Stability of Networked Feedback System with Frequency-wise Bounded Uncertainty Quartets

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***Abstract*— In this paper, we study the robust stability of a networked control system (NCS) with the communication channels described by cascaded two-port networks. Distortions and interferences are taken into account at each two-port network between a pair of relays during the communication. Dynamic uncertainties are modeled within the cascaded two-port networks, where four types of uncertainties appear at one two-port network and we call them the uncertainty quartet. Specifically, we consider frequency-wise bounded uncertainties in transmission matrices of two-port networks. A necessary and sufficient condition for the robust stability of the NCS is given in the form of an "arcsin" inequality, which states that the NCS will be stable whenever the uncertainties are well bounded. Key ideas of the result are based on the geometric insights into the graphs of systems and key techniques are based on the analysis of subspaces.**

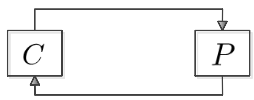

Fig. 1: A standard closed-loop system.

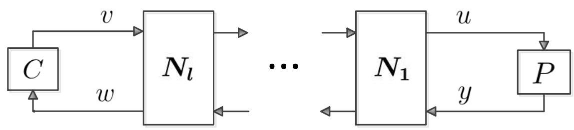

Fig. 2: An NCS with cascaded two-port connections.

I. Introduction

Uncertainties are everywhere. In accordance with different application scenarios, various types of uncertainty models have been proposed, such as dynamic uncertainties [1], [2], parametric uncertainties [2] and gap-type uncertainties [3]–[7]. A stable dynamic uncertainty is usually measured by its $\mathcal{H}_\infty$ norm, which is defined as the maximum amplitude of its transfer function on the imaginary axis. With this measure, several widely used uncertainty models have been proposed, including the additive/multiplicative uncertainties [1], the feedback uncertainty [2], the uncertainty quartet [8], [9] and so on. In terms of gap-type uncertainties, the gap (pointwise gap or ν-gap) metric defines a distance between the graphs of two linear time-invariant (LTI) systems, hence induces a measure of uncertainty.

Uncertainties in this study are assumed to be present within communication channels. Practically in most systems, the signals are transmitted through imperfect communication channels. Since the system stability and control performance heavily rely on the quality of communication channels, one should pay serious attention to possible communication uncertainties when modeling a feedback system, more specifically, a networked control system (NCS). An NCS differs from a standard closed-loop system as the information exchanged between the plant and controller is through communication networks [10]. The communication channels in an NCS can be modeled in a various ways so as to reveal actual situations. In this study, we give a two-port NCS model by extending the standard closed-loop system (Fig. 1) to the feedback system with cascaded two-port connections (Fig. 2). Based on the architecture of the two-port NCS, we measure the dynamic uncertainties, which are frequency-wise bounded, in the transmission matrices of the two-port networks. That is, the uncertainties are bounded by frequency-wise weighting functions differently at each two-port network. This uncertainty model applies to the situations when the plant and the controller cannot communicate directly and the signals can only pass through communication networks with several relays. Some direct motivating examples include satellite networks [11], wireless sensor networks [12] and other large-scale networks with multiple routing. Specifically, each sub-system between a pair of neighbouring relays represents a communication channel, which involves not only multiplicative distortions on the transmitted signal itself but additive interferences caused by the signal in the reverse direction. This phenomenon is usually encountered in a bidirectional wireless network subject to channel fading or under malicious attacks [13]. We wish to mention that instead of bounding the dynamic uncertainties uniformly with the $\mathcal{H}_\infty$ norm as in [8], [9], we bound the uncertainties frequency-wise because in many circumstances, we have prior knowledge about the uncertainties. For instances, we have estimations on the uncertainties in communication channels from pilot sequences or environment detection [14]. Moreover, stability conditions concerning the frequency-wise bounded uncertainties will generalize the results in [9].

So far, we have presented the architecture of our two-port NCS model and described the robust stabilization problem, where the uncertain cascaded two-port networks play a crucial role. A two-port network is not a new concept and

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has been studied over decades for different purposes. It was first introduced in electrical circuit theory [15]. Later, it was borrowed to characterize an LTI system with a chain-scattering representation in [16]. In the application of teleoperation in robotics, the two-port networks are used to model communication blocks between the human operator and the environment [17]. Such representations of two-port networks have also been used for studying feedback robustness from the perspective of the ν-gap metric [18]. Recently, approaches based on the two-port network to modeling communication channels in a networked feedback system are studied in [8] and [9].

