedback passivity index [4], [16], [26], [28] have been used to characterize the tradeoff between the surplus and deficit of passivity in open-loop systems. It is our belief that the concept of MIMO system phases is more suited to this task. Specifically,  $\frac{\pi}{2} - \Phi_\infty(G)$  gives a natural measure of passivity of system  $G$ , which we call the angular passivity index. The small phase theorem above implies that if the sum of the angular passivity indexes of  $G$  and  $H$  is positive, then  $G\#H$  is stable. In addition, one can see that  $\pi - \Phi_\infty(GH)$  yields a natural phase stability margin of  $G\#H$ .

It is well known that the condition given in the small gain theorem is necessary in the following sense [30]. Let  $G \in \mathcal{RH}_\infty^{m \times m}$ . Then,  $G\#H$  is stable for all  $H \in \mathcal{RH}_\infty^{m \times m}$  with  $\|H\|_\infty < \gamma$  if and only if  $\|G\|_\infty \leq \frac{1}{\gamma}$ .

Regarding the necessity of small phase theorem, we have the following conjecture.

*Conjecture 1 (Small phase theorem with necessity):* Let  $G \in \mathcal{RH}_\infty^{m \times m}$ . Then,  $G\#H$  is stable for all  $H \in \mathcal{RH}_\infty^{m \times m}$  with  $\bar{\phi}(H(j\omega)) < \frac{\pi}{2} + \angle m(j\omega)$ , where  $m$  is a scalar real rational function, if and only if  $\bar{\phi}(G(j\omega)) \leq \frac{\pi}{2} - \angle m(j\omega)$  for all  $\omega \in \mathbb{R} \cup \{\infty\}$ .

Evidently, this conjecture holds in the SISO case in view of the Nyquist stability criterion. A rigorous proof in the MIMO case is under our current investigation.

## VI. STATE-SPACE CONDITIONS FOR PHASE BOUNDED SYSTEMS

The  $\mathcal{H}_\infty$  norm of an LTI system can be determined by the well-known bounded real lemma. The efficient computation of  $\mathcal{H}_\infty$  norm is specifically useful as evidenced in small gain theorem and facilitates robust control design.

The bounded real lemma [30] states that for  $G \in \mathcal{RH}_\infty^{m \times m}$  with a minimal realization  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ ,  $\|G\|_\infty < \gamma$  if and only if there exists  $X > 0$  satisfying the LMI

$$\begin{bmatrix} A'X + XA & XB & C' \\ B'X & -\gamma I & D' \\ C & D & -\gamma I \end{bmatrix} < 0.$$

One would wish to see an analogous state-space condition for phase bounded systems. It is equally important to have an LMI characterization for a system  $G$  satisfying  $\Phi_\infty(G) < \alpha$ , where  $\alpha \in (0, \pi]$ . For this purpose, we obtain a sectorial real lemma in the case when  $\alpha \in (0, \frac{\pi}{2}]$ , which serves as a natural counterpart of the bounded real lemma.

*Theorem 2 (Sectorial real lemma):* Let  $G \in \mathcal{RH}_\infty^{m \times m}$  with a minimal realization  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  and  $\alpha \in (0, \frac{\pi}{2}]$ . Then  $\Phi_\infty(G) < \alpha$  if and only if there exists  $X > 0$  satisfying the LMI

$$\begin{bmatrix} A'X + XA & XB - e^{-j(\frac{\pi}{2}-\alpha)}C' \\ B'X - e^{j(\frac{\pi}{2}-\alpha)}C & -e^{j(\frac{\pi}{2}-\alpha)}D - e^{-j(\frac{\pi}{2}-\alpha)}D' \end{bmatrix} < 0.$$

The sectorial real lemma can be proved by exploiting the well-known KYP lemma [19] which builds the equivalence between frequency domain characterization over the entire frequency range and a finite dimensional LMI.

When  $\alpha = \frac{\pi}{2}$ , the above sectorized real lemma reduces to the strongly positive real lemma [25].

The case when  $\alpha \in (\frac{\pi}{2}, \pi]$  appears much more complicated. Nevertheless, for half-sectorial real systems, one only needs to concern the frequency domain characterization for positive frequency, i.e.,  $\{j\omega | \omega \in [0, \infty)\}$ . By virtue of the generalized KYP lemma [14], we are able to derive an LMI condition.

In contrast to the KYP lemma which copes with frequency domain inequalities over the entire frequency, the generalized KYP lemma has the capability to address the frequency domain inequalities over partial frequency ranges. In particular, it builds the equivalence between inequalities on curves in the complex plane and LMIs.

*Theorem 3:* Let  $G \in \mathcal{RH}_\infty^{m \times m}$  with a minimal realization  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  and  $\alpha \in (\frac{\pi}{2}, \pi]$ . Then  $G$  is half-sectorial and  $\Phi_\infty(G) < \alpha$  if and only if there exist Hermitian matrices  $X$  and  $Y$  satisfying either

$$Y > 0, \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}' \begin{bmatrix} 0 & X + jY \\ X - jY & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + M < 0,$$

where  $M = \begin{bmatrix} 0 & -e^{-j(\alpha - \frac{\pi}{2})} C' \\ -e^{j(\alpha - \frac{\pi}{2})} C & -e^{j(\alpha - \frac{\pi}{2})} D - e^{-j(\alpha - \frac{\pi}{2})} D' \end{bmatrix}$ , or

$$Y > 0, \begin{bmatrix} A & B \\ I & 0 \end{bmatrix}' \begin{bmatrix} 0 & X + jY \\ X - jY & 0 \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + N < 0,$$

where  $N = \begin{bmatrix} 0 & -e^{j(\alpha - \frac{\pi}{2})} C' \\ -e^{-j(\alpha - \frac{\pi}{2})} C & -e^{-j(\alpha - \frac{\pi}{2})} D - e^{j(\alpha - \frac{\pi}{2})} D' \end{bmatrix}$ .

## VII. CONCLUSION

In this paper, we define the phase responses of frequency-wise sectorial MIMO LTI systems. We obtain a small phase theorem for feedback stability, a natural counterpart of the well-known small gain theorem. In addition, we derive a sectorized real lemma for phase-bounded systems, a counterpart of the bounded real lemma.

This paper focuses on stable LTI systems. We are currently trying to extend phase analysis to marginally stable MIMO LTI systems with possible zeros on the imaginary axis. Such extension will further generalize the existing passivity theory. Moreover, we wish to study the synthesis problem, aiming at solving an  $\mathcal{H}_\infty$ -phase optimal control problem, a counterpart of the  $\mathcal{H}_\infty$ -norm optimal control problem. How to apply and extend phase theory to large scale dynamical networks is also an interesting future work.

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