Stabilization of Cascaded Two-Port Networked Systems Against Nonlinear Perturbations

Di Zhao, Sei Zhen Khong, and Li Qiu

***Abstract*— A networked control system (NCS) consisting of cascaded two-port communication channels between the plant and controller is modeled and analyzed. It is shown that the robust stability of the two-port NCS can be guaranteed when the nonlinear uncertainties in the transmission matrices are sufficiently small in norm. The stability condition, given in the form of "arcsin" of the uncertainty bounds, is both necessary and sufficient.**

I. Introduction

Feedback is widely used for handling modeling uncertainties in the area of systems and control. Within a feedback loop, communication between the plant and controller plays an important role in that the achieved control performance and robustness heavily rely on the quality of communication. In practice, communication can never be ideal due to the presence of channel distortions and interferences. In this study, we analyze the robust stability of a feedback system involving bidirectional uncertain communication modeled by cascaded two-port networks.

Most control systems can be regarded as structured networks with signals transmitted through channels powered by various devices, such as sensors or satellites. A networked control system (NCS) differs from a standard closed-loop system in that the information is exchanged through a communication network [1]. The presence of such a network may introduce disturbances to a control system and hence significantly compromise its performance.

In this study, we introduce an NCS model, extending the standard linear time-invariant (LTI) closed-loop system (Fig. 1) to the feedback system with cascaded two-port connections (Fig. 2). We assume that the controller and plant are LTI while the two-port networks involve nonlinear perturbations on their transmission matrices. In terms of communication uncertainties, we model the transmission matrices as $\boldsymbol{T} = \boldsymbol{I} + \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}$ is a bounded nonlinear operator. Our formulation of robust stabilization problem is mainly motivated by the application scenario of stabilizing a feedback system where the plant and controller do not possess an ideal communication environment and their input-output signals can only be sent through communication networks with several relays, as in, for example, teleoperation systems [?], satellite networks [?], wireless sensor networks [?] and so on. Moreover, each sub-system between two neighbouring relays, representing a communication channel, may involve not only multiplicative distortions on the transmitted signal itself but also additive interferences caused by the signal in the reverse direction, which is usually encountered in a bidirectional wireless network subject to channel fading or under malicious attacks [2].

Two-port networks are not a new concept and have been studied for decades for different purposes. Historically, two-port networks were first introduced in electrical circuits theory [3]. Later on they were utilized to represent LTI systems in the so-called chain-scattering formalism [4], which is essentially a two-port network. Two-port representations have also been used for studying feedback robustness from the perspective of the ν-gap metric [?]. Recently, approaches based on the two-port network to modeling communication channels in a networked feedback system is studied in [5] and [?]. There, uncertain two-port connections are used to introduce channel uncertainties, based on which we propose our cascaded two-port communication model with nonlinear perturbations in this paper.

One of the main contributions of our study is a clean result for analyzing the stability of a networked feedback system with nonlinear uncertainties in the cascaded two-port networks. A general approach to robust stabilization of LTI systems with structured uncertainties is μ analysis, which is known to be computationally intractable in general in the presence of multiple uncertainties [6]. Furthermore, the two-port uncertainties in this study are nonlinear, which brings in an additional obstacle. To overcome these difficulties, we take advantage of the special two-port structures and make use of geometric insights on system stability via an input-output approach. By generalizing the "arcsin" theorem in [7] for a standard closed-loop system, we are able to give a concise necessary and sufficient robust stability condition for the two-port NCS. Moreover, the stability condition is scalable and computationally friendly, in the sense that when the topology of the two-port NCS is changed, the stability condition can be efficiently updated based only on the modified components. In terms of designing an optimal controller, it suffices to solve an $\mathcal{H}_\infty$ optimization problem, which is mathematically tractable.