The main contribution of our study is a concise result on analysing the stability of a feedback system with cascaded communication uncertainties. As we know, a general approach to handling the robust stabilization problem with multiple sources of uncertainties is through μ analysis, which is computationally difficult and even NP hard in the case of multiple block-diagonal uncertainties [1], [19]. Actually, we can avoid these difficulties by investigating the special structures of the two-port networks and taking advantage of geometric insights into the angles and rotations of subspaces.

It is worth noting that there are previous works on robust stabilization of NCSs with special communication architectures and various uncertainty descriptions. For example, [17] considers teleoperation of robots through two-port communication networks with time delay, [20] considers a plant with parametric uncertainties over networks subject to packet loss, [21] considers a plant with polytopic uncertainties in its coefficients over a communication channel subject to channel fading and so on. The main differences of this work from the previous works are that we model the dynamic uncertainties using uncertainty quartets and bound them in a frequency-wise manner.

The rest of the paper is organized as follows. In Section II, we introduce the notation system and the model of cascaded two-port networks. In Section III, we state the main theorem of this study. In Section IV, we conclude this study and show possible extensions of the results.

II. Problem Formulation

A. Notation

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be the real or complex field and $\mathbb{F}^n$ be the linear space of n-dimensional vectors over the field $\mathbb{F}$. For matrix $A \in \mathbb{F}^{m\times n}$, its conjugate transpose is denoted by A^* and its k-th singular value is denoted by $\sigma_k(A)$, for $k = 1, 2, \ldots, \min\{m, n\}$, in a nonincreasing order. The largest singular value is specially denoted as $\bar{\sigma}(A) := \sigma_1(A)$, and the smallest is denoted as $\underline{\sigma}(A) := \sigma_{\min\{m,n\}}(A)$. The range of A is denoted as $\mathcal{R}(A)$.

We assume that every system in this paper is continuous-time LTI system represented by its Laplace transfer matrix. The symbol s of transfer matrices may be omitted for simplicity. Denote by $\mathcal{H}_2$ ($\mathcal{H}_\infty$) the standard Hardy 2-spaces (∞-spaces). Let $\mathcal{RH}_\infty$ be the set consisting of all real rational members of $\mathcal{H}_\infty$. Denote by $\mathcal{P}$ the field of real rational transfer functions. For transfer matrix $P \in \mathcal{P}^{m\times n}$, its conjugate is denoted as $P^\sim(s) = P^T(-s)$. We say a transfer matrix $P \in \mathcal{P}^{m\times n}$ is stable if $P \in \mathcal{RH}_\infty^{m\times n}$, where the superscripts may be omitted for simplicity if it can be inferred from contexts.

Two transfer matrices M and N in $\mathcal{RH}_\infty$ are (right) coprime if there exist transfer matrices X and Y in $\mathcal{RH}_\infty$ such that

$$XM + YN = I.$$

It is known [22] that every $P \in \mathcal{P}^{m\times n}$ admits a right coprime factorization:

$$P = NM^{-1},$$

where $M, N \in \mathcal{RH}_\infty$. Later, when we write $P = NM^{-1}$, we always assume M, N are right coprime.

For a possibly unstable system P, assume its input as u and its output as y, then its graph is defined by the following set:

$$\mathcal{G}_P := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} : \ u \in \mathcal{H}_2, \ y = Pu \in \mathcal{H}_2 \right\}.$$

Following some simple argument, we obtain that

$$\mathcal{G}_P = \begin{bmatrix} M \\ N \end{bmatrix} \mathcal{H}_2,$$

where $\begin{bmatrix} M \\ N \end{bmatrix}$ is called the graph symbol of P.

The standard closed-loop system in Fig. 1 is denoted as $[P, C]$, where $P \in \mathcal{P}^{p\times m}$ represents the plant and $C \in \mathcal{P}^{m\times p}$ the controller. Let $n := m + p$.

Under the mild condition that $[P, C]$ is well-posed, i.e., $I - CP$ has full normal rank, the well-known "Gang of Four" transfer matrix [23] can be represented as

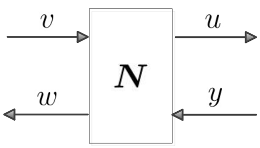
$$\text{GoF}(P, C) = \begin{bmatrix} I \\ P \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} I & -C \end{bmatrix}.$$

The closed-loop system $[P, C]$ is said to be stable if GoF(P, C) is stable, or GoF$(P, C) \in \mathcal{RH}_\infty^{n\times n}$. The quantity $\|\text{GoF}(P, C)\|_\infty^{-1}$ is called the robust stability margin of a stable closed-loop system $[P, C]$.