It is worth noting that there exist previous works on robust stabilization of NCSs with special architectures and various uncertainty descriptions. For example, [?] considers teleoperation of robots through two-port communication

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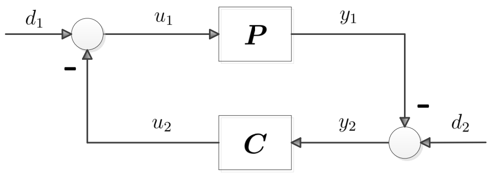


Fig. 1: A Standard Closed-Loop System

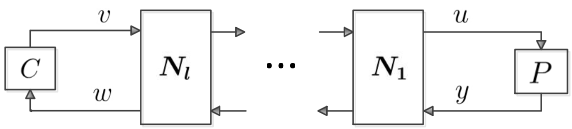


Fig. 2: Communication Channels Modeled by Cascaded Two-port Networks

networks with time-delay, [**?**] considers a plant with parametric uncertainties over networks subject to packet loss, [**?**] considers a plant with polytopic uncertainties in its coefficients over a communication channel subject to fadings and so on. The differences of our work from the previous ones are that our channel model characterizes bi-directional communication involving both distortions and interferences and these uncertainties may be nonlinear.

The rest of the paper is organized as follows. First in Section II, we define open-loop stability, closed-loop well-posedness, and closed-loop stability. Thereafter in Section III, we present the results on the robust stability of cascaded two-port networks. In Section IV, we conclude this study and summarize our contributions.

II. Preliminaries

A. Open-loop Stability

Let $\mathcal{H}_2^n := \{f : [0,\infty) \to \mathbb{R}^n \mid \|f\|_2^2 := \int_0^\infty |f(t)|^2\, dt < \infty\}$, where $|\cdot|$ denotes the Euclidean norm. Let $\mathcal{RH}_\infty$ consist of all the real rational members of $\mathcal{H}_\infty$, the Hardy ∞-space of functions that are holomorphic on the right-half complex plane.

Denote the time truncation operator at time $\tau \in [0,\infty)$ as $\boldsymbol{T}_\tau$, such that for $u(t) \in \mathcal{H}_2$,

$$(\boldsymbol{T}_\tau u)(t) = \begin{cases} u(t), & 0 \le t < \tau; \\ 0, & \text{Otherwise.} \end{cases}$$

A nonlinear system is represented by an operator $\boldsymbol{P} : \text{dom}(\boldsymbol{P}) \subset \mathcal{H}_2 \mapsto \mathcal{H}_2$ with domain $\text{dom}(\boldsymbol{P}) = \{u \in \mathcal{H}_2 \mid \boldsymbol{P}u \in \mathcal{H}_2\}$. We denote its image as $\text{img}(\boldsymbol{P})$. A physical system should additionally be causal, which is defined as follows [**?**].

Definition 1. *A nonlinear system* $\boldsymbol{P} : \text{dom}(\boldsymbol{P}) \subset \mathcal{H}_2 \mapsto \mathcal{H}_2$ *is said to be causal if for every* $\tau \in [0,\infty)$ *and* $u_1, u_2 \in \text{dom}(\boldsymbol{P})$*,*

$$\boldsymbol{T}_\tau u_1 = \boldsymbol{T}_\tau u_2 \Rightarrow \boldsymbol{T}_\tau \boldsymbol{P} u_1 = \boldsymbol{T}_\tau \boldsymbol{P} u_2$$

We assume $\boldsymbol{P}0 = 0$ throughout this study, which means every nonlinear system we consider has zero output whenever the input is zero. The finite-gain stability of a system is defined as follows [**?**].

Definition 2. *A causal nonlinear operator (system)* $\boldsymbol{P}$ *is said to be (finite-gain) stable if* $\text{dom}(\boldsymbol{P}) = \mathcal{H}_2$ *and its operator norm is bounded, that is*

$$\|\boldsymbol{P}\| := \sup_{0 \ne x \in \mathcal{H}_2} \frac{\|\boldsymbol{P}x\|_2}{\|x\|_2} < \infty.$$

B. Closed-loop Stability

We consider a standard closed-loop system in Fig. 1 with plant $\boldsymbol{P} : \text{dom}(\boldsymbol{P}) \subset \mathcal{H}_2^p \mapsto \mathcal{H}_2^m$ and controller $\boldsymbol{C} : \text{dom}(\boldsymbol{C}) \subset \mathcal{H}_2^m \mapsto \mathcal{H}_2^p$. In the following, the superscripts of $\mathcal{H}_2^m$ and $\mathcal{H}_2^p$ will be omitted for notational simplicity.

The graph of $\boldsymbol{P}$ is defined as

$$\mathcal{G}_{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{P} \end{bmatrix} \text{dom}(\boldsymbol{P})$$

and similarly the inverse graph of $\boldsymbol{C}$ is defined as

$$\mathcal{G}'_{\boldsymbol{C}} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{I} \end{bmatrix} \text{dom}(\boldsymbol{C}),$$

both of which are assumed to be closed in this study.

It can be seen in [**?**], [**?**], [8] that various versions of feedback well-posedness may be assumed based on different signal spaces and causality requirements. In this study, we adopt the well-posedness definition from [8] without appealing to extended spaces, by contrast to, for example, [**?**], [**?**].

Definition 3. *The closed-loop system* $[\boldsymbol{P}, \boldsymbol{C}]$ *is said to be well-posed if*

$$\begin{aligned} \boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}} &: \text{dom}(\boldsymbol{P}) \times \text{dom}(\boldsymbol{C}) \mapsto \mathcal{H}_2 \\ &:= \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{C} \\ \boldsymbol{P} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} u_1 \\ y_2 \end{bmatrix} \end{aligned}$$

is causally invertible on $\text{img}(\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}})$*.*

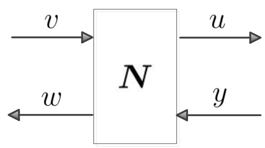
Correspondingly, the stability of the closed-loop system is defined as follows:

Definition 4. *A well-posed closed-loop system* $[\boldsymbol{P}, \boldsymbol{C}]$ *is (finite-gain) stable if* $\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}$ *is surjective and* $\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}^{-1}$ *is finite-gain stable.*

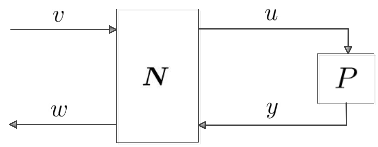
When $\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}$ is surjective, the parallel projection operators [**?**] along $\mathcal{G}_{\boldsymbol{P}}$ and $\mathcal{G}'_{\boldsymbol{C}}$, $\Pi_{\mathcal{G}_{\boldsymbol{P}} /\!/ \mathcal{G}'_{\boldsymbol{C}}}$ and $\Pi_{\mathcal{G}'_{\boldsymbol{C}} /\!/ \mathcal{G}_{\boldsymbol{P}}}$, can be defined respectively as

$$\begin{aligned} \Pi_{\mathcal{G}_{\boldsymbol{P}} /\!/ \mathcal{G}'_{\boldsymbol{C}}} &: \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathcal{H}_2 \mapsto \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \in \mathcal{G}_{\boldsymbol{P}} \\ &= \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{bmatrix} \boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{I} \end{bmatrix}, \end{aligned} \tag{1}$$

$$\begin{aligned} \Pi_{\mathcal{G}'_{\boldsymbol{C}} /\!/ \mathcal{G}_{\boldsymbol{P}}} &: \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in \mathcal{H}_2 \mapsto \begin{bmatrix} u_2 \\ y_2 \end{bmatrix} \in \mathcal{G}'_{\boldsymbol{C}} \\ &= \begin{bmatrix} -\boldsymbol{I} & 0 \\ 0 & \boldsymbol{I} \end{bmatrix} \boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}^{-1} + \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2}$$



(a) A Single Two-Port Network



(b) One-Stage Two-Port Connection

Fig. 3: Two-Port Networks: an Illustration

It follows that every $w \in \mathcal{H}_2$ has a unique decomposition as $w = m + n$ with $m = \Pi_{\mathcal{G}_{\boldsymbol{P}} /\!/ \mathcal{G}'_{\boldsymbol{C}}} w \in \mathcal{G}_{\boldsymbol{P}}$ and $n = \Pi_{\mathcal{G}'_{\boldsymbol{C}} /\!/ \mathcal{G}_{\boldsymbol{P}}} w \in \mathcal{G}'_{\boldsymbol{C}}$.

The next proposition bridges the finite-gain stability and the boundedness of parallel projections [?].

Proposition 1. *A well-posed closed-loop system $[\boldsymbol{P}, \boldsymbol{C}]$ is stable if and only if $\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}$ is surjective and $\Pi_{\mathcal{G}_{\boldsymbol{P}} /\!/ \mathcal{G}'_{\boldsymbol{C}}}$ or $\Pi_{\mathcal{G}'_{\boldsymbol{C}} /\!/ \mathcal{G}_{\boldsymbol{P}}}$ is finite-gain stable.*

For a finite-gain stable closed-loop system $[\boldsymbol{P}, \boldsymbol{C}]$, its stability margin is defined as $b_{\boldsymbol{P},\boldsymbol{C}} := \|\Pi_{\mathcal{G}_{\boldsymbol{P}} /\!/ \mathcal{G}'_{\boldsymbol{C}}}\|^{-1}$. It is shown in [?] that if either $\boldsymbol{P}$ or $\boldsymbol{C}$ is linear, then $b_{\boldsymbol{P},\boldsymbol{C}} = b_{\boldsymbol{C},\boldsymbol{P}}$.