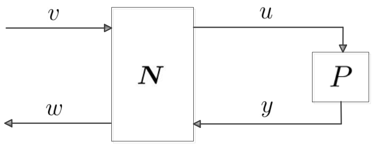
B. Two-Port Networks as Communication Channels

As mentioned, we utilize the two-port networks to characterize cascaded bidirectional communication channels with uncertainties. What is a two-port network? How to introduce uncertainties into the networks? We answer the questions in this part. First, we briefly introduce the two-port NCS model, which mostly follows from [9]. The network $\boldsymbol{N}$ in Fig. 3a has two external ports, with one port composed of v, w and the other of u, y, which is the reason it is called a two-port network. A two-port network $\boldsymbol{N}$ has various representations, from which we choose the transmission type representation. Define the transmission matrix T and the corresponding input-output relation as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \text{ and } \begin{bmatrix} v \\ w \end{bmatrix} = T \begin{bmatrix} u \\ y \end{bmatrix}. \quad (1)$$



(a) A single two-port network.



(b) A one-stage two-port connection.

Fig. 3: Two-port network $\boldsymbol{N}$.

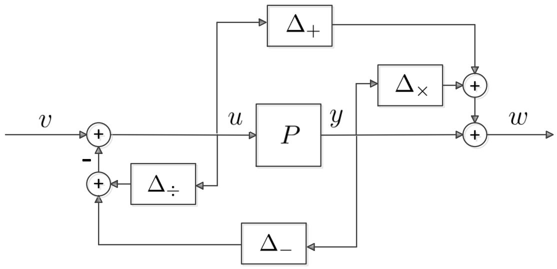


Fig. 4: Plant with the uncertainty quartet.

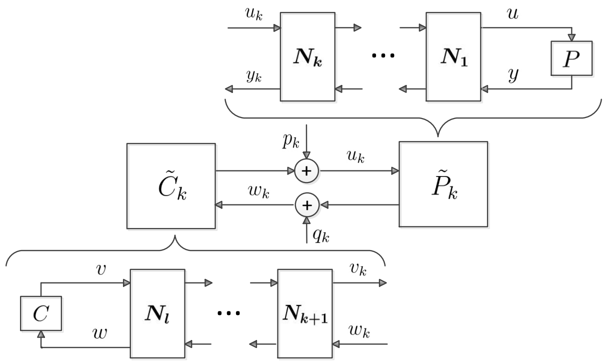


Fig. 5: Equivalent closed-loop system.

When the communication channel is perfect, i.e., communication without distortion or interference, the transmission matrix is

$$T = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}.$$

If the bidirectional channel admits both distortions and interferences, we can assume the transmission matrix to be

$$T = I + \Delta = \begin{bmatrix} I_m + \Delta_{\div} & \Delta_{-} \\ \Delta_{+} & I_p + \Delta_{\times} \end{bmatrix},$$

where $\Delta = \begin{bmatrix} \Delta_{\div} & \Delta_{-} \\ \Delta_{+} & \Delta_{\times} \end{bmatrix} \in \mathcal{RH}_\infty$.

The subscripts $\div, -, +, \times$ of Δ representing different types of uncertainties are firstly used in [24]. The four-block transfer matrix Δ is the uncertainty quartet [9]. As shown in Fig. 3b, we connect a two-port network $\boldsymbol{N}$ to the plant P and denote the transmission matrix of $\boldsymbol{N}$ as $T = I + \Delta$. It follows from [9] that a perturbed plant $\tilde{P}$ from v to w can be determined by a linear fractional transformation (LFT) as follows:

$$\begin{aligned} \tilde{P} &= \text{LFT}\left(\begin{bmatrix} I_m + \Delta_{\div} & \Delta_{-} \\ \Delta_{+} & I_p + \Delta_{\times} \end{bmatrix},\ P \right) \qquad (2) \\ &= [(I_p + \Delta_{\times})P + \Delta_{+}][I_m + \Delta_{\div} + \Delta_{-}P]^{-1}. \end{aligned}$$

Fig. 4 is the diagram showing how the uncertainties play different roles on perturbing the nominal plant P.

We will frequency-wise bound the uncertainty quartet by a scalar weighting function $W \in \mathcal{RH}_\infty$, i.e., $\bar{\sigma}[\Delta(j\omega)] \leq |W(j\omega)|$ for every $\omega \in \mathbb{R}$. It applies to the scenario when we have priori knowledge on the uncertainties Δ_k based on the frequency contents. For example, when we study a wireless communication channel, the frequencies that are occupied mostly will tend to suffer more interferences and some other frequencies may have better performance.