III. Networked Robust Stabilization with Cascaded Nonlinear Uncertainties

A. Two-Port Networks as Communication Channels

The use of two-port networks as a model of communication channels is adopted from [?], [5]. Two-port networks were first introduced and investigated in electrical circuits theory [3]. The network $\boldsymbol{N}$ in Fig. 3a has two external ports, with one port composed of v, w and the other of u, y, and is called a two-port network. A two-port network $\boldsymbol{N}$ may have various representations, out of which we choose the transmission type to model a communication channel. Define the transmission matrix $\boldsymbol{T}$ as

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_{11} & \boldsymbol{T}_{12} \\ \boldsymbol{T}_{21} & \boldsymbol{T}_{22} \end{bmatrix} \text{ and } \begin{bmatrix} v \\ w \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} u \\ y \end{bmatrix}. \tag{3}$$

When the communication channel is perfect, i.e., communication takes place without distortion or interference, the transmission matrix is simply

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I}_m & 0 \\ 0 & \boldsymbol{I}_p \end{bmatrix}.$$

If the bidirectional channel admits both distortions and interferences, we can let the transmission matrix take the form

$$\boldsymbol{T} = \boldsymbol{I} + \boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{I}_m + \boldsymbol{\Delta}_\div & \boldsymbol{\Delta}_- \\ \boldsymbol{\Delta}_+ & \boldsymbol{I}_p + \boldsymbol{\Delta}_\times \end{bmatrix},$$

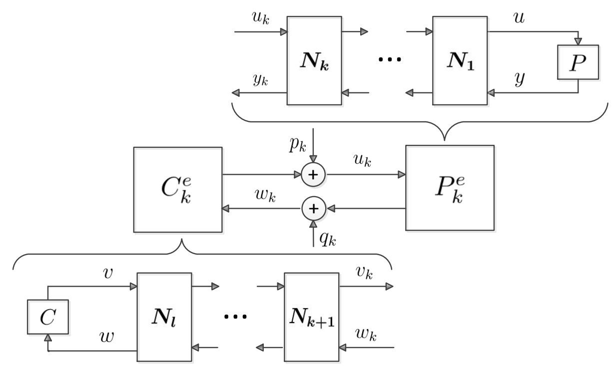


Fig. 4: Equivalent Plant and Controller

where $\boldsymbol{I} : \mathcal{H}_2 \mapsto \mathcal{H}_2$ is the identity operator and

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{\Delta}_\div & \boldsymbol{\Delta}_- \\ \boldsymbol{\Delta}_+ & \boldsymbol{\Delta}_\times \end{bmatrix} : \mathcal{H}_2 \mapsto \mathcal{H}_2$$

satisfies $\|\boldsymbol{\Delta}\| \le r < 1$, which ensures that $\boldsymbol{T}$ is stably invertible. The four-block matrix $\boldsymbol{\Delta}$ is called the uncertainty quartet.

B. Graph Analysis on Cascaded Two-Port NCS

Consider the transmission type representation of the two-port networks $\{\boldsymbol{N}_k\}_{k=1}^{l}$. If the k-th stage of the network admits a stable nonlinear uncertainty $\boldsymbol{\Delta}_k$, then the transmission matrix is given as $\boldsymbol{T}_k = \boldsymbol{I} + \boldsymbol{\Delta}_k$. Signals in Fig. 4 have the following relations:

$$\begin{bmatrix} u_k \\ y_k \end{bmatrix} = \left(\prod_{j=1}^{k} \boldsymbol{T}_{k+1-j} \right) \begin{bmatrix} u \\ y \end{bmatrix} = \left(\prod_{j=1}^{k} (\boldsymbol{I} + \boldsymbol{\Delta}_{k+1-j}) \right) \begin{bmatrix} u \\ y \end{bmatrix},$$

$$\begin{bmatrix} v_k \\ w_k \end{bmatrix} = \left(\prod_{j=k+1}^{l} \boldsymbol{T}_j^{-1} \right) \begin{bmatrix} v \\ w \end{bmatrix} = \left(\prod_{j=k+1}^{l} (\boldsymbol{I} + \boldsymbol{\Delta}_j)^{-1} \right) \begin{bmatrix} v \\ w \end{bmatrix}.$$

If we view P together with $\{\boldsymbol{N}_j\}_{j=1}^{k}$ as an equivalent plant $\boldsymbol{P}_k^e$ with uncertainties $\{\boldsymbol{\Delta}_j\}_{j=1}^{k}$, then the graph of $\boldsymbol{P}_k^e$ is given by

$$\mathcal{G}_{\boldsymbol{P}_k^e} = \left(\prod_{j=1}^{k} (\boldsymbol{I} + \boldsymbol{\Delta}_{k+1-j}) \right) \mathcal{G}_P. \tag{4}$$

Similarly, if we view C together with $\{\boldsymbol{N}_j\}_{j=k+1}^{l}$ as an equivalent controller $\boldsymbol{C}_k^e$ with uncertainties $\{\boldsymbol{\Delta}_j\}_{j=k+1}^{l}$, then the graph of $\boldsymbol{C}_k^e$ is

$$\mathcal{G}'_{\boldsymbol{C}_k^e} = \left(\prod_{j=k+1}^{l} (\boldsymbol{I} + \boldsymbol{\Delta}_j)^{-1} \right) \mathcal{G}'_C. \tag{5}$$

For convenience, we regard $k = 0$ as the situation when P is isolated from the two-port networks and $k = l$ when C is isolated.

C. Robust Stability Condition

With the equivalent plant and controller representations derived aforehand, next we extend the definition on the stability of the two-port NCS in [**?**] to the nonlinear case.

As shown in Fig. 4, we denote the k-th input pair as $I_k := [p_k, q_k]^T$, the k-th output pair as $O_k := [u_k, w_k]^T$ and the set of all outputs as $O := [u_1, w_1, u_2, w_2, \ldots, u_l, w_l]^T$. By the feedback well-posedness assumption, the map from input I_k to output O exists and we denote it as $\boldsymbol{A}_k : I_k \in \mathcal{H}_2 \mapsto O \in \mathcal{H}_2$.

Definition 5. *The two-port NCS in Fig. 4 is said to be stable if the operator $\boldsymbol{A}_k$ is finite-gain stable for every $k = 0, 1, \ldots, l$.*

Given the stability definition, we present next the main robust stability theorem involving nonlinear perturbations in a two-port NCS.

In the following we assume that every closed-loop system $[\boldsymbol{P}, \boldsymbol{C}]$ is well-posed and $\boldsymbol{F}_{\boldsymbol{P},\boldsymbol{C}}$ is surjective. Hence from Proposition 1, the stability of $[\boldsymbol{P}, \boldsymbol{C}]$ is equivalent to the finite-gain stability of $\Pi_{\mathcal{G}_{\boldsymbol{P}} /\!/ \mathcal{G}'_{\boldsymbol{C}}}$. Let nominal LTI closed-loop system $[P, C]$ be stable.

Theorem 1. *The two-port NCS is finite-gain stable for all $\{\boldsymbol{\Delta}_k\}_{k=1}^{l}$ subject to $\|\boldsymbol{\Delta}_k\| \leq r_k$ if and only if*

$$\sum_{k=1}^{l} \arcsin r_k < \arcsin b_{P,C}. \tag{6}$$

The development of the full proof of the above theorem can be referred to [**?**]. We omit it here due to space limitation. From the above theorem, we know the stability margin $b_{P,C}$ is the same as that in a standard closed-loop system with "gap" uncertainties [6], [7], [9], hence the synthesis problem of a two-port NCS can be solved by an $\mathcal{H}_\infty$ optimization. In addition, the synthesis is irrelevant to detailed requirements of communication channels between the plant and controller, such as the number of two-port connections and how the uncertainty bounds are distributed among all the channels, which provide more flexibility on the selection of the communication channels.

IV. Conclusion

We investigate networked robust stabilization problem concerning LTI systems perturbed by nonlinear uncertainties. A necessary and sufficient stability condition is given in the form of an "arcsin" inequality. As far as controller synthesis is concerned, the problem can be solved through an $\mathcal{H}_\infty$ optimization regarding the closed-loop stability margin.

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