We have shown how to analyse one-stage two-port network. In order to analyse the cascaded two-port networks, we investigate the graphs of the plant and controller simultaneously. As illustrated in Fig. 2, the plant $P = NM^{-1}$ and controller $C = VU^{-1}$ communicate with each other through cascaded two-port networks. Considering the input and output of P, we can represent every element in the graph of P using its graph symbol as follows:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} x,$$

for every $x \in \mathcal{H}_2$.

Focusing on the k-th two-port network, we can equivalently transform the diagram in Fig. 2 to that in Fig. 5. If the k-th stage of the network admits an uncertainty $\Delta_k \in \mathcal{RH}_\infty$, then the transmission matrix is given as $T_k = I + \Delta_k$. Signals in Fig. 5 admit the following relations:

$$\begin{bmatrix} u_k \\ y_k \end{bmatrix} = \left(\prod_{j=1}^{k} T_{k+1-j} \right) \begin{bmatrix} u \\ y \end{bmatrix} = \left(\prod_{j=1}^{k} (I + \Delta_{k+1-j}) \right) \begin{bmatrix} u \\ y \end{bmatrix},$$

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = \left(\prod_{j=k+1}^{l} T_j^{-1} \right) \begin{bmatrix} v \\ w \end{bmatrix} = \left(\prod_{j=k+1}^{l} (I + \Delta_j)^{-1} \right) \begin{bmatrix} v \\ w \end{bmatrix}.$$

With these relations, we can determine the equivalent plants $\tilde{P}_k$ and equivalent controllers $\tilde{C}_k$ from their graphs.

The stability of the NCS is defined with an input-output manner, as follows:

Definition 1. *The NCS in Fig. 5 is said to be stable if for all input signals* $\{p_k\}_{k=1}^{l}$ *and* $\{q_k\}_{k=1}^{l} \in \mathcal{H}_2$, *it holds that the signals on all ports, namely,* $\{u_k\}_{k=1}^{l}$, $\{y_k\}_{k=1}^{l}$, $\{v_k\}_{k=1}^{l}$ *and* $\{w_k\}_{k=1}^{l}$ *are in* $\mathcal{H}_2$.

III. Robust Stability of Two-port NCS

Our main result is a necessary and sufficient condition for the networked feedback system to be robustly stable with frequency-wise bounded uncertainties.

Theorem 1. *Assume the nominal system $[P, C]$ is stable and $W_k \in \mathcal{RH}_\infty$. Then the NCS in Fig. 2 is robustly stable for all $\Delta_k \in \mathcal{RH}_\infty$ with $\bar{\sigma}[\Delta_k(j\omega)] \leq |W_k(j\omega)|$, $\forall \omega \in \mathbb{R}$, $k = 1, 2, \ldots, l$ if and only if for every $\omega \in \mathbb{R}$, it holds*

$$\sum_{k=1}^{l} \arcsin |W_k(j\omega)| < \arcsin\{\bar{\sigma}[\mathrm{GoF}(P, C)(j\omega)]\}^{-1}. \quad (3)$$

Similar to the stability margin we define in Section II, here we have $\{\bar{\sigma}[\mathrm{GoF}(P, C)(j\omega)]\}^{-1}$ as the frequency-wise stability margin. The larger the margin is at some frequency ω, the more robust the feedback system will be against the uncertainties at that frequency.

A similar result is given in [9], which states that given $r_k \in [0, 1)$, the two-port NCS is stable for all $\Delta_k \in \mathcal{RH}_\infty$ with $\|\Delta_k\|_\infty \leq r_k$ if and only if

$$\sum_{k=1}^{l} \arcsin r_k < \arcsin \|\mathrm{GoF}(P, C)\|_\infty^{-1}. \quad (4)$$

Actually, Theorem 1 generalizes the above result, which can be seen by taking $W_k = r_k \in [0, 1)$ in Theorem 1 and taking minimum at the right side of equation (3) over $\omega \in \mathbb{R}$.

The full proof of the main theorem is omitted due to the space limitation.

IV. Conclusion

We propose an NCS model with frequency-wise bounded uncertainty quartets in cascaded two-port communication channels. This model fits well the situation when we have prior knowledge on the transmission uncertainties. A necessary and sufficient condition is given, in terms of frequency-wise bounds $W_k \in \mathcal{RH}_\infty$, for the NCS to be robustly stable. We may extend the current work to include plant/controller uncertainties.

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